JOSE FAUSTINO SANCHEZ CARRION NATIONAL UNIVERSITY

FACULTY OF CIVIL ENGINEERING

PROFESSIONAL SCHOOL OF CIVIL ENGINEERING

THESIS:

GLOBAL STRUCTURAL ANALYSIS OF TALL BUILDINGS BY THE CONTINUOUS METHOD AND TRANSFER MATRIX METHOD USING ENERGY FORMULATION

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF

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PRESENTED BY

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UNDERGRADUATE THESIS

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DEDICATION

To Jesus Christ, KING of kings and LORD of lords. "*Commit your works to the Lord, and your plans will be established"* (Proverbs 16-3).

With much affection and immense love, to my daughter Zoé Juliette, with the hope that this research project will serve as a stimulus for her personal and professional development. *"Be the change you want to see in the world"* Mahatma Ghandi.

To my parents Julio and Lucia, for being the support of my life throughout all these years. I deeply appreciate the advice at each stage of my personal and professional development, for the care and love that you have always shown me and above all for your patience and sacrifice.

To my siblings Yoselin, Marisol, Jeanpierre, Luana and nephews Elian and Gael; for the trust they selflessly place in me and for the moments lived at each stage of their lives.

With much love and deep respect to my grandmother Cirila and to the memory of my grandfather Corpus, for the love and example they gave me in my childhood.

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Mao Cristian, Pinto Cruz

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Last but not least, to my alma mater, Jose Faustino Sanchez Carrion National University, for the university education provided.

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NOMENCLATURE

E	Modulus of elasticity of the material
G	Shear modulus of the material
K_b	Flexural stiffness
K_s	Shear stiffness
K_{b1}	Global bending stiffness
K_{b2}	Stiffness to local bending
K_{s1}	Global shear stiffness
K_{s2}	Local shear stiffness
K_θ	Torsional stiffness
G_{eq}	Equivalent stiffness of the connecting beam
I_w	Moment of inertia of the shear wall
I_c	Moment of inertia of the column
I_b	Moment of inertia of the beam
I_w	Warping torsion constant
J	Saint-Venant torsion constant
$\bar{J}$	Constant that represents the effect of the connection beam
A_w	Cross-sectional area of the shear wall
A_c	Cross-sectional area of the column
A_0	Area enclosed by the mean center lines of the thin wall sections
k	Form factor
h	Floor height

h_b	Depth of the beam
l	Length to axes of the column or shear wall
l^*	Free length between shear walls or columns
s	Vertical distance between connecting beams
v_i	Core thickness
w	Number of shear walls
c	Number of columns
b	Number of beams
t_w	Shear wall width
t_c	Column width
t_b	Beam width
h_b	Depth of the connecting beam
c_i	Distance from the centroid of the area of the columns or shear walls to the centroid of each column or shear wall
S_1	Length of the left shear wall
S_2	Left shear wall length
$M_{(x)}$	Bending moment
$V_{(x)}$	Shear force
$M_{(0)}$	Bending moment at the ends of the shear wall
ρ	Mass density

ABSTRACT

The structural analysis of a tall building can be solved by a local or global analysis. It is known that the global response is not the simple superposition of responses of the individual structural systems and is important to carry out a global structural analysis of the tall building that considers the complex three-dimensional interaction between the structural systems. Although technological advancement has made a full structural analysis using commercial finite element packages easy to obtain, at an early stage of the project structural engineers need to make quick decisions and the use of complex 3D models can be time consuming, be impractical and expensive. In contrast, the use of approximate methods such as the continuous method and the transfer matrix method substituting a tall building as an equivalent replacement beam drastically reduces the degrees of freedom of the structure, involves a minimal amount of time, and allows concentrate the analysis on the most important structural features and ignore those that have no significant (and sometimes no) influence on the structural response.

The main objective of this research project is to develop and propose an analytical procedure for the global structural analysis of the tall building that involves a static, dynamic and stability analysis based on the continuous method and the transfer matrix method using an energy formulation. The mathematical formulation using the continuum method provides closed form solutions for the static, dynamic and stability structural analysis of regular tall buildings and the joint use of the continuum method and the transfer matrix method allows to evaluate the structural analysis of tall buildings that present structural variability in height. The simplified model is used to calculate the lateral displacement profile, the maximum displacement, the interstory drifts and the global drift in the static case; to calculate the frequencies and periods in the dynamic case and to calculate the global critical buckling load in the stability case.

This investigation is divided into two parts. The first part presents the mathematical procedure that leads to closed solutions of the static, dynamic and stability structural analysis of replacement beam models suitable for each structural element such as frames, shear walls, coupled shear walls, cores and strategies are also analyzed to represent the tall building by a single replacement beam with its characteristic stiffnesses. The second part presents an analysis of the

accuracy and reliability of the models developed by comparing the results of the continuous method and the transfer matrix method with the finite element method. The results of the investigation demonstrate excellent accuracy and reliability of the application of the continuous models developed for the structural systems and for the tall building.

As a general conclusion, the analytical procedure proposed in this thesis for the global structural analysis of the tall building has proven to be a very reliable procedure in its precision and involves a reduced processing time, which makes it convenient to be implemented in engineering offices such as an excellent alternative for structural analysis of tall buildings at a preliminary stage and as a verification method at the final stage of the project.

Keywords: Tall building, replacement beam, continuous method, transfer matrix method, energy formulation, Hamilton's principle, static structural analysis, dynamic structural analysis, structural stability analysis.

RESUMEN

El análisis estructural de un edificio alto puede resolverse mediante un análisis local o global. Se conoce que la respuesta global no es la simple superposición de respuestas de los sistemas estructurales individuales y es importante realizar un análisis estructural global del edificio alto que considere la compleja interacción tridimensional entre los sistemas estructurales. A pesar de que el avance tecnológico ha contribuido a que un análisis estructural completo utilizando paquetes comerciales de elementos finitos sea fácil de obtener, en una etapa temprana del proyecto los ingenieros estructurales necesitan tomar decisiones rápidas y el uso de modelos tridimensionales complejos pueden demandar mucho tiempo y resultar poco práctico y costoso. Por el contrario, el uso de métodos aproximados como el método continuo y el método de matriz de transferencia que sustituye un edificio alto como una viga de reemplazo equivalente, reduce drásticamente los grados de libertad de la estructura, involucra una mínima cantidad de tiempo y permite concentrar el análisis en las características estructurales más importantes e ignorar aquellas que no tienen una influencia importarte (y a veces nula) en la respuesta estructural.

El objetivo principal de este proyecto de investigación es desarrollar y proponer un procedimiento analítico para el análisis estructural global del edificio alto que involucra un análisis estático, dinámico y de estabilidad basado en el método continuo y el método de matriz de transferencia utilizando una formulación energética. La formulación matemática utilizando el método continuo proporciona soluciones de forma cerrada para el análisis estructural estático, dinámico y de estabilidad de edificios altos regulares y la utilización conjunta del método continuo y el método de matriz de transferencia permite evaluar el análisis estructural de los edificios altos que presentan variabilidad estructural en altura. El modelo simplificado se utiliza para calcular el perfil de desplazamiento lateral, el desplazamiento máximo, las derivas de entrepiso y la deriva global en el caso estático; para calcular las frecuencias y los periodos en el caso dinámico y para calcular la carga crítica global de pandeo en el caso de estabilidad.

Esta investigación se divide en dos partes. La primera parte presenta el procedimiento matemático que conduce a soluciones cerradas del análisis estructural estático, dinámico y de estabilidad de modelos de vigas de reemplazo adecuados a cada elemento estructural como

pórticos, muros de corte, muros de corte acoplado, núcleos y también se analiza estrategias para representar al edificio alto mediante una sola viga de reemplazo con sus rigideces características. La segunda parte presenta un análisis de precisión y fiabilidad de los modelos desarrollados comparando los resultados del método continuo y método de matriz de transferencia con el método de elementos finitos. Los resultados de la investigación demuestran una excelente precisión y fiabilidad de la aplicación de los modelos continuos desarrollados para los sistemas estructurales y para el edificio alto.

Como conclusión general, el procedimiento analítico propuesto en este proyecto de investigación para el análisis estructural global del edificio alto ha demostrado ser un procedimiento muy confiable en su precisión e involucra un reducido tiempo de procesamiento, lo que lo hace conveniente para implementarse en las oficinas de ingeniería como una excelente alternativa para el análisis estructural de edificios altos en una etapa preliminar y como un método de verificación en la etapa final del proyecto.

Palabras claves: Edificio alto, viga de reemplazo, método continuo, método de matriz de transferencia, formulación energética, principio de Hamilton, análisis estructural estático, análisis estructural dinámico, análisis estructural de estabilidad.

INTRODUCTION

Numerous approximate and exact analysis methods have been developed in the literature to assess the overall structural analysis of tall buildings. In this research project an analytical procedure is presented that allows the static, dynamic and stability structural analysis of the tall building to be carried out in a practical way and in a shorter time using the continuous method and the transfer matrix method with an energetic formulation.

The continuous method leads to closed solutions of structural analysis for buildings that are regular in height; that is, whose structural properties do not change along the height of the building. However, not all tall buildings are regular in height for structural, aesthetic, and cost reasons. The joint application of the continuous method and the transfer matrix method allows evaluating the structural analysis of tall buildings that present structural variability in height. In this way, the analytical procedure developed in this research project allows to evaluate the global structural analysis of tall buildings that are regular and irregular in height.

This research project is divided into seven chapters, where each one is dedicated to a particular aspect of the research; however, chapter four "results" contains all the mathematical support of this research.

Chapter 1 covers the problem statement, which describes the problematic reality, the formulation of the problem, the objectives, the justification and the delimitation of the research study.

Chapter 2 contains the theoretical framework, which describes the background of the research where an exhaustive study of international research related to the analysis of tall buildings has been carried out through the continuous method and the transfer matrix method, given the theoretical bases that provide an overview of the tall building and the structural systems to be studied, the existing replacement beam models in the literature are described and the concepts associated with the study of this research project are described.

Chapter 3 describes the methodology applied to this research project, which contains the design, the population, the sample, the data collection and information processing techniques.

Chapter 4 presents the results of the research on the global structural analysis of the replacement beams and the tall building and is divided into five parts:

- Static analysis of individual structural systems. The mathematical development of the static analysis of thirteen uniform and staggered replacement beams that represent the behavior of structural systems is presented. The static analysis has as main objective to calculate the lateral displacement profile, the maximum displacement, the drifts of stories and the global drift of the replacement beams subjected to a general lateral load.

- Dynamic analysis of individual structural systems. The mathematical development of the dynamic analysis of thirteen uniform and stepped replacement beams that represent the behavior of the structural systems is presented. The main objective of the dynamic analysis is to calculate the frequency, the period, the eigenvalues and the mode shapes of the replacement beams subjected to a vertical load that can be uniform or variable in height.

- Stability analysis of individual structural systems. The mathematical development of the stability analysis of thirteen uniform and stepped replacement beams that represent the behavior of the structural systems is presented. The stability analysis has as its only primary objective to calculate the global critical buckling load of the replacement beams subjected to a vertical load that can be uniform or variable in height.

- Global analysis of the tall building. The mathematical development to represent the entire tall building by a single suitable replacement beam is presented and its characteristic stiffnesses are calculated. The analysis is applied directly to buildings that are symmetrical in plan (they do not present torsion effects); However, in the case of asymmetrical buildings (they present torsion effects), the analogy known as "Vlasov analogy" is used.

- Numerical applications. To demonstrate the efficiency of the proposed analytical procedure, the global analysis of structural systems and tall buildings is developed. The comparison of the results of the approximate method and the finite element method demonstrate the efficiency of the proposed formulation.

Chapter 5 presents the discussion of the research results.

Chapter 6 is dedicated to the conclusions and a description of the results achieved, referring to personal contributions and possible future research.

Finally chapter 7 contains the extensive source of information. It was important to give ample space to the bibliographical references in order to provide the reader with an efficient starting point to deepen future research topics.

1 STATEMENT OF THE PROBLEM

1.1 DESCRIPTION OF THE PROBLEMATIC REALITY

Tall buildings have become increasingly popular in densely populated cities and have represented the symbol of urban development of nations in many countries around the world; this popularity is mainly due to the rapid growth of economic activities, high demand for housing and limited land. Even with the technological advances in computer analysis, such as increasingly powerful computers and sophisticated software packages, high computational effort and high economic resources are required to perform the structural analysis of a tall building. Furthermore, the horizontal stiffness of a tall building cannot be considered as the simple sum of the individual stiffnesses of the structural elements because the overall stiffness of the tall building ensures that the structural elements work together and develop a complex structural interaction. As a consequence, it is of great interest to develop a structural analysis methodology with a global approach where the tall building can be idealized as a continuous beam and where the stiffnesses and kinematic fields associated to the continuous beam can represent as real as possible the structural characteristics and behavior of the tall building.

The global structural analysis of tall buildings can be solved by two different types of methods: the exact method (full model) and the approximate method (condensed model). The exact method is based on a mathematical model as accurate as possible considering many individual structural elements, material properties, and geometric and stiffness characteristics, resulting in a highly redundant indeterminate structure. On the other hand, the approximate method must be based on the most important structural features and ignore those that do not have a significant (and sometimes zero) influence on the structural response. The finite element method is an example of the exact method (full model). In contrast, one of the most commonly used approximate methods (condensed model) is the continuous method and the transfer matrix method.

The continuous method (CM) assumes that all horizontal elements connecting the vertical components are effectively connected over the height of the building to produce a continuous connecting means; i.e., the connecting beams are replaced by a system of uniformly distributed plates (Figure 1). As a consequence, the three-dimensional (3D) structure leads to a replacement beam (RB) that is characterized by equivalent stiffnesses and kinematic fields (Figure 2). It has been widely used in the literature to analyze structures whose structural properties do not vary with building height. The transfer matrix method (TMM) has been widely used to solve differential equations with discontinuities, applied to the structural field it allows to analyze continuous systems with varying and/or discontinuous structural properties with the height of the building by transforming the boundary conditions into initial conditions and thus allows to express the equations as a function of the initial constants.

At an early stage of the structural project, engineers need to make quick decisions and often opt for complex three-dimensional models that are impractical. Analyzing tall buildings using the continuous method and the transfer matrix method is justified because it drastically reduces the degrees of freedom of the structure. Any errors in the structural modelling and the introduction of the applied loads will lead to erroneous and inaccurate results of the analysis; moreover, in complex structures, depending on the experience of the structural engineer, it becomes difficult to investigate and find the errors within the massive output data of the discrete method results (finite element method). In addition to this point, the structural analysis by the continuous method allows for verifying the results obtained from the discrete method, which is advantageous because both methods follow a different mathematical nature.

Perhaps the best way to analyze structures is to employ both methods: in the preliminary design phase, the continuous method and the transfer matrix method can quickly help identify key project parameters and establish structural dimensions. In contrast, in the final design phase, the discrete method allows for more detailed analysis through more accurate calculations.

In this sense, the use of approximate methods such as the continuous medium method and the transfer matrix method allows the analysis of the structures with a global approach in a relatively simple way. It allows the structural engineer to understand the correct complex behaviour of tall buildings and to know which key parameters and characteristics have a dominant role in the behaviour of the tall building.

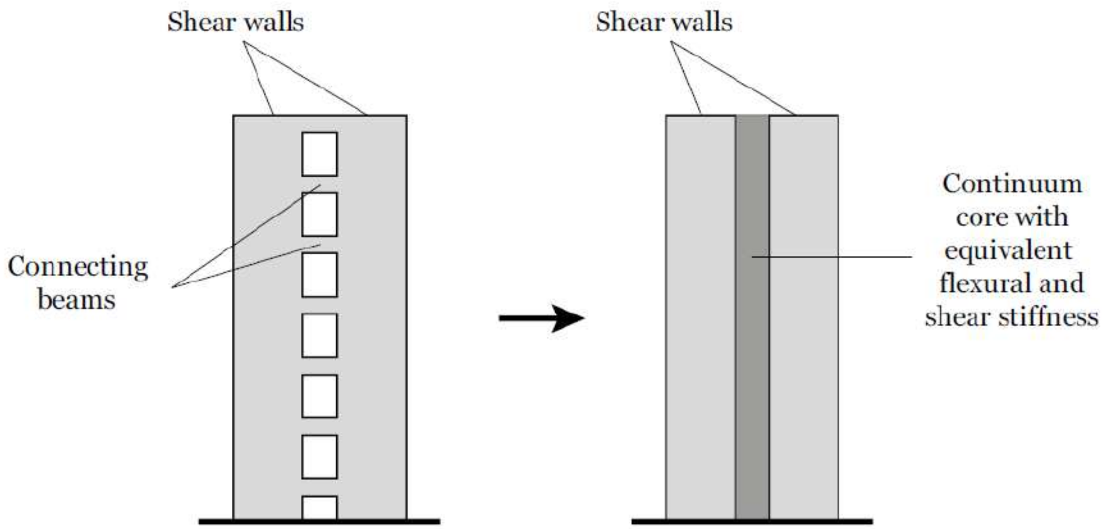


Figure 1. Coupled shear wall. a) With discrete connecting beams, b) With continuous connecting beams (Migliorati & Mangione, 2015).

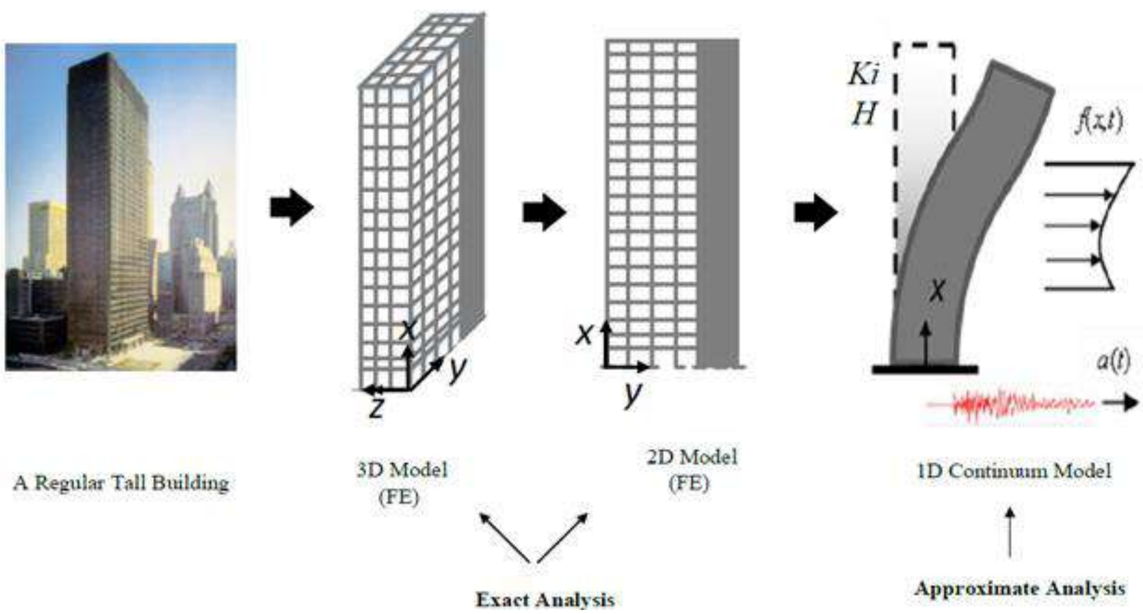


Figure 2. RB idealization process from 3D model to 1D model (Moghadasi, 2015).

Within this context, this research thesis focuses on five main topics within the global structural analysis of tall buildings: the development of continuous models that will lead to a replacement beam (RB), the development of a methodology for the static structural analysis of the tall building, the development of a methodology for the dynamic structural analysis of the tall building, the development of a methodology for the structural stability analysis of the tall building, and the definition of damage indicators for the assessment of the vulnerability of the tall building.

1.2 FORMULATION OF THE PROBLEM

1.2.1 General problem

- Will it be possible to develop a methodology for global structural analysis of tall buildings by the continuous method and the transfer matrix method using an energy formulation?

1.2.2 Specific problems

- Will it be possible to develop a methodology for global static structural analysis of the tall building by the continuous method and the transfer matrix method using an energy formulation?
- Will it be possible to develop a methodology for dynamic global structural analysis of the tall building by the continuous method and the transfer matrix method using an energy formulation?
- Will it be possible to develop a global structural analysis methodology of tall building stability by the continuous method and the transfer matrix method using an energy formulation?

1.3 RESEARCH OBJECTIVES

1.3.1 General Objective

- Develop a methodology for global structural analysis of tall buildings by the continuous method and the transfer matrix method using an energy formulation.

1.3.2 Specific objectives

- Develop a methodology for global static structural analysis of the tall building by the continuous method and the transfer matrix method using an energy formulation.
- Develop a methodology for dynamic global structural analysis of the tall building by the continuous method and the transfer matrix method using an energy formulation.
- Develop a methodology for global structural analysis of tall building stability by the continuous method and the transfer matrix method using an energy formulation.

1.4 JUSTIFICATION FOR THE RESEARCH

- **Theoretical justification.** One of the problems faced by the structural engineer when analyzing tall buildings is the problem of horizontal lateral loads. While it is true that these lateral loads are small compared to gravity loads, the transfer of these loads to the foundation requires special design work. Tall buildings are very sensitive to dynamic vibrations due to their height and slenderness; and, therefore, to perform accurate structural analysis using the finite element method requires a great technological and economic effort. This research project proposes an accurate and reliable approximate method based on the continuum method and the transfer matrix method using an energy formulation to calculate lateral and torsional deflections, periods, uncoupled and coupled frequencies, and critical loads in tall buildings.

- **Methodological justification.** To meet the objectives of this research project, the continuous method and the transfer matrix method with an energy formulation will be used. The continuous method assumes that all horizontal elements connecting the vertical components are effectively connected over the height of the building to produce a continuous connecting means, i.e., the connecting beams are replaced by a uniformly distributed sheeting system. As a consequence of the continuous method, the three-dimensional (3D) structure leads to a replacement beam (RB) which is characterized by equivalent properties K_i that attempt to adequately represent the actual stiffness of the structural system. It is important to mention that in order to obtain more accurate RB systems for the structural analysis of tall buildings, a mathematical model with additional kinematic fields and stiffness properties to those existing in the literature is going to be used. In order to develop a comprehensive methodology and to take into account the vertical discontinuities existing in many tall buildings, it was decided not to limit to tall buildings with regular structural systems and the transfer matrix method was implemented to the continuous method.

- **Practical justification.** The author hopes that the results of this research project will help engineering offices specialized in the development of structural projects in tall buildings, to minimize cost and time in computational technology and human resources by focusing primarily on the choice of optimal structural systems for each structural project. In addition, it is expected that the methodology developed will be taken into account so that the academic community can continue developing more structural analysis using approximate methods focused on other structural problems of interest.

- **Social justification.** The results of this research are primarily beneficial to the structural engineer; however, society in general can also benefit when a large-scale evaluation of a structural strengthening plan for existing structures is required prior to a natural disaster such as earthquakes.

1.5 DELIMITATIONS OF THE STUDY

The development of the research project comprises the following elements: the geographic space, the subjects participating in the research and the content.

- **Geographic space**. The research has a worldwide geographical space due to the fact that currently all countries have tall buildings. The research project is limited to the study of linear systems, the need to study the nonlinear response of structures is recognized; however, previous studies have shown that a greater increase in seismic forces is perceived in the linear range with respect to the nonlinear range. As a consequence, the linear approach can be considered conservative for nonlinear systems. The slabs are considered to be rigid in their plane and only transfer horizontal forces, i.e., they do not transfer vertical or bending forces. In addition, the research project focuses on a global analysis that is not limited to structural systems that are regular, allowing the analysis of structural systems with vertical discontinuities in the building height.

- **Subjects who will participate in the study**. The study population is made up of tall buildings. In this regard, there is no universally recognized definition of a tall building, because height is a relative parameter. For the purposes of the research project, the minimum height to be considered is that corresponding to a 4-story building, because in order to use the continuous method it is necessary to have enough connecting beams to consider a continuous connection between the vertical components.

- **Contents**. To carry out the operationalization of the variables, we will work with the independent variable "continuous method and transfer matrix method" and with the dependent variable "global structural analysis of tall buildings".

2 STATE OF THE ART

2.1 RESEARCH BACKGROUND

2.1.1 International Research

To summarize a complete chronology on the study of the structural analysis of tall buildings using the continuous method and the transfer matrix method would be too extensive. Therefore, a review of the research most relevant to the topic of study of the present research project will be made.

The use of the continuous medium technique in structural engineering dates back to the work of Jacobsen, L. (1930), who modeled the underlying soil as a shear beam, with the objective of evaluating the site response. Somewhat later, Biot, M. (1933) and Westergaard, H. (1933) considered the same continuous model to estimate the building behavior.

The continuous connection method was probably created by Chitty, L. (1947), who proposed the first formulation of the continuous connection method using a shear beam and a bending beam coupled by rigid bars. He investigated parallel beams interconnected by cross bars, subjected to a uniform lateral load, and established the differential equation that solves the problem. In a later work Chitty, L. and Wan, W. (1948) applied the continuous medium technique to analyze tall buildings subjected to a wind load uniformly distributed in height.

Rosenblueth, E. and Holtz, I. (1960), used a shear beam to relate slope to bending moment and drift. They solved the shear distribution between the wall and the portal frame using a method of successive approximations.

Vlasov, V. (1961), was the originator of the theory of sectoral areas. He was the first to combine the thin-walled open section theory with the continuum approach to analyze the torsional behavior of three-dimensional shear walls, defining the tendency of the bimoment action as a result of this deformation. Based on this theory, many researchers introduced continuous formulations for the case of closed and open sections.

Khan, F. and Sbarounis, J. (1964), used the coupling of a shear beam and a bending beam and solved the interaction between shear walls and frames by a solution in which the shear wall is treated as the primary system and the frame as the secondary system, or vice versa. The resulting deformations of the primary system are imposed on the secondary system. The resistance forces induced in the secondary system are taken as the correction load in the primary system. This process is repeated successively until convergence of equilibrium and compatibility of deformations is achieved.

Glück, J. (1970), presented a three-dimensional continuous method for structures consisting of shear walls and portal frames arranged asymmetrically in the floor plane. He used the continuous approach and the theory of thin-walled sections of Vlasov, V. (1961). Based on compatibility and equilibrium conditions, he derived a set of coupled differential equations with translational and rotational displacement functions. However, this analysis did not include the effect of axial deformations of shear walls and portal frames.

Glück, J. and Gellert, M. (1972), developed a more complete three-dimensional analysis of an asymmetric tall building including the influence of axial deformations in the portal frames and shear walls. They derived the inhomogeneous second-degree differential equations of the shear forces in the sheeting system. With the known basis functions, they established all the forces and internal displacements of the individual reinforcement elements.

Tso, W. and Bismas, J. (1973), developed a method for the three-dimensional analysis of nonplanar coupled shear walls of arbitrary cross section and considered the axial deformations of the shear walls. Based on compatibility and equilibrium conditions, they derived a set of three coupled differential equations, which can be reduced to a single equation with rotational deformation as a variable.

Heidebrecht, A. and Stafford, B. (1973), represented the shear wall by a bending beam and the portal frame by a shear beam, and connected them by an axially rigid linking means distributed along the height of the building. The columns of the portal frames were considered axially rigid. From that continuous representation, they proposed the solution for the deflections of uniform shear wall - portal frame structures based on the differential equation governing the system.

Reinhorn, A. (1978), developed an approximate analytical model for the static structural analysis of structural elements based on the continuous method and included axial deformations through only three horizontal deformations (two translations and one rotation), for the case of staggered buildings he used the transfer matrix method. The developed model was part of a general perturbative model, where the perturbative solution allowed to verify and correct errors if necessary. In addition, it investigated the influence of static loads and the effect of torsional translational coupling on the dynamic response.

Nollet, M. (1979), provided a detailed exposition on the behavior of continuous and discontinuous shear-frame wall structures, considering the influence of horizontal interaction between shear walls and frames to stiffen the structure. He developed continuous solutions that allow generalizations on the behavior of a wide range of shear wall-frame structures. For stepped shear wall - portal frame structures, it is found that the walls can be reduced without significantly modifying the overall horizontal interaction and lateral stiffness.

Stafford, S., Kuster, M. and Hoenderkamp, J. (1981), generalized the continuous medium technique that had been applied earlier to coupled shear wall structures so that it could be applied to any type of flexural and shear cantilevers. They defined the characteristic parameters αH and κ^2, where κ^2 includes consideration of axial deformation of the vertical elements.

Hoenderkamp, J. (1983), extended the continuous solution for asymmetric structures, proposed a generalized solution that included the axial deformations of shear walls and portal frames. The coupled torsion-bending differential equations were decoupled using an orthogonal transformation.

Miranda, E. (1999), used the continuum model to estimate the maximum lateral displacement demands on tall buildings that respond primarily in a fundamental mode when subjected to seismic motions. This method allows rapid estimation of the maximum roof displacement and maximum interstitial drift for a given acceleration time history or for a given displacement response spectrum. The procedure is based on a simplified model of multistory buildings consisting of a combination of a flexural cantilever beam and a shear cantilever beam. The simplified model is used to investigate the relationship between spectral displacement and roof displacement and the ratio of maximum interstitial drift to roof drift ratio. However, it neglected axial deformations, which are important to consider in the structural analysis of slender buildings.

Shiu Cho, N. (1999), based on the continuum method developed a general approximate solution based on the Galerkin method to the eigenvalue problem of complex structures in triple coupled vibrations. In addition, he developed a parametric study that allows to visualize how the coupled frequencies and mode shapes are related to the key parameters of the dynamic behavior of the building; constructing several design plots useful for engineering offices.

Kuang, J. and Ng, S. (2000), proposed a method to determine the interconnected modes and periods of asymmetric structures. The Galerkin approach was used to obtain the model. In an asymmetric structure that was examined in order to show the accuracy of the method, it was observed that the mode and periods obtained by the proposed method were close enough to the periods and modes found with finite elements.

Wang, Q., Wang, L. and Liu, Q. (2001), based on the continuous method and the transfer matrix method, investigated the effect of shear wall height on the dynamic behavior of portal shear wall structural systems. It was shown that the shear wall height does not influence the dynamic behavior except in very special cases and that it is not necessary to extend the shear wall over the entire height.

Hans, S. (2002), developed an experimental program on buildings before and after demolition of a 16-story shear wall building with the objective of gathering information to integrate it into a seismic vulnerability diagnosis of existing buildings. Results of the information collected allowed characterizing the dynamic behavior of the buildings by means of simple shear beam, bending beam and Timoshenko beam models. In addition, it demonstrated on the basis of the information collected that the discrete-means homogenization method provides a theoretical justification for the use of continuous beam models to characterize the dynamic behavior of real structures.

Potzta, G. (2002), developed a whole building replacement beam model using a sandwich beam with an energy approach and derived the three characteristic stiffnesses of the sandwich by applying a sinusoidal displacement and balancing the total deformation energy of the building with the sum of the deformation energies of each structural scheme. They used this replacement beam model for wind, earthquake, and building stability analyses.

Rafezy, B. (2004), presented two global analysis approaches for the calculation of frequencies of tall buildings. Both methods assume rigid floor diaphragms and require knowledge of the static eccentricity of the building at each floor level. Because the methods for calculating static eccentricity are complicated, a practical calculation method and a small parametric study are presented. An accuracy analysis confirms that the proposed methods can yield results of sufficient accuracy for engineering calculations.

Takabatake, H. and Satoh, H. (2006) proposed an analytical method that replaces the building by a continuous equivalent rod for the dynamic analysis of tall buildings consisting of doubly symmetric frame tubes with or without bracing. The solution of the differential equations are based on the finite difference method; the suitability of the method was verified with four different types of buildings analyzed with the finite element method. In addition, the effect of soil-structure interaction is discussed using the proposed method.

Espezúa, C. (2009), used an analysis method based on the continuous medium technique to study the static and dynamic behavior of tall buildings against earthquakes. The approximation of the method was compared with the results of a finite element analysis with the SAP 2000 program, obtaining values with an acceptable approximation for engineering terms.

Jigorel, S. (2009), developed different continuous models using the discrete periodic means homogenization method to represent the dynamic behavior of buildings. He highlighted a new generic equation from which the other particular behaviors are derived, finding a new parameter that measures the contrast of shear stiffnesses between shear walls and floors.

Bozdoğan, K. (2010), used the continuum method and the transfer matrix method for static, dynamic and stability analysis of the tall building whose geometric, material and loading properties vary along the height modeling the building as a sandwich replacement beam. For the case of asymmetric structures, it neglected the shear stiffness of the walls and the axial deformations of the portal frames and coupled shear walls.

Chesnais, C. (2010), studied the dynamic behaviors of a family of lattice structures, formed by a network of beams using the method of homogenization of discrete periodic media (HMPD) allowing to construct an equivalent continuous medium at macroscopic scale that allows to represent buildings when the cell size is very small compared to the wavelength. He developed different generalized continuous models and generalized the sandwich beam model by including the local shear stiffness for the case of buildings with shear walls of significant camber.

Pârv, B. (2012), developed calculation programs based on global analysis of tall buildings and spatial analysis based on matrix formulations using Matlab language. In addition, he performed a sensitivity analysis to develop a structural optimization program using genetic algorithms.

Lavan, O. (2012), adopted a continuum approach to model the structure and rigorously evaluate the efficiency of viscous dampers connecting two walls to result in viscously coupled shear walls. He found that under certain considered approximations, the damping ratio of the system is a simple compact convenient parameter that controls the reduction of the response of an undamped system. Furthermore, it reveals the efficiency of the added damping in reducing not only displacements, interstory drifts, and wall moments, but also absolute accelerations, wall shear, total shear, and total overturning moments.

Cammarano, S. (2014), proposed a synthetic three-dimensional approach based on the continuous method and Vlasov's theory of sectorial areas. This approximate approach is adaptable to the static and dynamic analysis of uniform or staggered tall buildings in building height. In addition, it performs an experimental test to measure the effect of thin-walled beams subjected to torsion.

Huang, K. (2009), found that pushover analysis underestimates drift in the upper stories and is deficient in predicting overturning moments, shear forces because they neglect high modes of vibration. To overcome that problem he developed a simplified continuum model for seismic analysis of tall shear wall - portal frame structures designed for wind loads. In addition, he verified the accuracy of the model by investigating three tall portal frame-shear wall buildings, with satisfactory results compared to the pushover analysis.

Tuncay, A. (2014), developed a continuous method to determine the effects of non-uniform Vlasov torsion caused by horizontal loads in tall buildings. As a result of a sensitivity analysis, a good accuracy of the model was obtained and showed that non-uniform torsion is of great importance, which should not be neglected in the analysis of tall buildings that is very common in design offices.

Moghadasi, H. (2015), proposed two replacement beams based on the continuous method for structural analysis of tall buildings. The first beam consists of parallel coupling of two Timoshenko beams and takes into account the four characteristic stiffnesses of a tall building, and is applicable to all structural systems. The second beam consists of the parallel coupling of an extensible Timoshenko beam and a continuous core as a supporting rotation constraint. Due to the complexity of the coupled differential equations, he developed a one-dimensional finite element formulation evaluating the static and dynamic responses. In addition, using a discrete system of coupled shear walls he theoretically established the distributed internal viscous damping of the Kelvin-Voigt type with the bending and shear mechanisms.

Lavan, O. and Abecassis, D. (2015), studied the seismic behavior of a continuous shear wall - portal frame system in the context of retrofitting existing portal frame structures. They first identified the controlling non-dimensional parameters of such systems. This is followed by a rigorous and extensive parametric study revealing the pros and cons of the new system versus wall-frame systems. The effects of the control parameters on the behavior of the new system are analyzed and discussed.

Aydin, S. (2016), developed a methodology to calculate the critical buckling loads of buildings on elastic and rigid foundations by solving the stability equations expressed by differential equations with the Differential Transform Method.

Migliorati, L. and Mangione, M. (2015), using the continuum method modeled each structural system differently and studied their three-dimensional combination. They developed a coupled continuous Timoshenko - Vlasov model and a discrete model to account for local bending effects. Due to the complexity of the coupling between the differential equations, they formulated a one-dimensional finite element model for static and dynamic analysis of tall buildings.

Puthanpurayil, A., Lavan, O., Carr, A. and Dhakal, R. (2016), adopted the local continuous damping model for simian analysis by applying the Galerkin procedure. Two local continuous damping models used in the linear dynamic analysis regime are adapted and extended to the nonlinear dynamic analysis scenario. In addition, schemes for implementing the models using the classical Newmark framework were presented. They showed that all the proposed models appear to produce more reliable results than the global models without increasing the computational demand.

Anesi, R. (2018), developed a simple methodology to address the problem of defining the dominant action between wind and earthquake, with special reference to the case of structures located in a low seismicity zone, as well as to refine this procedure by adopting an analytical continuous replacement beam model consisting of the parallel coupling of a bending and shear beam. As a result of three reference structures, a greater influence of the earthquake on the dominant action definition was highlighted when higher order modes are taken into account.

Kara, D. (2019), in order to study soil-structure interaction, investigated the dynamic behavior of buildings standing on five different soil classes. He determined that the shear beam model representing soil provides consistent and engineering acceptable results. In addition, this model is suitable for understanding the soil-structure interaction behavior with fewer parameters than those used with the finite element method.

Zalka, K. (2020), based on the continuous approach developed closed form equations for two categories of analysis: a) An individual analysis and b) A three-dimensional analysis (global approach), where he developed closed form equations for displacements presenting two methodologies: the simple method and the precise method (using the interaction between bending and shear deformations), stability, frequency and critical load of whole buildings. In addition, he introduced the "global critical load ratio" which acts as a generic characteristic with which the designer can monitor the overall performance of the whole building.

Dinh, H. (2020), based on the homogenization method of discrete periodic means (HMPD) established a practical method that estimates the dynamic behavior of buildings using general beam models and integrated these models to include viscous dampers in the analysis. He concluded that the addition of viscous dampers only modifies the shear parameter in the generic beam models.

Franco, C. (2021), based on the homogenization method (HM) and the multifiber finite element method, proposed a strategy to improve the integration of local and global scales in the definition of damage indicators in building response. He implemented the homogenization method in complex multi-frame structures, described single-story numerical modeling (MEM), and proposed a novel strategy that could be used in the future as damage criteria.

Gungor, Y. and Bozdogan, K. (2021), using the continuous method and differential transformation method adapted a Timoshenko type replacement beam for dynamic analysis of steel plate shear wall systems (SPSW). In addition, based on the dynamic characteristics they performed a response spectrum analysis by finding the displacement, shear force and bending moments.

2.1.2 National Research

An exhaustive search for similar research in Peru has been carried out. No research related to this research project has been found.

2.2 THEORETICAL BASIS

Since time immemorial, man has wanted to build structures beyond his means to demonstrate power and wealth; to honor religious leaders and beliefs; and even simply as an objective of competition between owners, families, architects and builders.

Structures such as the Tower of Babel (Figure 3) to which the Bible refers: "And they said, Let us build us a city and a tower, whose top may reach unto heaven; and let us make us a name, lest we be scattered abroad upon the face of the earth" (Genesis (11:4), 2019). The sons of men seemed to be affronting or rivaling God, for they wanted to build a tower whose top would reach to heaven. Moreover, they hoped to make a name for themselves that would be remembered by men through time, leaving as a legacy this monument symbol of their pride, their ambition and their folly. However, to this day there is no book of history in which a single name of these builders of the tower of Babel is remembered. Paradoxically, Babel means confusion; this should remind us that those who are ambitious for a great name, ordinarily come out with a bad name.

Figure 3. The Tower of Babel (Brueghel, 1563)

The factors that contributed decisively to the development of tall buildings occurred in the 19th century. The first was the creation of the safe elevator in 1853 by American inventor Elisha Graves Otis (figure 4), who developed a safety device that prevented traditional elevators from falling when the cable broke, shortly after which the first passenger elevator opened to the public in the E. V. Haughwout building in New York. Haughwout building in New York; the second factor was the devastating fire in Chicago in 1871 (figure 5), where contrary to common sense the city experienced exponential growth, only nine years later, the land available for the construction of new buildings could not meet the demand which led as the only option to build in height.

According to Dario Trabucco: "The poorest people used to live on the highest floors, but the elevator changed this scenario and high floors soon became fashionable as they offered more natural light, cleaner air and less traffic noise". New construction methods made it possible to reach ever greater heights, the skyscraper was born, thus beginning the race for the tallest building.

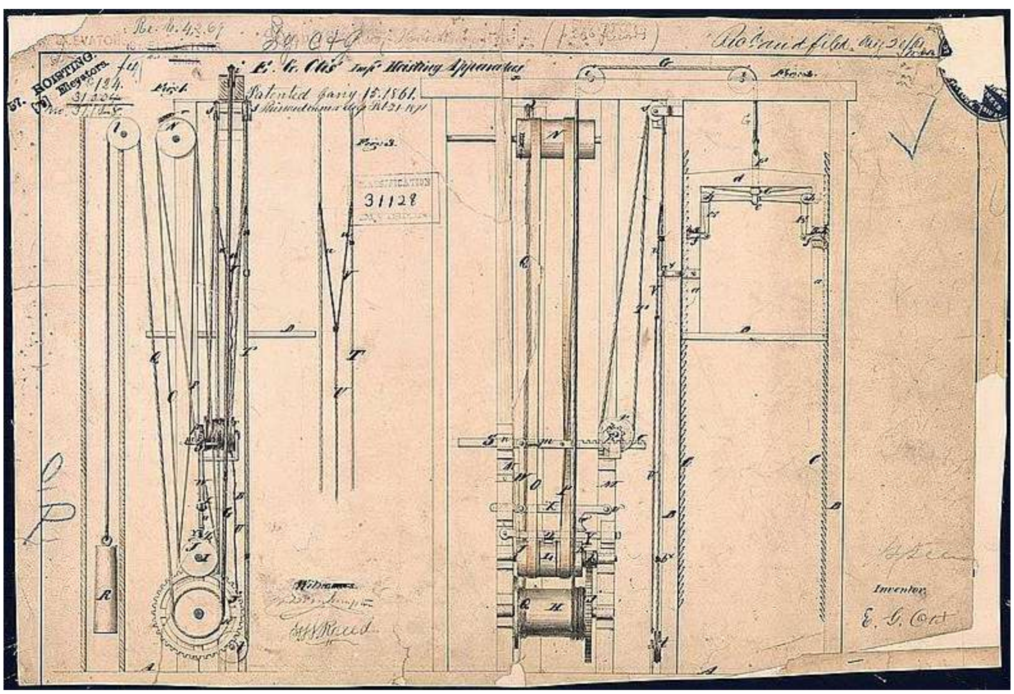


Figure 4. Patent drawing of the elevator (Otis, 1861).

Figure 5. Chicago in flames - The race for lives over the Randolph Street Bridge (Chapin, 1871).

Today, the world is witnessing rapid population growth. In the last 200 years, the world's population has grown from 1 billion in 1800 to 7.9 billion by the beginning of 2022. This pattern of development is based on an inexhaustible supply of arable land, water and energy that cannot be sustained in the coming years. While it is true that buildings are not the only source of environmental pollution, where we build, how we build and how we move are the main causes of climate change. One way to solve this challenge is to design intelligent forms of human settlement that are dense, compact and highly livable. A clear example of sustainability and zero pollution is the Q'eswachaka bridge (Figure 6), located over the Apurimac River in Cusco Peru. De Wolf (2015) states, "the materials grow naturally, are locally sourced, and their construction by hand does not pollute, so it is a historical process that should inspire today's engineers."

However, the rapid growth in height of tall buildings appears to be directly related to economic downturns. In 1999, economist Andrew Lawrence (Thornton, 2005) created the "skyscraper index", which aimed to show that the construction of the world's tallest skyscrapers coincides with economic cycles. He concluded that the construction of the world's tallest building is a good indicator for determining the onset of major economic crises.

Figure 6. Q'eswachaka bridge at present (Palomo, 2020).

Looking at Table 1, it is clear to state that the index's ability to predict economic collapse is astounding. For example, the Panic of 1907 was foreshadowed by the Singer Building and the Metropolitan Life Building; the Great Depression was accurately foreshadowed by the Crysler Building, Empire State Building and the 40 Wall Tower; the stagflation suffered by the United States between 1970 and 1982 were surprisingly foreshadowed by the World Trade Center (one and two) and the Sears Tower; the completion of the Petronas Tower in 1997 marked the beginning of the extreme slump in Malaysia's stock market, the rapid depreciation of its currency and widespread social unrest, spreading these economic problems to all economies in the region (Asian contagion) and Dubai's Burj Khalifa which was completed in 2010 shortly after the country went into financial crisis. However, there are important exceptions to the index's ability to predict an economic downturn, clear examples being the construction of the Woolworth Building (apparently not a complete exception due to World War I not providing enough time for the economic depression to deepen) and Japan's continued economic recession since 1990. This does not suggest that the heights of tall buildings should be limited to avoid economic crises, as proposed by Thornton (2005) the institutions that regulate debt financing should be re-evaluated or changed to more efficient and stabilizing institutions.

Studies show that vanity is the main justification that leads investors to risk resources to the construction of very tall buildings. In 1998, yet to become the former President of the United States of America, Donald Trump, stated as justification for the construction of his Trump Tower a financially meaningless phrase: "I think New York should have the biggest building in the world" (Lawrence, 1998).

Tabla.1 *Skyscrapers and economic crises (Thornton, 2005)*

Completed	Building	Location	Height	Stories	Economics Crisis
1908	Singer	New York	612 ft.	47	Panic of 1907
1909	Metropolitan Life	New York	700 ft.	50	Panic of 1907
1913	Woolworth	New York	792 ft.	57	-
1929	40 Wall Street	New York	927 ft.	71	Great Depression
1930	Chrysler	New York	1,046 ft.	77	Great Depression
1931	Empire State	New York	1,250 ft.	102	Great Depression
1972/73	World Trade Center	New York	1,368 ft.	110	1970s stagflation
1974	Sears Tower	Chicago	1,450 ft.	110	1970s stagflation
1997	Petronas Tower	Kuala Lumpur	1,483 ft.	88	East Asian
2012	Shanghai	Shanghai	1,509 ft.	94	China?

Knutsen, C. (2011) in his doctoral thesis conjectured that skyscrapers could give us clues about who built them. He states, "autocratic regimes tend to build such more excessive buildings and, in contrast to democracies, tend to build skyscrapers regardless of whether the country is urbanized or not." Gjerlow, H. and Knutsen, C. (2017) state, "autocracies build more skyscrapers than democracies and autocracies build more wasteful skyscrapers than democracies." They further noted that subsidizing such projects will often detract resources from more mundane investments in local roads, schools, or health clinics throughout the country; this is important in poor and developing countries (such as Peru) where resources are scarce and where the population tends to limit the construction of very tall buildings.

It is worth noting that the Council on Tall Buildings and Urban Habitat (CTBUH), classifies the tallest buildings in the world by height and pinnacle, noting that several buildings appear higher in the classification than they would be due to their spires, masts and extra structures that they add to the buildings with the sole purpose of gaining height. In this regard Gjerlow, H. and Knutsen, C. (2017), state, "Vanity height, are more present in autocracies than in democracies." As an example, the tallest building in the world today, the 828-meter tall Burj Khalifa, located in the United Arab Emirates, has an excessive unoccupied area of 30% and what would likely be the next tallest building in the world, the 1000-meter-tall Jeddah Tower (potentially realizing Ludwig Mies van der Rohe's 1920s "Impossible Dream" and Frank Lloyd Wright's 1956 "Mile High Illinois" (Skelton, 2016)), located in Saudi Arabia, will have 37% of the total height unoccupiable. Analyzing these data, it is hard not to wonder about the exaggerated role vanity plays in the most spectacular building decisions of recent years.

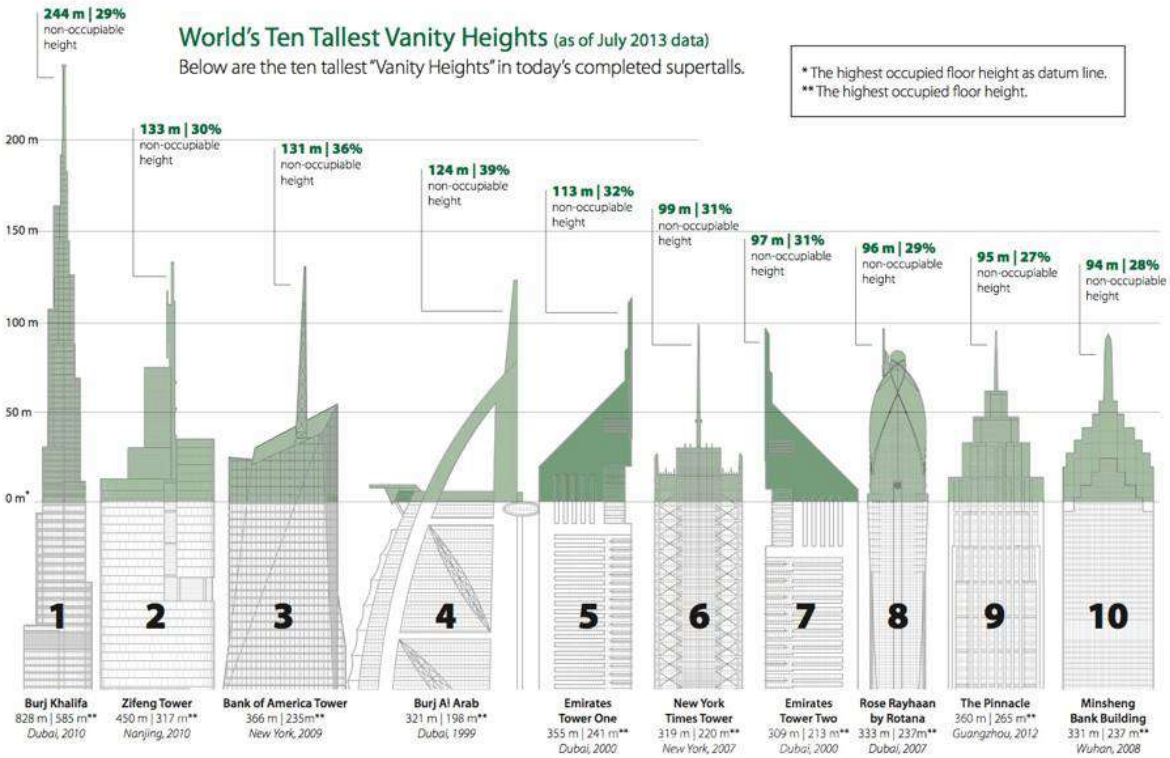


Figure 7. The 10 tallest buildings in the world with the highest number of meters of vanity (CTBUH, 2013).

- **Definition of Tall Building**

There is no universally recognized definition of a tall building, because height is a relative parameter. Historically, low-rise buildings have been defined as those with less than 8 stories, mid-rise buildings as those with between 8 and 20 stories, and tall buildings as those with more than 20 stories.

When we analyze in more detail what seems obvious to us, certain doubts begin to arise. If one were to ask a person what a tall building is, perhaps his immediate answer would be precisely that: "A tall building is a tall building, that is, a building with many floors". The question that should really be asked is: What is a tall building in a historical, regional and global context? Nowadays, people can hardly call a 20-story building a tall building anymore, if it is to be compared to the tallest buildings in the world. The definition that comes closest to this clarification is Stafford and Coull (1991), which states that it is not safe to indicate how many stories are needed to define a tall building because this is conditioned by the historical period in which the structure is located, as well as the type of buildings that are present in the city where the building is located.

According to the Council on Tall Buildings and Urban Habitat (CTBUH), the authority that announces the title of "World's Tallest Building," the classification of tall structures is subjective and depends on the height of a building in relation to the context in which it is located, its proportion (or slenderness) and adopted height-related technologies. In that sense, for a building to be considered a tall building, it must have one of the following characteristics:

- ✓ **Height relative to context**: when a building is clearly taller than the average value of the heights of the surrounding buildings.
- ✓ **Proportion**: when the building is slender enough to give the impression of verticality of a tall building.
- ✓ **Tall building technologies**: the building contains technologies that are a product of building height, such as specific vertical transportation technologies and structural bracing against wind.

As defined by the Council on Tall Buildings and Urban Habitat (CTBUH), a building that is 14 stories tall or has a height of 50 m or more is typically considered a tall building. Structures whose height exceeds 300 m are classified as supertall and those exceeding 600 m are classified as megatall. The same principle of measuring the height of a straight building applies to leaning buildings, meaning that the height is measured vertically from the base to the top.

Günel, M. and Ilgin, H. (2014) defined tall buildings according to specialty: by structural designers as buildings that require an unusual structural system and where wind loads are prominent in the analysis and design; by architectural designers as buildings that require interdisciplinary work, particularly with structural designers, and with experts in the fields of aerodynamics, mechanics, and urban planning that affect design and use; and by civil engineers as buildings that need unusual and sophisticated construction techniques.

From the above, from a structural engineer's point of view, we can define a structure as a tall building when the first priority in structural analysis and design consideration is the lateral stability system, because its structural analysis and design are mainly affected by lateral loads such as wind and earthquake.

We cannot forget that, although the height of tall buildings is an important parameter because it determines the lateral forces distributed in height, slenderness is perhaps the fundamental parameter in terms of structural engineering because it conditions the distribution of loads on the structural elements. Defined as the ratio between height and structural width, it conditions many aspects related to the effects of horizontal actions.

- **Structural Systems of Tall Buildings**

The key characteristic of a tall building from a structural analysis and design point of view is its lateral stability structural system. There have been several attempts over the years to classify the structural systems that are appropriate for tall buildings.

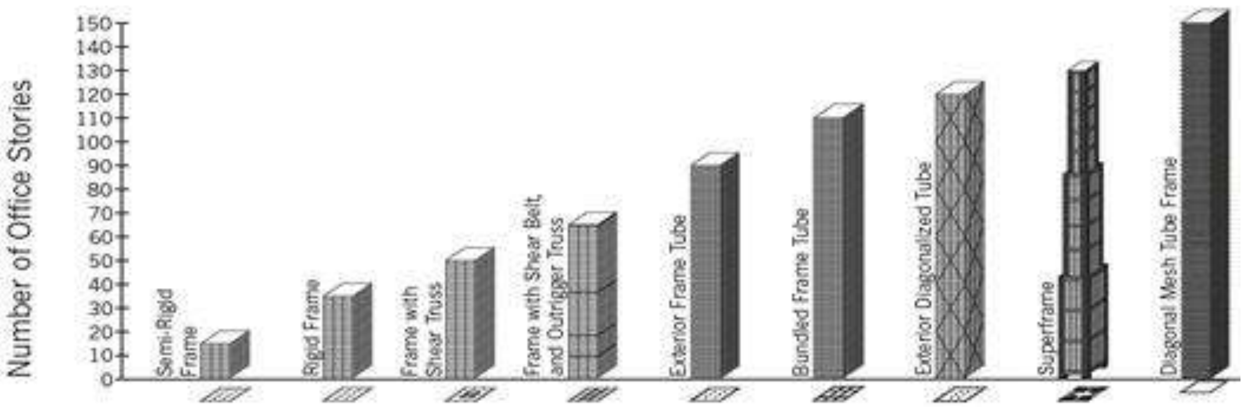
Fazlur Khan (1969), considered as "the Einstein of structural engineering", "the best structural engineer of the 20th century" and "the father of tubular systems", classified structural systems for tall buildings in relation to their heights with structural efficiency considerations in the form of diagrams: "Heights for Structural Systems". He later developed new diagrams based on the structural material used: structural steel, reinforced concrete and composite systems.

Although Fazlur Khan initially worked on prestressed structures, when faced with the challenge of analyzing and designing tall buildings, he focused on them with a passion. With an intuitive understanding of the technical aspects of structures, Khan set out to find the right structural system for tall buildings.

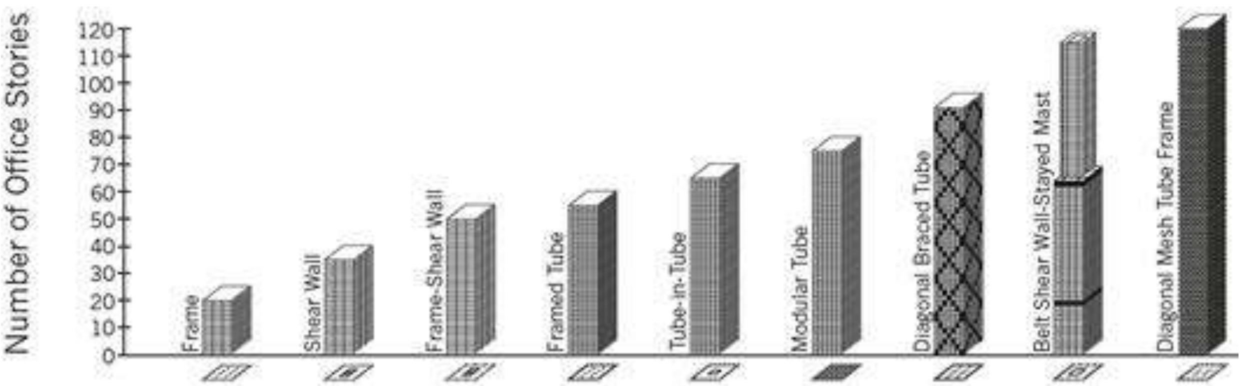
He argued that the rigid portal frame that had dominated tall building design and construction for so long was not the only structural system suitable for tall buildings and that structural systems could be analyzed three-dimensionally, rather than as a series of flat systems. The viable structural systems he mentioned are: Rigid portal frame, shear walls, shear wall - portal frame and tubular systems.

Figure 8. Fazlur Khan and Bruce Graham (from left to right) next to the Hancock Center model (Khouyali, 2021).

There are several ways to combine structural systems to achieve adequate performance. For example, at the beginning of the 20th century, the structural system that governed the analysis and design of tall buildings was the rigid reinforced concrete portal frame system; this system consists of columns and beams that are rigidly connected at their nodes, which provides the advantage of reducing the bending moment and buckling length of the columns. However, this system does not provide adequate stiffness which limited the height of tall buildings. To overcome this problem, the shear wall and portal frame structural system was developed, which combined the advantages of the rigid portal frame with the shear wall; with this new structural system, it was possible to achieve sufficient horizontal stiffness while retaining the flexibility of the spaces to achieve greater heights. As cities grew, this new height limit was no longer sufficient. To build even taller structures, the central structural system was invented, which is often combined with other more basic structural systems such as portal frames or bracing around the perimeter of the building to provide lateral stability to the building.



(a)



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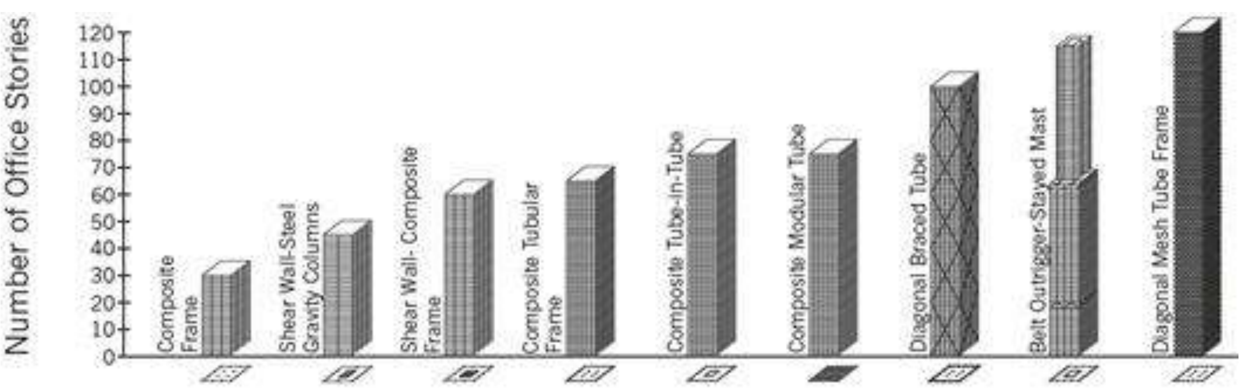
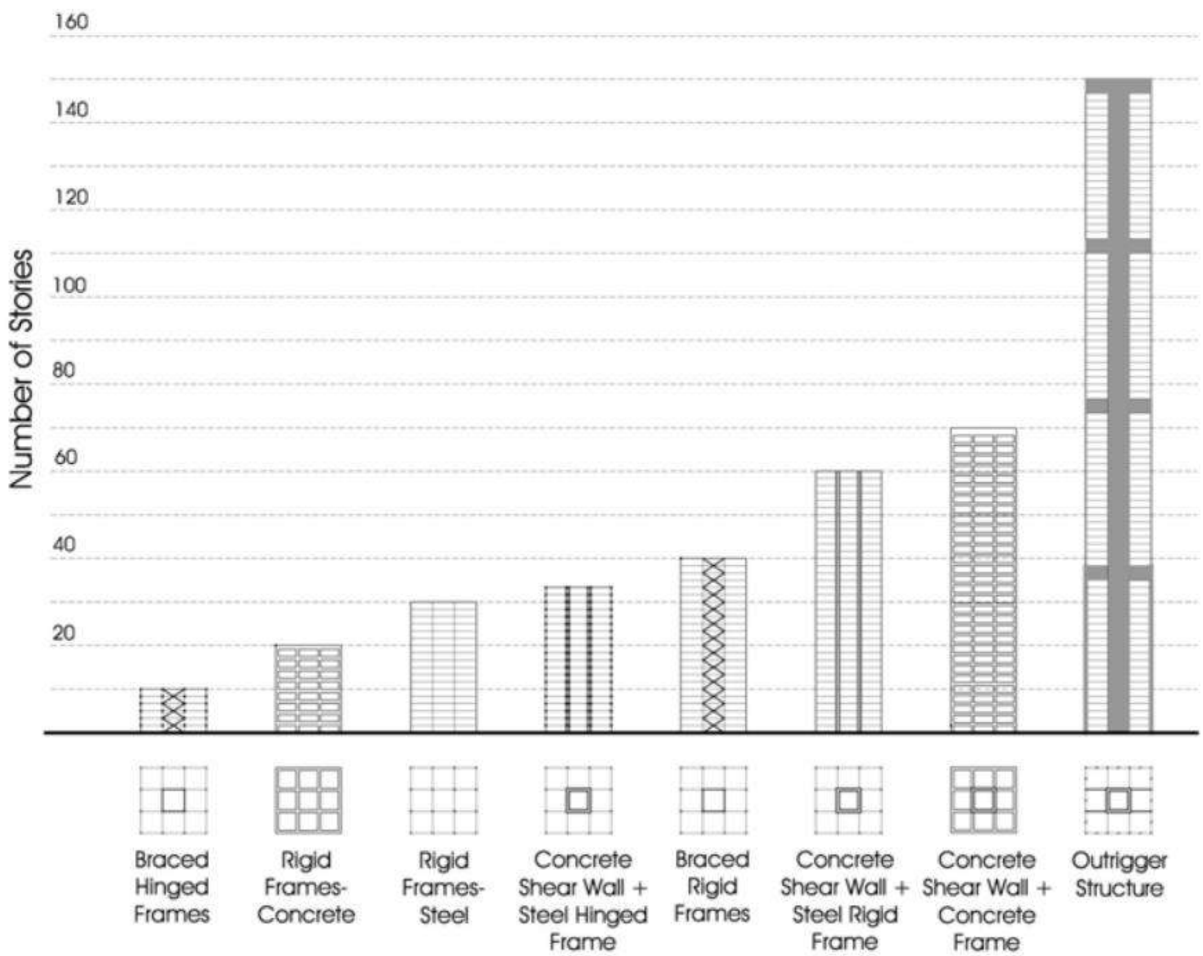


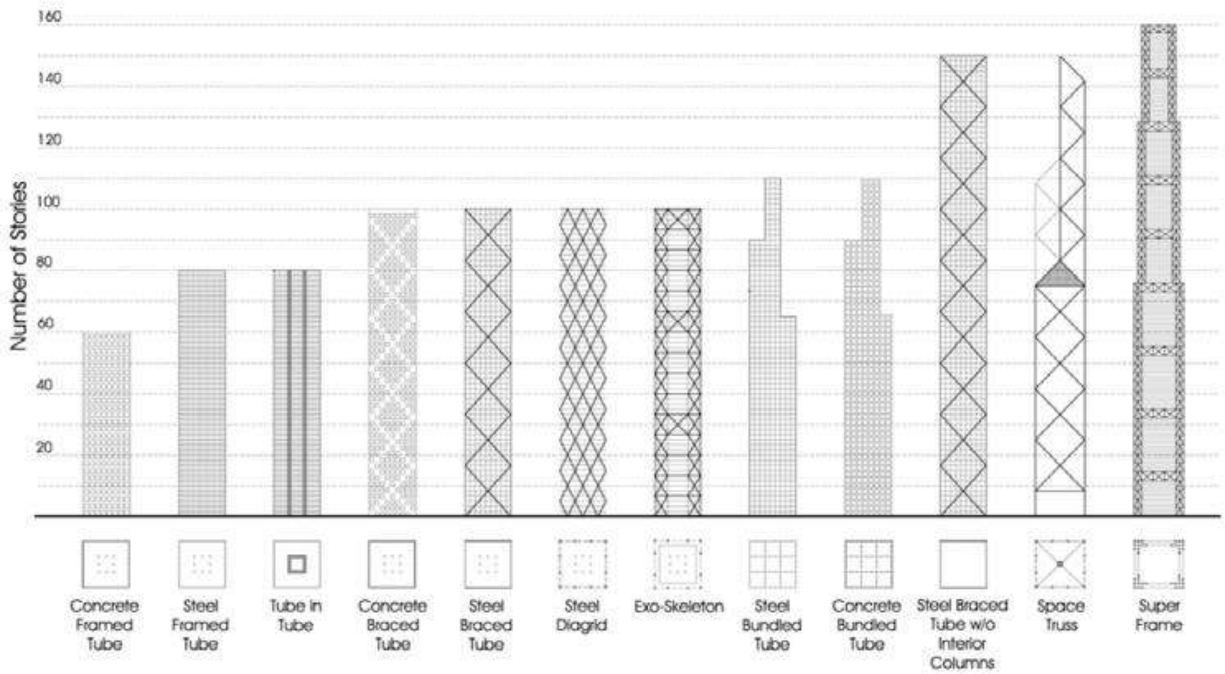
(c)

Figure 9. Classification of structural systems of tall buildings according to Fazlur Khan. (a) Steel structural systems, (b) Reinforced concrete structural systems, (c) Composite structural systems (structural steel + reinforced concrete) (Sarkisian, 2016) (Sarkisian, 2016).

In 2007, Mir M. and Kyoung, S. (2007) developed a new classification based on lateral load bearing capacities. He divided the structural systems of tall buildings into two broad categories: interior structures and exterior structures. A system is classified as an interior structure when the majority of the lateral load resisting system is located inside the interior of the building, similarly, if the majority of the lateral load resisting system is located on the perimeter of the building, a system is classified as an exterior structure. It is important to mention that it is desirable to place as many resisting elements as possible as far as possible close to the perimeter of the building to efficiently resist lateral and torsional forces together.



(a)



(b)

Figure 10. Classification of tall building structures according to Mir M. Ali. (a) Interior systems, (b) Exterior systems (Mir & Kyoung, 2007).

There are several factors to consider when selecting a structural system for tall buildings: safety, occupant comfort, economy, intended function, architectural considerations, internal traffic flow, height and load intensity.

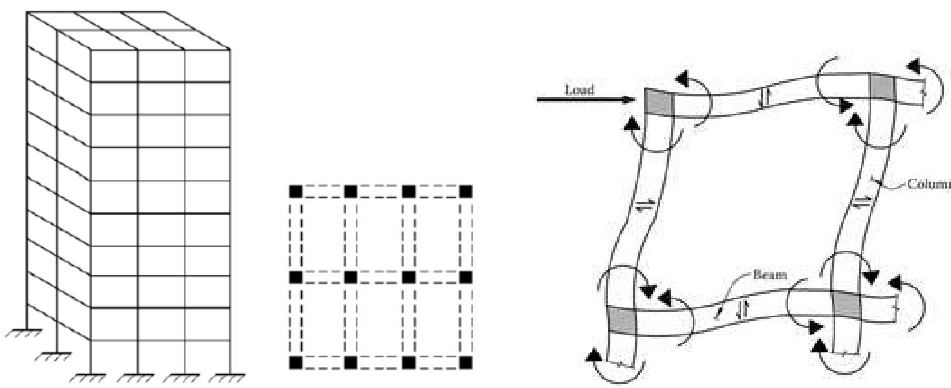
It is important to mention that only systems that are suitable and economical for tall buildings will be investigated in this research project. Therefore, systems specified for very tall buildings are not the subjects of this research project. The structural systems considered in this section are the following: moment resisting portal frames, shear walls, coupled shear walls, dual systems (portal frame + shear wall) and cores.

- **Frame System**

The most common construction materials are steel and concrete. In this system, resistance to lateral loads is provided by the interaction of the beams and columns, i.e., by the bending and shear stiffness of the network of beams and columns (Figure 11). In general it works better in concrete than in steel because in steel the nodes are usually considered as semi-rigid, while in concrete they are usually considered as rigid, this characteristic in turn seems to be a disadvantage, because the portal frame requires rigid connections which are usually expensive. It is required that, in the rigid node, the bending resistance of the columns be at least 20% greater than the bending resistance of the beams, to ensure that in the presence of cyclic loads (such as earthquake) the plastic hinges are generated in the beams and not in the columns.

This structural system is generally chosen when the horizontal forces are not predominant compared to the vertical forces, because otherwise, this would imply an excessive increase in the dimensions of the structural elements. When designed for strength considerations alone, lateral drift causes discomfort to occupants and damage to non-structural elements, and the P-Delta effect causes additional bending in the building.

The efficient design height without additional lateral load resisting systems is 30 stories in steel structures and 20 stories in reinforced concrete structures. Column spacing generally varies from approximately floor-to-floor height to twice the floor-to-floor height; in general the spacing ranges from 4.5 meters to 9 meters.



(a) (b)

Figure 11. Rigid frame system. (a) Three-dimensional structure, (b) Deformation and interaction of beams and columns (Taranath, 2016).

Although the portal frame is usually the first option for tall buildings, since most tall buildings have it as a base; once a certain height is reached, lateral forces make the frame insufficient to work alone, an efficient way to overcome the height is to have braced elements completely changing its behavior since the building would behave as a truss, where the columns would be the chords, the beams the uprights and the bracings the diagonals that would transmit shear forces. This solution is interesting because the beams would not have a significant participation, which would allow to uniform its section in height and to design it only for gravity loads.

- **Shear Wall System**

They are used in reinforced concrete structures. They can consist of simple (solid) shear walls and those with openings (coupled shear walls). Coupled shear walls (Figure 12) have been one of the most popular systems used for the construction of tall buildings to resist lateral forces such as wind and earthquake. In structures with residential programs, shear walls can be distributed throughout the floor plane resisting all loads in the building without columns. In some cases, these shear walls are eccentrically located in the floor plane producing significant torsion when the building is subjected to lateral loads due to the eccentricity generated between the center of mass and the center of stiffness.

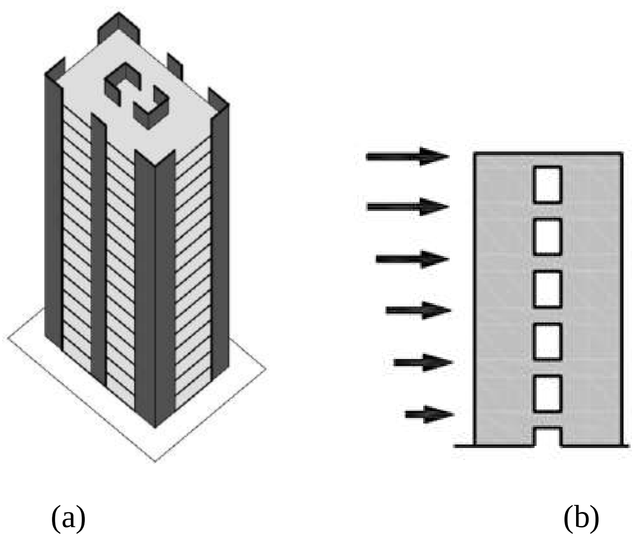


Figure 12. Shear wall system.(a) Simple (solid) shear wall, (b) Shear wall with openings (coupled shear walls) (Taranath, 2016).

Buildings designed with shear walls are generally stiffer than rigid portal frame systems, thus reducing the possibility of excessive lateral deformations, and consequently, damage. They are referred to as shear walls because they absorb much of the total lateral shear force. Although the name is appropriate, shear behavior must be controlled, especially in the face of cyclic loading (inelastic behavior). In practice this is easily achieved because shear walls provide excellent stiffness, strength and ductility.

It is important to choose optimal shear wall locations. As many shear walls as possible should be located on the periphery of the building to obtain better torsional resistance and it is important that the shear walls carry a significant fraction of the gravity load to reduce the bending demand on the wall and reduce stresses in the foundation.

The efficient design height is 35 stories for reinforced concrete structures and spacing locations are generally 9 m apart. Link beams, which interconnect shear wall segments where openings are required, are generally maximized to obtain the greatest shear and flexural strength.

- **Shear Wall - Frame System**

The above structural systems can be adopted together to increase the overall horizontal stiffness of the building and reduce lateral displacements. This effectiveness is due to the different characteristic deformation with which each of the subsystems responds in the presence of lateral loads. A portal frame deforms predominantly in shear, due to the bending of the web of beams and columns, while the shear wall responds with deformation in bending. As a consequence, when both subsystems work together, the shear wall supports the portal frame at the bottom, while the portal frame supports it at the top; thus the system exhibits a very impressive performance against lateral loads by reducing the overall deformation of the resisting system.

The efficient design height is 50 stories for reinforced concrete structures and spacing locations are generally between 4.5 m and 9 m apart. Link beams, which interconnect shear wall segments where openings are required, are generally maximized to obtain the greatest shear and flexural strength.

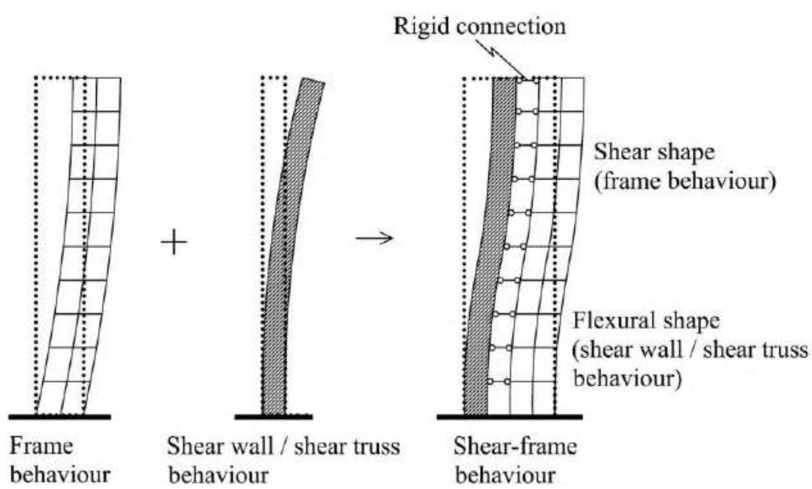


Figure 13. Behavior of the portal system - shear wall (Cammarano, 2014).

- **Limitation of Structural Systems at Present**

Although today's structural systems already allow engineers to analyze, design and construct very tall buildings, there is still a limitation in terms of structural systems. Central core systems have sufficient horizontal stiffness to reach very tall heights, however, these central cores also take up a large amount of space on each floor. For aerodynamic reasons and to keep the structures stable, very tall buildings often reduce the building perimeter with increasing height. Then a problem appears, the core area increases with height and the building perimeter decreases with height; that is, after a certain height, buildings can no longer lift people to the top, as the core area required for the elevators will be even larger than the floor area. A clear example of this problem is the Burj Khalifa building, the tallest building in the world with a height of 828 m, where the actual occupied height is only 584 m. Therefore, one of the limitations of very tall buildings is that people cannot reach the top of the buildings.

- **The Building is as Tall as a Cantilevered Cantilever Beam.**

The fundamental conceptual simplification of the tall building is a vertical cantilever beam, and as a consequence, globally it is a statically determined beam where the approximate total forces are known a priori. This means that, at any height of the building, the total forces are generally known. As such, when faced with lateral forces, the total forces acting on the cantilever beam are in the form of shear forces and overturning moments resulting in shear and bending deformations. Gravity loads are the sum of everything above a given elevation, wind shears and overturning moments are integrated from top to bottom, and even seismic forces can be approximated in this manner.

It is a structural irony that the taller the building, the purer the beam must be and, somehow, the simpler its solution. Illogical as it may seem smaller buildings can be conceptually more complex than tall buildings. Although they are cantilevered from the ground, the structural system is often a series of parallel systems or individual elements that behave with complicated three-dimensional interaction. On the other hand, by taking something as large and complex as a tall building and imagining it as a simple cantilevered beam, the designer can design it in a rational and approximate manner.

Many successful tall building designs are based on the analogy that emerges between a tall building and a tree. Like a tall building, a tree is a very slender structure, with a trunk that emerges from the ground until it becomes a series of branches that cantilever from the trunk.

Any structure that carries lateral forces to the ground must resist two structural phenomena: shear and bending. The taller and slimmer the building, the more efficient the shear resistance system must be, because it is essential to carry lateral loads to the vertical elements which, in turn, resist the overturning forces in the cantilever. A rigid shear system is necessary for the entire building to act as one giant beam rather than a collection of individual elements or subsystems. Because it is not possible to create a completely rigid shear system, there is a phenomenon called "shear lag." This occurs when the overturning stresses are not distributed linearly, resulting in less effective use of the vertical elements to resist the overturning moments in the structure.

Although efficient and rigid shear resistance systems can reduce shear deflections to a small portion of the target deflections, it is not practical to do the same for bending deflections. Deflection can generally only be reduced at the cost of increasing the size of columns and/or walls. Beam deflection can be reduced, for example, by half by doubling the cross-sectional area. The great expense of reducing deflections by increasing the cross-sectional area of the vertical element imposes very practical limits on reducing deflection due to bending.

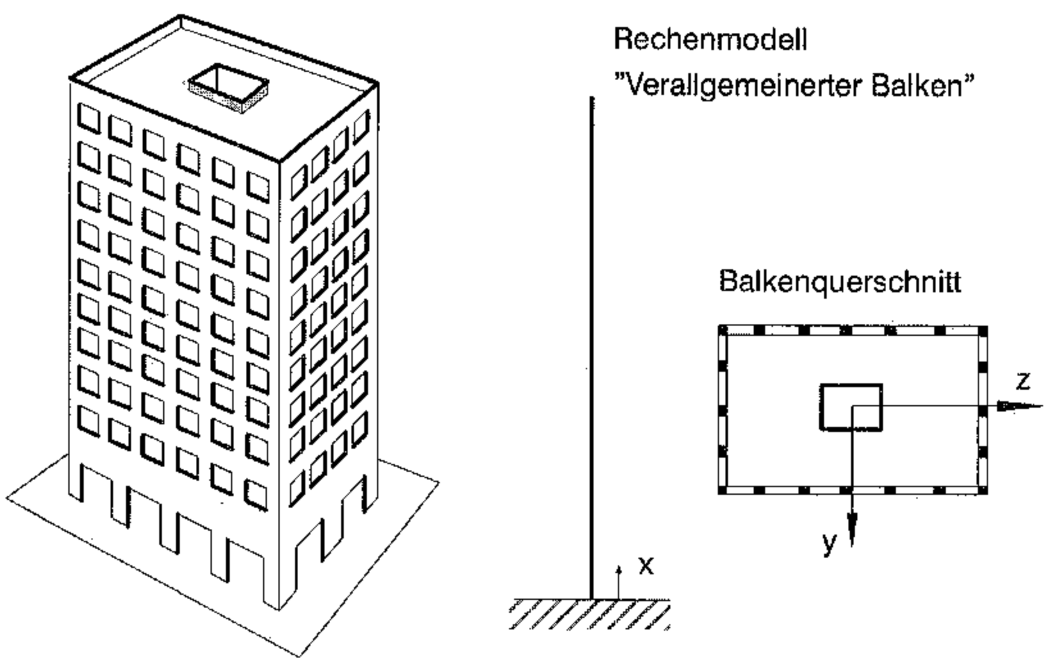


Figure 14. Tall building considered as a cantilevered cantilever beam (Schmidts, 1998).

- **Global Structural Analysis of Tall Buildings**

A global structural analysis of tall buildings involves numerical modeling and static, dynamic and stability characterization.

First of all, numerical modeling becomes indispensable for an evaluation of the structural response. With the technological progress of recent years, finer methods such as the finite element method (FEM) have been developed and used to simulate the structural behavior of the building; however, due to the high computational and resource costs, interest has always been devoted to the development of simplified models that allow fast, low-cost analyses with an engineering acceptable accuracy. Two approaches have been proposed: idealizing the building as a system of one-dimensional concentrated masses and idealizing the building as a continuous system.

A system of one-dimensional concentrated masses is connected by massless rods characterized by the stiffness of the floor. This model is widely known in earthquake engineering as the "shear beam model", where infinite bending stiffness is assumed and the vertical structural elements are considered to be inextensible; however, considering the vertical elements to be inextensible is not valid for slender buildings whose axial deformation is not negligible. In order to model the building as a system of concentrated masses and to take into account its most important characteristic stiffnesses, this research project will make use of the transfer matrix method to solve firstly buildings with vertical discontinuities that do not allow the development of a closed form equation and secondly uniform buildings in height as a verification to the continuous model with closed formulas.

A continuous system connects two replacement beams by means of inextensible elements (rigid links) that transmit only horizontal loads. These models are widely used when sensitive parametric analysis or rapid estimation of building response is required. It is ideal for modeling buildings that do not have vertical discontinuities because they allow the development of closed formulas that are easy and quick to apply. Its formulation is fully analytical allowing to easily identify the key structural parameters that govern the building behavior drastically reducing the computational cost.

Secondly, the static, dynamic and stability characterization allows to understand the behavior of the building. The static characterization allows the calculation of the static horizontal displacements of the building and consequently the drifts, thus allowing to verify the compliance with the current regulations and above all to evaluate the performance of the building. The dynamic characterization allows the modal identification, thus providing the frequencies, periods and participatory mass factors for a spectral modal analysis; having as results global indicators such as dynamic displacements and drifts useful to determine the damage level of a structure. Stability characterization allows determining the global critical load of the building as a performance indicator; as mentioned by Zalka (2020) any weakness detected during stability analysis also leads to unfavorable behavior in the dynamic and static analysis of the building.

- **Continuous Method**

The continuous method assumes that all horizontal elements connecting the vertical components are effectively connected over the height of the building to produce a continuous connecting means, i.e., the connecting beams are replaced by a system of uniformly distributed sheets. As a consequence of the continuous method, the three-dimensional (3D) structure leads to a replacement beam (RB) which is characterized by equivalent properties Ki that try to represent as best as possible the actual stiffness of the structural system.

The basic assumptions of the method consider that the structural elements are elastic and linear, the diaphragms are considered rigid in their plane and only transfer horizontal forces, the midpoints of the connecting beams are considered points of contraflexure, the discrete shear forces in the connecting beams are replaced by an equivalent continuous shear flow along the midpoint of the connecting plates, the Bernoulli-Navier hypothesis is valid for the connection beams, the minimum number of floors of the building is four, the P-delta effects are negligible, the connection beams do not deform axially and we will replace the whole building (consisting of discrete elements) by a continuous beam and then analyze this continuous beam as a replacement of the building.

- **Continuous Replacement Beam (RB) Models**

The structural nature of a tall building is three-dimensional; however, representing the tall building by a suitable replacement beam is only possible if the complex combination between the structural systems can be drastically simplified while maintaining the behavior of the structure and with reasonable accuracy results. Therefore, it is important to choose suitable replacement beams for each structural system, which can adequately represent the predominant modes of behavior; and then combine them to account for the complex interaction between the structural systems.

The complete number of kinematic fields of plane systems connected to each other in parallel depend on the number of kinematic fields associated with each element (Figure 15). If each element contains three kinematic fields (transverse u_i, rotational ϑ_i and axial w_i), then the number of kinematic fields of the whole system is three times the number of elements. Since the elements are connected by inextensible rigid links, it is possible to assume an identical horizontal displacement field for the whole system ($u_1 = u_2 \ldots u_{i-1} = u_i = u$), while the other kinematic fields may be different for each element.

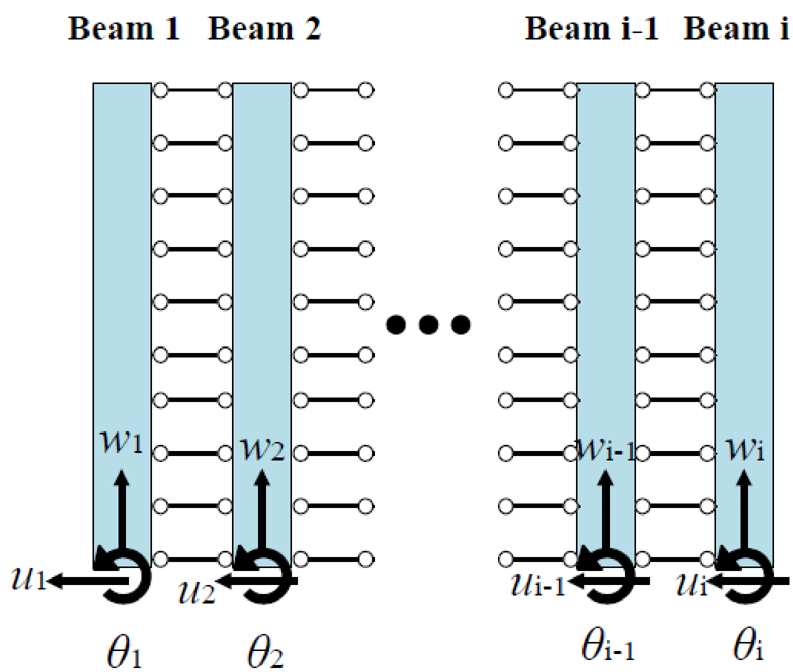


Figure 15. Continuous system consisting of several beam elements aligned in parallel (Moghadasi, 2015).

Depending on the structural characteristics of the structural systems composing the building it is possible to use various RB models. The two main characteristics that define the appropriate idealization are:

- The equivalent stiffness: bending stiffness K_f and shear stiffness K_f.
- The kinematic fields: transverse u and rotational θ and φ.

With respect to RB models applied in building analysis and the kinematic field point of view, current continuous models can be generally classified into three categories: one-field models, two-field models and two-field models.

a) Models of a Field

The transverse deformation field is u and they are the simplest to formulate as RB systems.

- Bending beam (EBB)

Suitable for a first approach to the structural modeling of tall buildings, it is characterized by K_b flexural behavior and stiffness, as a consequence it is suitable for modeling slender shear walls. The potential energy associated with this model is:

$$\nu_{EBB} = \frac{1}{2}\int_0^H K_b {u''_{(x)}}^2 \, dx \tag{1}$$

- Shear beam (SB)

Characterized by a shear behavior and stiffness K_s. Suitable for modeling shear frames. The potential energy associated with this model is:

$$\nu_{SB} = \frac{1}{2}\int_0^H K_s {u'_{(x)}}^2 \, dx \tag{2}$$

- Two-beam coupling (CTB)

It consists of the parallel coupling of a bending beam (EBB) and a shear beam (SB), they are connected by a continuous medium transmitting only horizontal forces and both beams experience a single kinematic field u. They are primarily suitable for modeling portal frames; and in lesser application for modeling coupled shear walls and shear wall - portal frame systems. As a particular case the CTB model approximates a bending beam and a shear beam when K_b and K_s tend to infinity respectively. The potential energy associated with this model is:

$$\nu_{CTB} = \frac{1}{2}\int_0^H K_b u''^2_{(x)} \, dx + \frac{1}{2}\int_0^H K_s u'^2_{(x)} \, dx \tag{3}$$

b) Two-Field Model

The transverse deformation field is u and the rotation field is θ.

- Timoshenko Beam (TB)

It is characterized by a series coupling between a bending beam (EBB) and a shear beam (SB). The potential energy associated with this model is:

$$\nu_{TB} = \frac{1}{2}\int_0^H \left\{K_b {\theta'_{(x)}}^2 + K_s\left[\theta_{(x)} - u'_{(x)}\right]^2\right\} dx \tag{4}$$

It is important to mention that compared to the bending beam model (EBB), the Timoshenko beam (TB) model can be used to more accurately model a shear wall. This helps to account for shear deformation, where such deformation can be significant in relatively non-slender and ordinary walls.

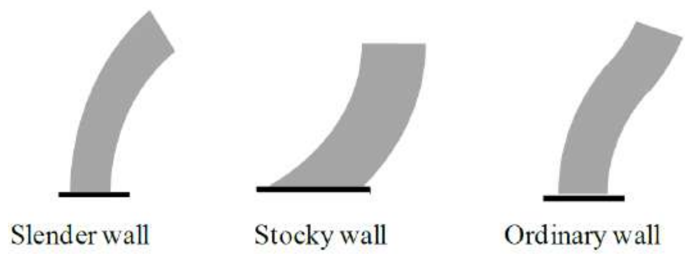


Figure 16. Schematic deformations of thin wall, non-thin wall and ordinary wall (Moghadasi, 2015).

- Sandwich Beam (SWB)

It is characterized by a parallel coupling between a Timoshenko beam (TB) and a bending beam (EBB). In the literature it is considered the most complete model using a single kinematic field u, because it is characterized by three different stiffnesses: local bending stiffness (K_{b1}), global bending stiffness (K_{b2}) and shear stiffness (K_{s1}). This model is appropriate to correctly model portal frames and coupled shear walls. The potential energy associated with this model is:

$$\nu_{SWB} = \frac{1}{2}\int_0^H \left\{K_{b1}\theta'^2_{(x)} + K_{s1}\left[\theta_{(x)} - u'_{(x)}\right]^2\right\} dx + \frac{1}{2}\int_0^H K_{b2} u''^2_{(x)} dx \tag{5}$$

The RB sandwich beam model has been extensively studied in the literature because it is possible to represent all structural schemes by means of its three characteristic stiffnesses.

Potzta, G. (2002) developed a whole building RB model using a SWB beam with an energy approach and derived the three characteristic stiffnesses of the SWB by applying a sinusoidal displacement and balancing the total deformation energy of the building with the sum of the deformation energies of each structural scheme. They used this SWB for wind, earthquake and building stability analyses. Bozdogan, K. (2010) developed the lateral static, dynamic and stability analysis using a SWB beam and the transfer matrix method. Zalka, K. (2020) derived his solutions using the behavior of a portal frame as a basis because it has each of the three characteristic stiffnesses of a SWB. He provided a complete treatment for all structural schemes (lateral deflection, rotational, frequency and stability) and showed that these areas are closely related to each other.

- Generalized Sandwich Beam (GSWB)

A generalized SWB model, it is characterized by a parallel coupling between a Timoshenko beam (TB) and the coupling of two beams (CTB); that is, the parallel coupling of two beams characterized by the series coupling of their bending stiffness (K_{b1}, K_{b2}) and shear stiffness (K_{s1}, K_{s2}). If the shear stiffness of the CTB (K_{s2}) is neglected, the GSWB results in a SWB. The potential energy associated with this model is:

$$\nu_{GSWB} = \frac{1}{2}\int_0^H \left\{K_{b1}{\theta'_{(x)}}^2 + K_{s1}\left[\theta_{(x)} - u'_{(x)}\right]^2\right\} dx + \frac{1}{2}\int_0^H \left[K_{b2}{u''_{(x)}}^2 + K_{s2}{u'_{(x)}}^2\right] dx \tag{6}$$

c) Three-Field Model

These models were proposed by Moghadasi, H. (2015). The transverse deformation field is u and the rotation field is θ and φ.

- Generalized Sandwich Beam (GSB)

It is characterized by a parallel coupling between two Timoshenko beams (TB); that is, the coupling of two beams characterized by a series coupling of their bending stiffness (K_{b1}, K_{b2}) and shear stiffness (K_{s1}, K_{s2}). Moghadasi, H. (2015) presented this GSB model and solved the static and dynamic analysis. Due to the complexity of the partial differential equations of motion, he discretized the differential equations to solve it numerically by one-dimensional finite element model by transforming the GSB (distributed parameter cantilever) to a multi-degree of freedom system. The potential energy associated with this model is:

$$\nu_{GSB} = \frac{1}{2}\int_0^H \left\{K_{b1}{\theta'_{(x)}}^2 + K_{s1}\left[u'_{(x)} - \theta_{(x)}\right]^2\right\} dx + \frac{1}{2}\int_0^H \left\{K_{b2}{\psi'}^2 + K_{s2}\left[u'_{(x)} - \psi_{(x)}\right]^2\right\} dx \tag{7}$$

- Three-field extensible CTB

Moghadasi, H. (2015) presented this three-field CTB model to adequately represent a coupled shear wall. This three-field CTB model is characterized by the parallel coupling between a tensile Timoshenko beam (TB) and a continuous core with a support rotation constraint (RC). The TB represents the condensed equivalence of the two walls and is characterized by its bending stiffness K_{b1}, its shear stiffness K_{s1} and its axial stiffness K_{a1}; and the RC represents the continuous effect of the connecting beam and is characterized by its shear stiffness K_{s2}. The potential energy associated with this model is:

$$v_{CTB-3} = \frac{1}{2}\int_0^H \left\{K_{b1}{\theta'_{(x)}}^2 + K_{a1}{w'_{(x)}}^2 + K_{s1}\left[u'_{(x)} - \theta_{(x)}\right]^2\right\} dx + \frac{1}{2}\int_0^H K_{s2}\gamma_c^2 dx \tag{8}$$

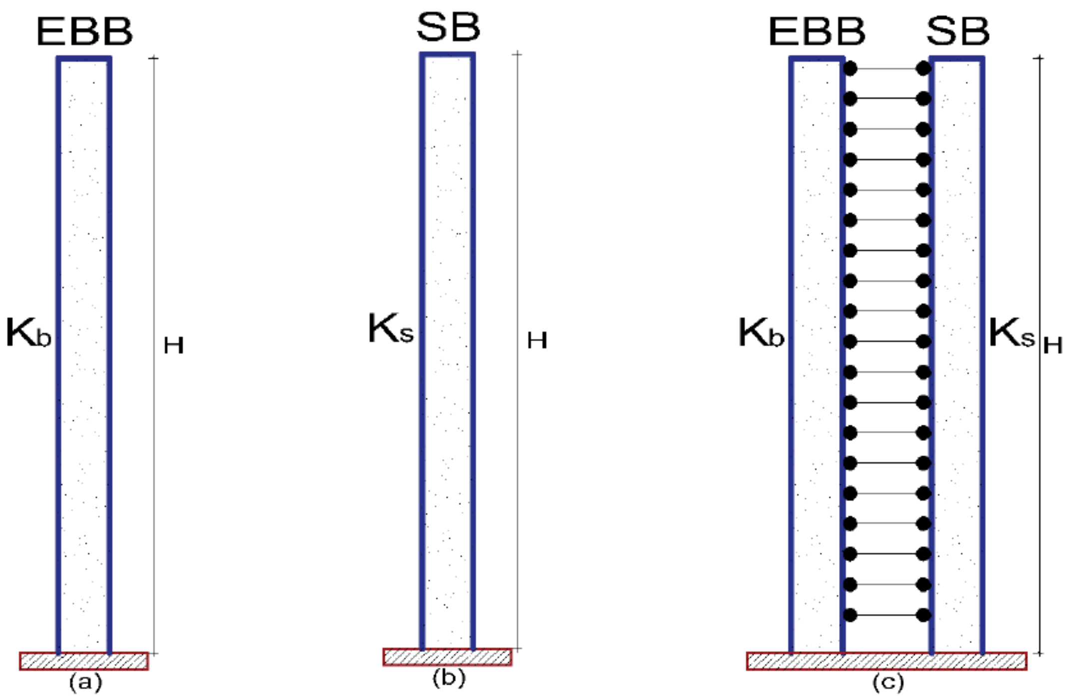


Figure 17. Models of a field: a) EBB beam, b) SB beam and c) CTB beam.

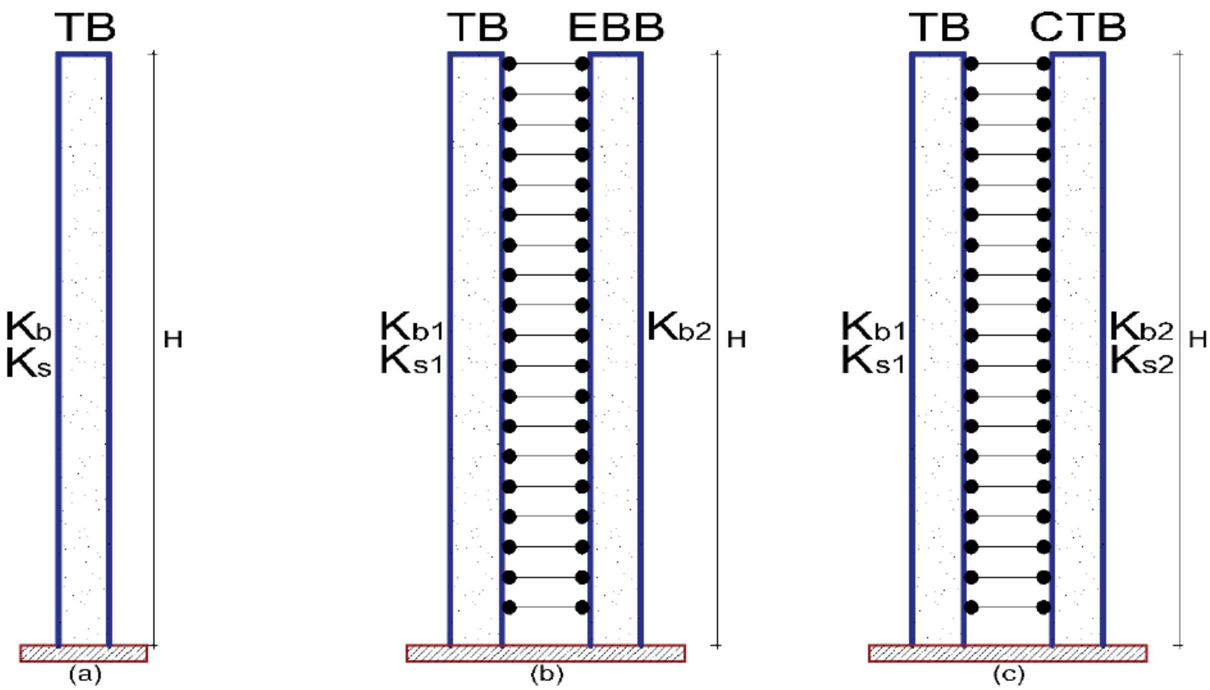


Figure 18. Two-field models. a) TB beam, b) SWB beam and c) GSWB beam.

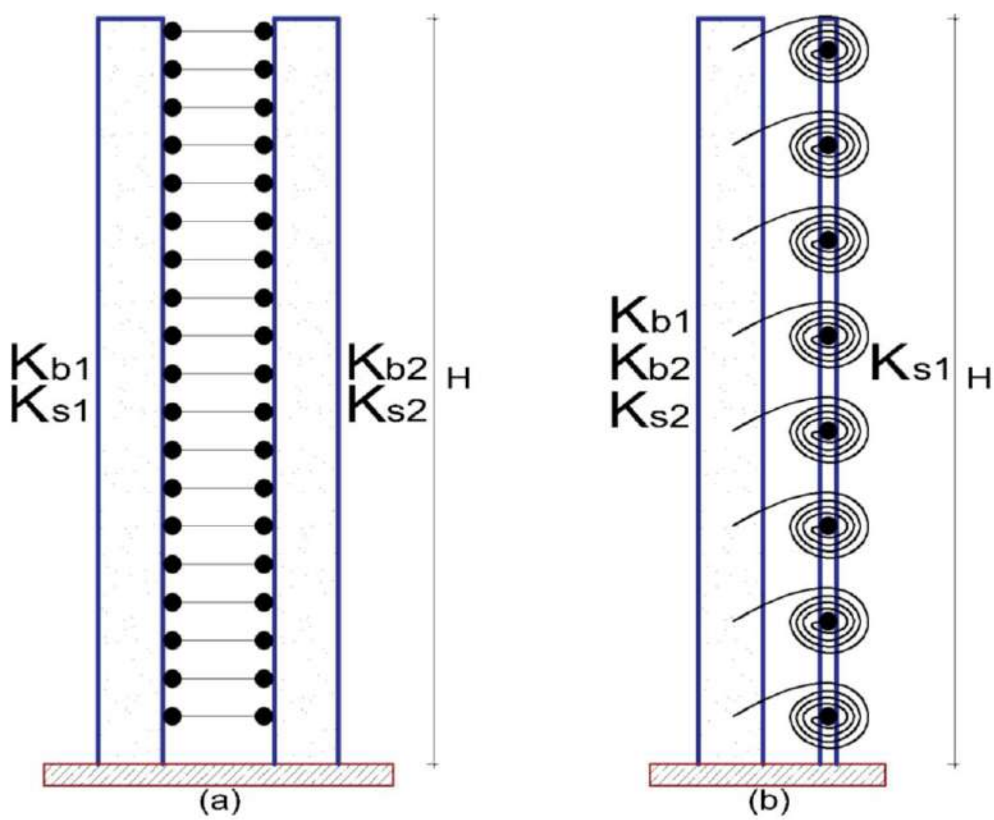


Figure 19. Three-field models. a) GSB beam and b) three-field extensible CTB beam.

- **Transfer Matrix Method**

There are several problems in engineering that are specified with boundary conditions, working with a high number of constants is not practical and is subject to possible calculation errors. In order to reduce calculation errors and work with the minimum number of constants, the transfer matrix method was implemented to mathematically reduce the boundary conditions problem to an initial conditions problem.

The transfer matrix method involves constructing a relationship between the end nodes of a structural element. Its application in a building with structural properties that are uniform in each substructure is appropriate because it is possible to calculate the transfer matrix of each substructure and then assemble it into a single global transfer matrix of the entire structure. By eliminating internal nodes by condensation, the size of the transfer matrix is minimized and kept constant in the calculation process and equal to the order of the differential equation of the structure.

- **Structural Loads**

Tall and high-rise buildings are subjected primarily to vertical loads (dead and live) and horizontal loads (wind and seismic). As the height of the building increases, the effect of horizontal loads also increases. Therefore, for tall buildings, it is important to choose structural systems that have sufficient horizontal stiffness.

From a structural design point of view, tall buildings show greater sensitivity to wind-induced lateral loads than medium- and low-rise buildings. While seismic loads increase with building weight, wind loads increase with building height. This results in the consequence that in certain cases, lateral drift due to wind is more critical than drift due to earthquake.

2.3 DEFINITION OF BASIC TERMS

In relation to the terminology applied to this research project, it is necessary to clarify that the term "system" is applied to refer to the complete three-dimensional structure, while "element" to the different flat structures studied, such as: frames, shear walls, coupled shear walls, cores etc.

- Replacement beam: one-dimensional continuous equivalent resulting from the application of the continuous method to the structural analysis of a tall building (Moghadasi, 2015).

- Continuous method: Approximate method in which the discrete structure is represented by a continuous one, replacing the horizontal connecting beams and the frame by an equivalent continuous medium (Glück, Gellert, & Danay, 1972). The method requires characteristic stiffnesses, kinematic fields and external loads for use in the structural analysis of a tall building uniform in height.
- Transfer matrix method: Method used in mathematics to solve differential equations containing discontinuities (Bozdogan, 2010). The method requires characteristic stiffnesses, kinematic fields and external loads for use in the structural analysis of a tall building with structural discontinuities in height.
- Equivalent stiffness: Stiffness corresponding to a type of deformation of the structure (local bending, global bending, local shear and global shear).
- Kinematic field: Independent degrees of freedom of a replacement beam (displacement and rotation).
- External loads: External loads on the structure (lateral loads, distributed mass density and point loads).
- Global structural analysis of tall buildings: Structural analysis of the building where the building is considered as a complete unit. It includes static, dynamic and stability structural analysis of the building.
- Static analysis: The building is subjected to external lateral loads and as a consequence displacements, interstory drifts and rotations are obtained.
- Dynamic analysis: The building is subjected to free vibration and as a result the frequencies and vibration periods are obtained.
- Stability analysis: The building is subjected to uniformly distributed external vertical and/or point loads and as a result the critical buckling load of the building is obtained.

2.4 OPERATIONALIZATION OF THE VARIABLES

Tabla.2 *Description of variables and indicators.*

Variable	Dimensions	Indicators	units
V1	Characteristic stiffness	Local bending stiffness	tn
		Global bending stiffness	tn
		Local shear stiffness	tn
Continuous method / Transfer matrix method	Kinematic field	Global shear stiffness	tn
		RB displacement	m
		RB rotation	rad
	Load	Lateral loads	tn/m
		Distributed mass density	kg-masa/m
		Point load	tn
V2	Static analysis	Lateral displacement	m
		Floor drift	m/m
Global structural analysis of tall buildings		Rotation	rad
	Dynamic analysis	Frequency	hz
		Period	s
	Stability analysis	Critical load	tn

3 METHODOLOGY

3.1 METHODOLOGICAL DESIGN

It is a basic type of research, because it seeks to generate scientific knowledge to enrich theoretical and scientific knowledge, oriented to the knowledge of principles and laws (Valderrama, 2006). In this research project, the differential equations that govern the static, dynamic and stability behavior of novel replacement beams have been established and solved, generating new theoretical knowledge.

The method used is deductive, because particular cases are analyzed from which conclusions of a general nature are drawn (Valderrama, 2006). In this research project to formulate and develop the generalized replacement beam, theories and indications left by other authors who studied the subject have been used.

The design is of a non-experimental type, because it is a systematic and empirical investigation, in which the independent variables are not manipulated, because they are already given (Valderrama, 2006). In this research project, the precision and reliability analysis was carried out keeping the independent variables constant and only the height of the tall building was varied to vary the global parameters.

In a first stage, the work will have a qualitative approach, because the phenomenon is described, understood and interpreted through the researcher's experience (Valderrama, 2006). In the research project, the qualitative approach will allow obtaining models of replacement beams for the structural systems (frames, shear walls, coupled shear walls and cores) and for the tall building.

In a second stage, the work will have a quantitative approach, because theories are developed and tested; in addition, the collection, analysis and processing of data is used to answer the formulation of the problem (Valderrama, 2006). In the research project, the quantitative approach will allow an analysis of precision and sensitivity to evaluate the efficiency of the proposed methodology.

3.2 POPULATION AND SAMPLE

3.2.1 Population

- The study population comprises all tall buildings (structural systems).

3.2.2 Sample

- The study sample comprises a total of 1017 structural systems: 90 shear walls, 558 portal frames, 333 coupled shear walls, 36 tall buildings.

3.3 DATA COLLECTION TECHNIQUES

The data for the analysis of accuracy and reliability of the proposed methodology were based on the criteria of the author of the thesis and some structural plants used in the existing literature on the structural analysis of tall buildings.

3.4 INFORMATION PROCESSING TECHNIQUES

The computational technique was used to process the information. The instrument used was a personal computer, Microsoft Excel spreadsheet software and SAP 2000 and ETABS 2016 finite element programs.

4 RESULTS

4.1 STATIC ANALYSIS OF INDIVIDUAL STRUCTURAL SYSTEMS

The horizontal stiffness of a tall building is composed of the contribution of different structural systems: shear walls, portal frames, coupled shear walls and cores. The contribution of each structural system to the horizontal stiffness of the building is distributed proportionally to its individual stiffness, but the nature of their behavior is somewhat different; therefore, it is essential for the engineer to know the behavior of each structural system in order to achieve a structure with optimum horizontal stiffness.

It is common to assume a triangular load distribution for earthquake static analysis and a uniformly distributed load for wind static analysis. However, for the purpose of generalizing the analysis, two cases are proposed:

- **Case 1**: A continuous analysis is considered because the method used is based solely on the continuous method; a general lateral load distributed over the height of the element is assumed. This load is dependent on a parameter a that controls its shape in height. This general lateral load model was proposed by Miranda (1999).

  $$f_{(x)} = \frac{W_{max}}{1 - e^{-a}}\left(1 - e^{-a+a\frac{x}{H}}\right) \tag{9}$$

  Where W_{max} is the intensity of the load distributed at the top of the model $(z = 0)$ and a is a dimensionless parameter that controls the shape of the lateral load. As shown in Figure 20, the extreme values of $a \to \infty$ and $a \to 0$ correspond to uniform and triangular load distributions, respectively.

  The main disadvantage is that it is only applicable to structures where the cross-section is uniform in height and the lateral load is continuous. The main advantage is that continuous closed-form solutions are obtained, allowing parametric analysis.

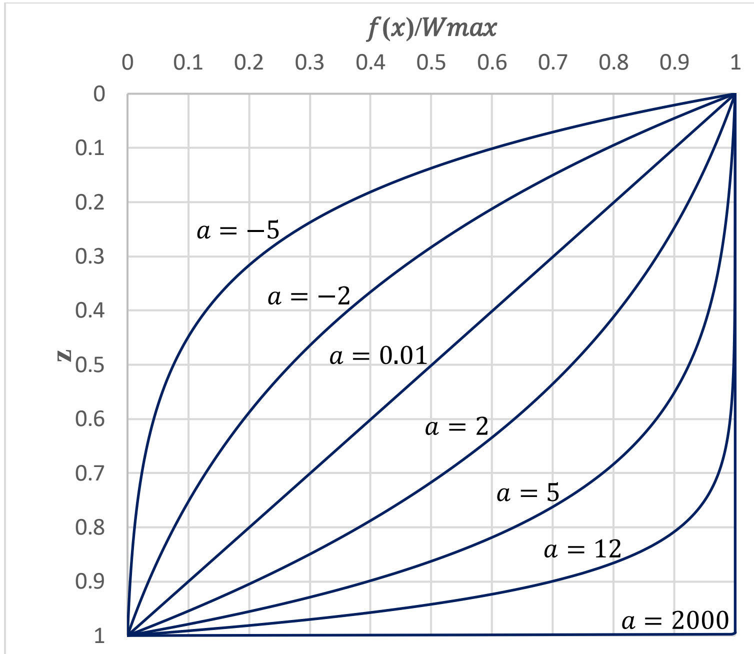


Figure 20. Effect of dimensionless parameter a on the shape of lateral load distribution for case 1.

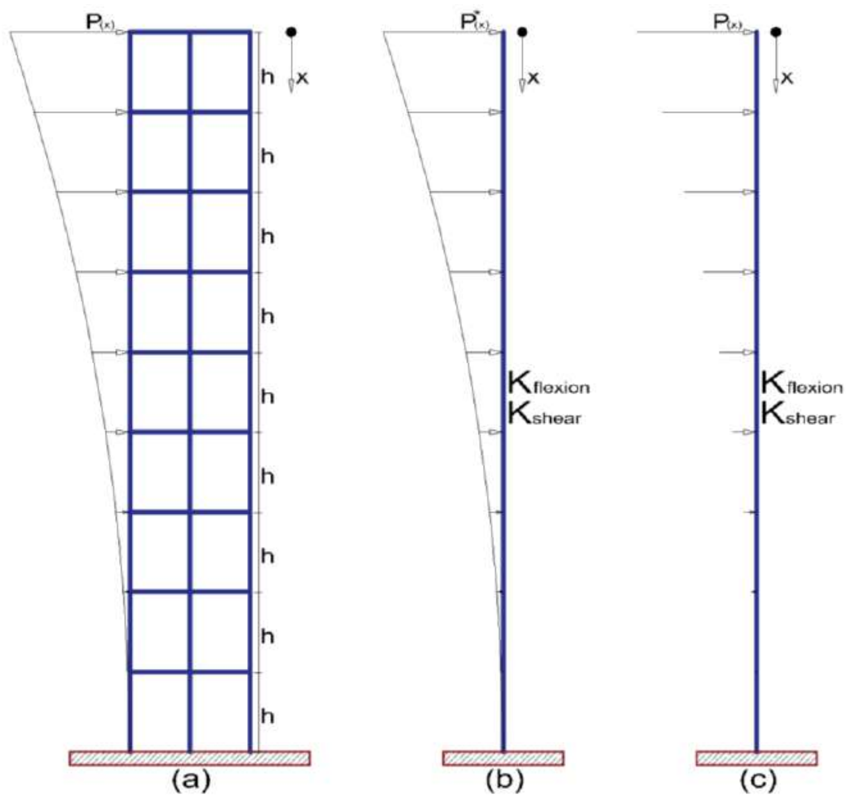


Figure 21. Structure subjected to lateral load. a) Structure and original load, b) case 1: replacement beam with continuous load and c) case 2: replacement beam with concentrated load.

- **Case 2:** A discrete analysis is considered because the methods used are the continuous method and the transfer matrix method; an arbitrary horizontal point load applied at floor level is assumed. These point loads can be the result of point loads directly applied at floor level or the result of loads transmitted from the facade to the floor slabs.

  The main disadvantage is that closed continuous solutions that allow parametric analysis are not obtained. The main advantage is that it allows to analyze structures whose cross section is not continuous in height and/or for structures where the loads are applied at floor level, whether their cross section is uniform or not; i.e., it is considered a case of general analysis because it even serves as a verification of case 1.

4.1.1 Bending Beam of a Field (EBB)

This model is typically used for slender shear walls and/or columns of flat slab buildings, i.e., where the shear stiffness is infinite and therefore, their shear deformation can be considered negligible. The model takes into account a transverse motion u and a bending stiffness.

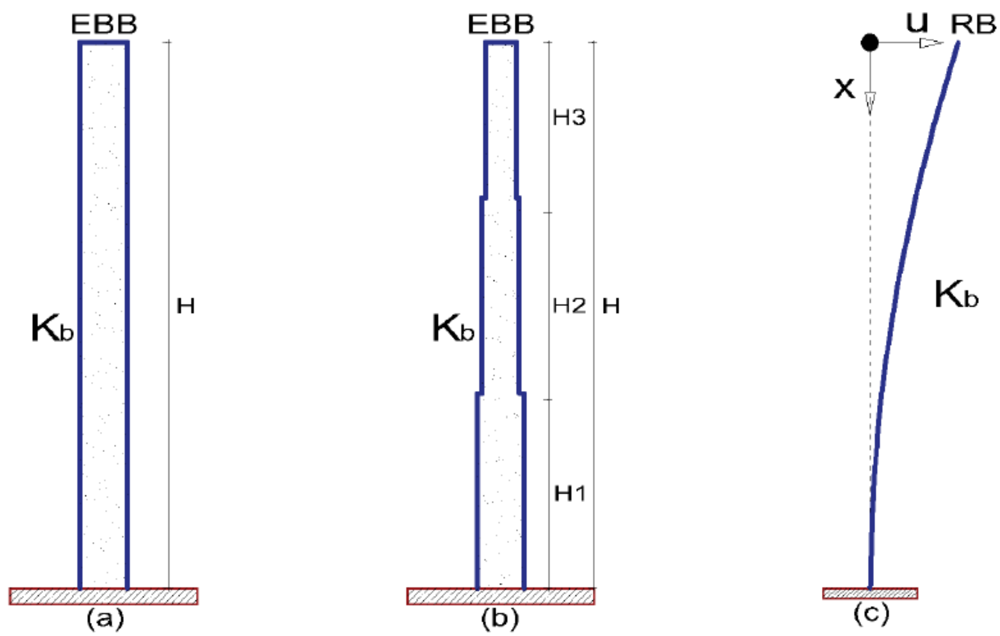


Figure 22. Euler Bernoulli Beam of a field (EBB). a) Case 1, b) Case 2 and c) equivalent RB.

4.1.1.1 Case 1

The potential energy of the EBB model of a field is:

$$V = \frac{1}{2}\int_0^H K_b {u''_{(x)}}^2\, dx \tag{10}$$

Where:

$$K_b = \sum EI_i \tag{11}$$

The work done by the external force is:

$$W = \int_0^H f_{(x)} u_{(x)}\, dx \tag{12}$$

Consequently, the total potential energy of the EBB beam of a field subjected to a general lateral load distribution is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H K_b {u''_{(x)}}^2 \, dx - \int_0^H f(x) u_{(x)} \, dx \tag{13}$$

Closed-form solutions of the model on which a transverse load acts are achieved by solving the differential system arising from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\delta\mathcal{U} = \int_0^H K_b u''_{(x)} \delta u''_{(x)} dx - \int_0^H f(x) \delta u_{(x)} \, dx - \int_0^H u_{(x)} \, \delta f(x) dx \tag{14}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\delta\mathcal{U} = \left[K_b u''_{(x)} \delta u'_{(x)}\right]_0^H - \left[K_b u'''_{(x)} \delta u_{(x)}\right]_0^H + \int_0^H [K_b u'''' - f(x)] \delta u_{(x)} dx - \int_0^H u_{(x)} \, \delta f(x) dx \tag{15}$$

Equating the terms to zero results in the following equation:

$$K_b u''''_{(x)} - f(x) = 0 \tag{16}$$

And boundary conditions:

$$\begin{Bmatrix} u''_{(0)} = 0 \\ u'''_{(0)} = 0 \end{Bmatrix} \tag{17}$$

A fourth order differential equation is obtained. Normalizing the differential equation by the variable $z = x/H$:

$$u''''_{(z)} = \frac{H^4}{K_b} f(z) \tag{18}$$

Assuming a general lateral load (Miranda E. , 1999):

$$f_{(x)} = \frac{W_{max}}{1 - e^{-a}} \left(1 - e^{-a + a\frac{x}{H}}\right) \rightarrow f_{(z)} = \frac{W_{max}}{1 - e^{-a}} (1 - e^{-a + az}) \tag{19}$$

Replacing it in the differential equation:

$$u_{(z)}'''' = \lambda(1 - e^{-a+az}) \tag{20}$$

Where:

$$\lambda = \frac{W_{max}H^4}{K_b(1 - e^{-a})} \tag{21}$$

The expression for $u_{(z)}$ is proposed:

$$u_{(z)} = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + \frac{\lambda}{24} z^4 - \frac{\lambda}{a^4} e^{-a+az} \tag{22}$$

The constants are obtained by evaluating the relevant boundary conditions:

$$\begin{Bmatrix} u_{(1)} = 0 \\ u'_{(1)} = 0 \\ u''_{(0)} = 0 \\ u'''_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} C_0 = \lambda\left[\left(\frac{1}{8} + \frac{1}{a^4} - \frac{1}{a^3}\right) + e^{-a}\left(\frac{1}{2a^2} + \frac{1}{3a}\right)\right] \\ C_1 = \lambda\left[\left(-\frac{1}{6} + \frac{1}{a^3}\right) - e^{-a}\left(\frac{1}{a^2} + \frac{1}{2a}\right)\right] \\ C_2 = \lambda\frac{e^{-a}}{2a^2} \\ C_3 = \lambda\frac{e^{-a}}{6a} \end{Bmatrix} \tag{23}$$

The lateral displacement is obtained by rewriting the expression for $u_{(z)}$:

$$\begin{aligned} u_{(z)} = \frac{W_{max}H^4}{K_b(1 - e^{-a})}\Big\{&\frac{1}{24}z^4 + \frac{e^{-a}}{6a}z^3 + \frac{e^{-a}}{2a^2}z^2 + \left[\left(-\frac{1}{6} + \frac{1}{a^3}\right) - e^{-a}\left(\frac{1}{a^2} + \frac{1}{2a}\right)\right]z \\ &+ \left[\left(\frac{1}{8} + \frac{1}{a^4} - \frac{1}{a^3}\right) + e^{-a}\left(\frac{1}{2a^2} + \frac{1}{3a}\right)\right] - \frac{1}{a^4}e^{-a+az}\Big\} \end{aligned} \tag{24}$$

The maximum displacement is obtained by evaluating $u_{(z)}$ at 0:

$$u_{(0)} = \frac{W_{max}H^4}{K_b(1 - e^{-a})}\left\{\left(\frac{1}{8} + \frac{1}{a^4} - \frac{1}{a^3}\right) + \left(\frac{1}{2a^2} + \frac{1}{3a} - \frac{1}{a^4}\right)e^{-a}\right\} \tag{25}$$

The interstory drift can be obtained by deriving $u_{(z)}$ once:

$$\Delta_s = \frac{W_{max}H^4}{K_b(1 - e^{-a})}\left\{\frac{1}{6}z^3 + \frac{e^{-a}}{2a}z^2 + \frac{e^{-a}}{a^2}z + \left[\left(-\frac{1}{6} + \frac{1}{a^3}\right) - e^{-a}\left(\frac{1}{a^2} + \frac{1}{2a}\right)\right] - \frac{1}{a^3}e^{-a+az}\right\} \tag{26}$$

The global drift is obtained as the quotient between maximum displacement and total height:

$$\Delta_g = \frac{W_{max} H^3}{K_b(1 - e^{-a})} \left\{ \left(\frac{1}{8} + \frac{1}{a^4} - \frac{1}{a^3} \right) + \left(\frac{1}{2a^2} + \frac{1}{3a} - \frac{1}{a^4} \right) e^{-a} \right\} \tag{27}$$

For the case of a uniformly distributed lateral load $(a \to \infty)$:

$$\begin{Bmatrix} u_{(z)} = \dfrac{W_{max} H^4}{K_b} \left(\dfrac{1}{24} z^4 - \dfrac{1}{6} z + \dfrac{1}{8} \right) \\ u_{(0)} = \dfrac{W_{max} H^4}{8K_b} \\ \Delta_s = \dfrac{W_{max} H^4}{6K_b} (z^3 - 1) \\ \Delta_g = \dfrac{W_{max} H^3}{8K_b} \end{Bmatrix} \tag{28}$$

According to the analysis of equations and graphs:

- ✓ The stiffness is inversely proportional to the fourth power of its height.
- ✓ The lateral displacement profile shows a behavior in favor of the lateral load (concavity to the right) with a tendency to reduce to an infinite slope at the base.
- ✓ The interstory drift profile shows a behavior against lateral loading (concave to the left) with maximum efficiency in the lower zone of the beam with a tendency to zero at the base.
- ✓ The normalized lateral displacement profile and the normalized interstory drift profile are practically identical for all cases and independent of the parameter a.

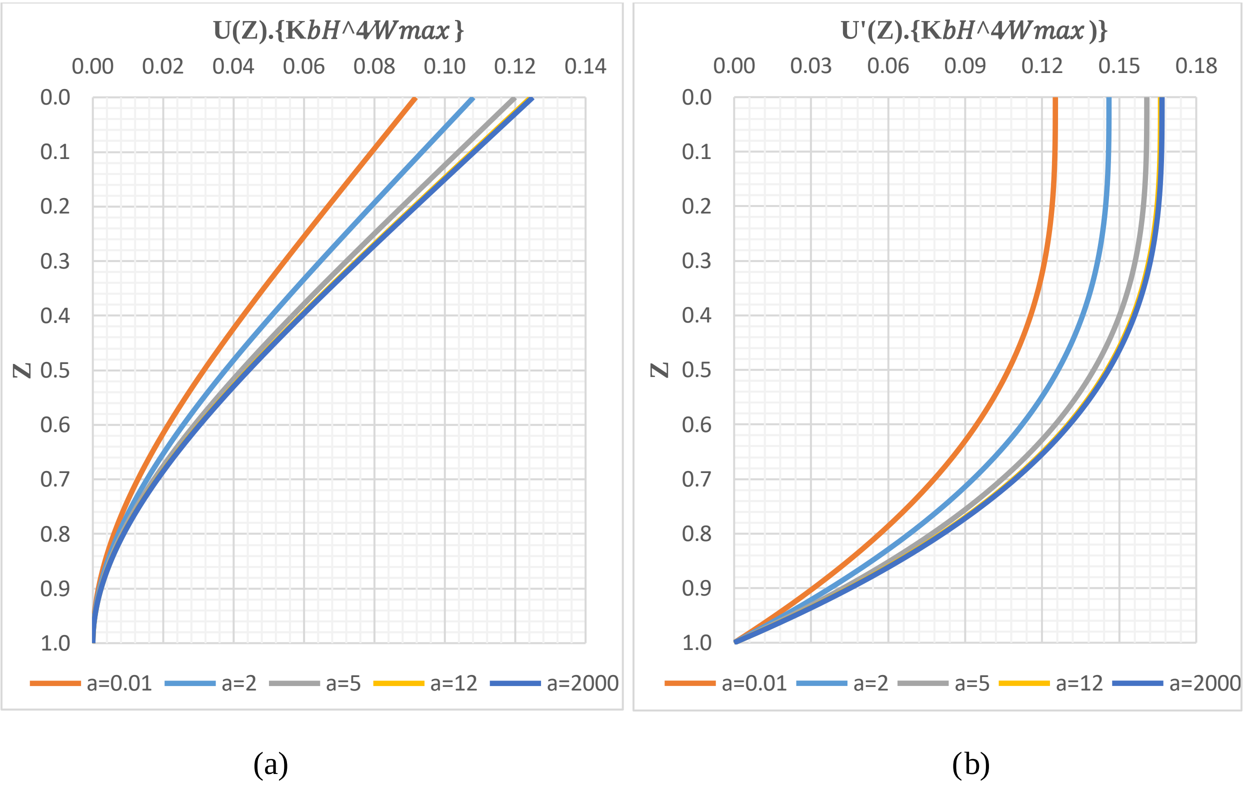


Figure 23. Effect of parameter a. a) Lateral displacement and b) Interstory drift.

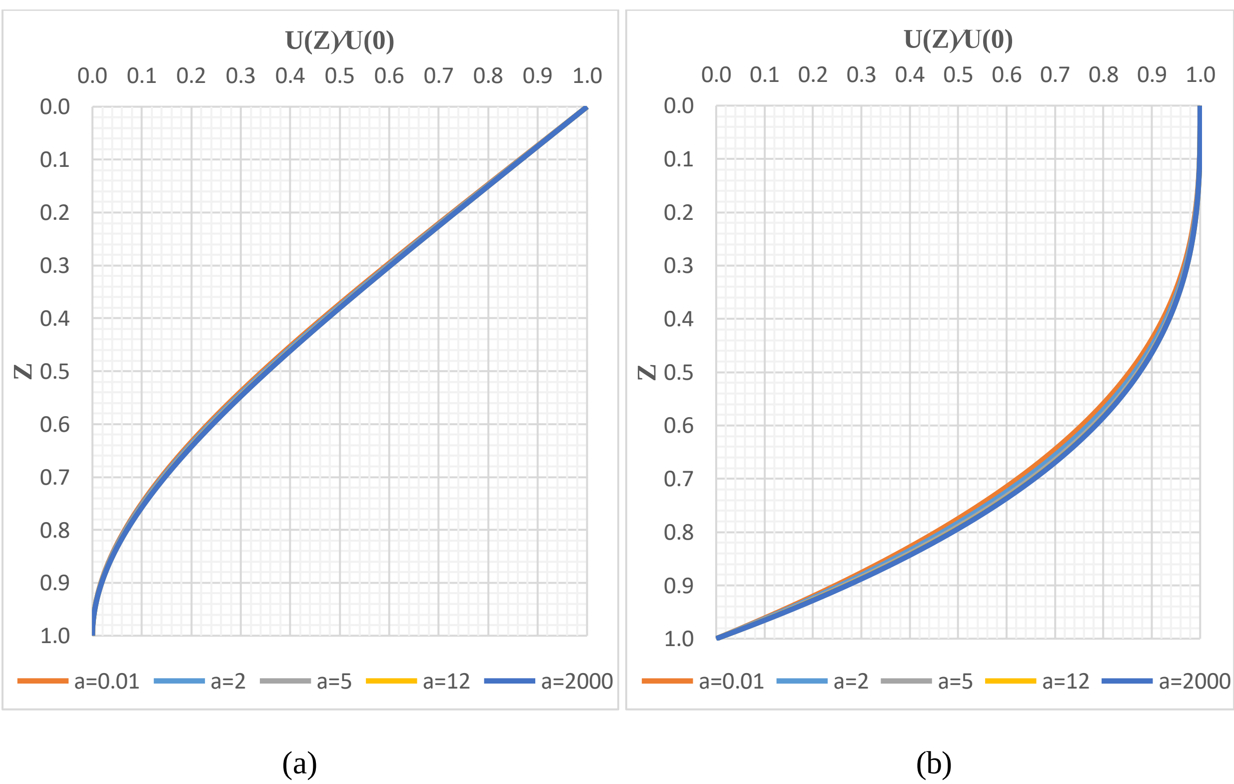


Figure 24. Effect of parameter a. a) Normalized lateral displacement and b) Normalized interstory drift.

4.1.1.2 Case 2

- **Calculation of The Transfer Matrix**

According to the fourth order differential equation:

$$u''''_{(x)} = \frac{1}{K_b} f(x) \tag{29}$$

Assuming that the external loads act on the floors and not along the floor height, it is possible to write the equation as follows:

$$u''''_{(z)} = 0 \tag{30}$$

The expression for $u_{(z)}$ and $u'_{(z)}$ is proposed:

$$\begin{Bmatrix} u_{(z)} = C_0 + C_1 z + C_2 z^2 + C_3 z^3 \\ u'_{(z)} = C_1 + 2C_2 z + 3C_3 z^2 \end{Bmatrix} \tag{31}$$

Internal forces such as bending moment and shear force result:

$$\begin{Bmatrix} M_{(z)} = K_b u''_{(z)} = (2K_b)C_2 + (6K_b z)C_3 \\ V_{(z)} = K_b u'''_{(z)} = (6K_b)C_3 \end{Bmatrix} \tag{32}$$

Writing the equations in matrix form:

$$\begin{Bmatrix} u_i(z_i) \\ u'_i(z_i) \\ M_i(z_i) \\ V_i(z_i) \end{Bmatrix} = K_i(z_i) \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{Bmatrix} \tag{33}$$

Where:

$$K_i(z_i) = \begin{bmatrix} 1 & z_i & z_i^2 & z_i^3 \\ 0 & 1 & 2z_i & 3z_i^2 \\ 0 & 0 & 2K_b & 6K_b z_i \\ 0 & 0 & 0 & 6K_b \end{bmatrix}_i \tag{34}$$

Evaluating at the base of the i-th floor; i.e., for $z_i = h_i$:

$$\begin{Bmatrix} u_i(h_i) \\ u_i'(h_i) \\ M_i(h_i) \\ V_i(h_i) \end{Bmatrix} = K_i(h_i) \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{Bmatrix} \rightarrow \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{Bmatrix} = K_i^{-1}(h_i) \begin{Bmatrix} u_i(h_i) \\ u_i'(h_i) \\ M_i(h_i) \\ V_i(h_i) \end{Bmatrix} \tag{35}$$

Replacing the vector of coefficients:

$$\begin{Bmatrix} u_i(z_i) \\ u_i'(z_i) \\ M_i(z_i) \\ V_i(z_i) \end{Bmatrix} = K_i(z_i) K_i^{-1}(h_i) \begin{Bmatrix} u_i(h_i) \\ u_i'(h_i) \\ M_i(h_i) \\ V_i(h_i) \end{Bmatrix} = T_i(z_i) \begin{Bmatrix} u_i(h_i) \\ u_i'(h_i) \\ M_i(h_i) \\ V_i(h_i) \end{Bmatrix} \tag{36}$$

Where:

$$T_i(\mathrm{z}) = K_i(z_i) K_i^{-1}(h_i) \tag{37}$$

If we evaluate the forces and displacements at the top of the i-th floor, we have:

$$\begin{Bmatrix} u_i(0) \\ u_i'(0) \\ M_i(0) \\ V_i(0) \end{Bmatrix} = T_i(0) \begin{Bmatrix} u_i(h_i) \\ u_i'(h_i) \\ M_i(h_i) \\ V_i(h_i) \end{Bmatrix} \tag{38}$$

This equation shows the relationship of forces and displacements between the top and bottom of the i-th floor.

- **Static Analysis Under Static Point Loads Applied at Floor Level**

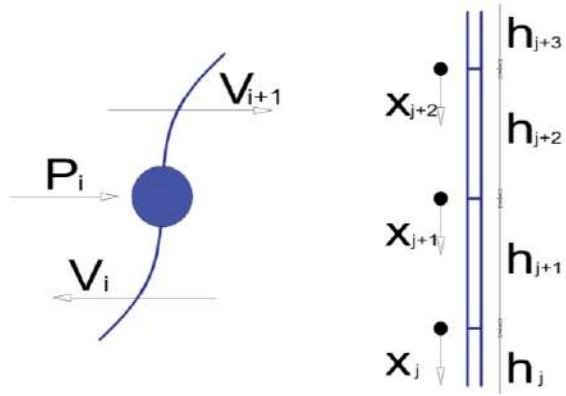
The relationship between forces and displacements between two consecutive floors is obtained by taking into account the transfer matrix and the vector of external point forces.

For the j-th floor:

$$V_{j+1} = V_j - P_j \tag{39}$$

i.e.,

$$\begin{Bmatrix} u_{i+1}(0) \\ u_{i+1}'(0) \\ M_{i+1}(0) \\ V_{i+1}(0) \end{Bmatrix} = T_i(0) \begin{Bmatrix} u_i(0) \\ u_i'(0) \\ M_i(0) \\ V_i(0) \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P_i \end{Bmatrix} \tag{40}$$



(a) (b)

Figure 25. a) Static equilibrium at the j-th level and b) Structural segments of variable properties.

For the first floor:

$$\begin{Bmatrix} u_1(0) \\ u_1'(0) \\ M_1(0) \\ V_1(0) \end{Bmatrix} = T_1(0) \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} - T_1(0) \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P_0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P_1 \end{Bmatrix} \tag{41}$$

For the second floor:

$$\begin{Bmatrix} u_2(0) \\ u_2'(0) \\ M_2(0) \\ V_2(0) \end{Bmatrix} = T_2(0)T_1(0) \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} - T_2(0)T_1(0) \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P_0 \end{Bmatrix} - T_2(0) \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P_1 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P_2 \end{Bmatrix} \tag{42}$$

For the third floor:

$$\begin{Bmatrix} u_3(0) \\ u_3'(0) \\ M_3(0) \\ V_3(0) \end{Bmatrix} = T_3(0)T_2(0)T_1(0) \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} - T_3(0)T_2(0)T_1(0) \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P_0 \end{Bmatrix} - T_3(0)T_2(0) \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P_1 \end{Bmatrix} - T_3(0) \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P_2 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P_3 \end{Bmatrix} \tag{43}$$

For the n-th floor (top of the beam):

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = T_n(0) \dots T_2(0) T_1(0) \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} - T_n(0) \dots T_2(0) T_1(0) \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P_0 \end{Bmatrix}$$

$$- T_n(0) \dots T_2(0) \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P_1 \end{Bmatrix} - T_n(0) \dots T_3(0) \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P_2 \end{Bmatrix} - \dots - T_n(0) \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P_{n-1} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P_n \end{Bmatrix} \tag{44}$$

Expressing the equation between product and sum symbols:

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_k(0) \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} - \prod_{k=1}^{n} T_k(0) \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P_0 \end{Bmatrix} - \prod_{k=2}^{n} T_k(0) \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P_1 \end{Bmatrix} - \prod_{k=3}^{n} T_k(0) \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P_2 \end{Bmatrix}$$

$$- \prod_{k=4}^{n} T_k(0) \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P_3 \end{Bmatrix} - \dots - \prod_{k=n}^{n} T_k(0) \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P_{n-1} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P_n \end{Bmatrix} \tag{45}$$

i.e.,

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_k(0) \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} - \sum_{s=0}^{n} \left[\prod_{k=s+1}^{n} T_k(0) \right] F_s - F_n \tag{46}$$

Defining two additional parameters:

$$\begin{Bmatrix} \mathrm{t} = \prod_{k=1}^{n} T_k(0) \\ f = - \sum_{s=0}^{n} \left[\prod_{k=s+1}^{n} T_k(0) \right] F_s - F_n \end{Bmatrix} \tag{47}$$

Replacing these two parameters:

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} + f \tag{48}$$

This equation expresses the relationship between the part forces and displacements of the top and the base of the beam. An important point to note is that the size of the transfer matrix is 6x6 and remains constant for all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ u'_{(1)} = 0 \\ u''_{(0)} = 0 \\ u'''_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1(h_1) = 0 \\ u'_1(h_1) = 0 \\ M_n(0) = 0 \\ V_n(0) = 0 \end{Bmatrix} \tag{49}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ u'_n(0) \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} \tag{50}$$

By clearing the bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} + \begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix} \rightarrow \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} = -\begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix}^{-1} \begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix} \tag{51}$$

The lateral displacement and its derivative at the top of the model:

$$\begin{Bmatrix} u_n(0) \\ u'_n(0) \end{Bmatrix} = \begin{bmatrix} t_{1,3} & t_{1,4} \\ t_{2,3} & t_{2,4} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{52}$$

Substituting internal forces:

$$\begin{Bmatrix} u_n(0) \\ u'_n(0) \end{Bmatrix} = -\begin{bmatrix} t_{1,3} & t_{1,4} \\ t_{2,3} & t_{2,4} \end{bmatrix} \begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix}^{-1} \begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{53}$$

4.1.2 Shear Beam of a Field (SB)

This model is usually used to model portal frames having beams and columns with infinite bending stiffness and negligible axial deformations of the columns. The model takes into account a transverse motion u and a shear stiffness.

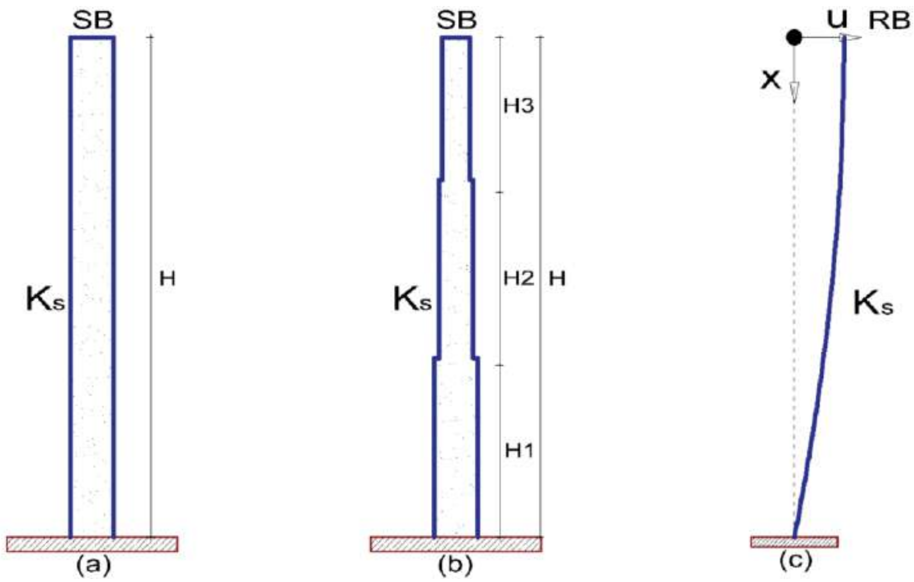


Figure 26. Shear beam of a field (SB). a) Case 1, b) Case 2 and c) equivalent RB.

4.1.2.1 Case 1

The total potential energy of the SB beam of a field subjected to a general lateral load distribution is expressed as:

$$V = \frac{1}{2}\int_0^H K_s {u'_{(x)}}^2 \, dx \tag{54}$$

Where:

$$\left\{K_s = \left(K_b^{-1} + K_c^{-1}\right)^{-1} = \frac{K_b\, K_c}{K_b + K_c}, K_b = \sum \frac{12EI_{b,i}}{lh}, K_c = \sum \frac{12EI_{c,i}}{h^2}\right\} \tag{55}$$

The work done by the external force is:

$$W = \int_0^H f_{(x)} u_{(x)} \, dx \tag{56}$$

Consequently, the total potential energy of the SB beam of a field subjected to a general lateral load distribution is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H K_s {u'_{(x)}}^2\, dx - \int_0^H f(x) u_{(x)}\, dx \tag{57}$$

Closed-form solutions of the model on which a transverse load acts are achieved by solving the differential system arising from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\delta\mathcal{U} = \int_0^H K_s u'_{(x)} \delta u'_{(x)} dx - \int_0^H f(x) \delta u_{(x)}\, dx - \int_0^H u_{(x)}\, \delta f(x) dx \tag{58}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\delta\mathcal{U} = \left[K_s u'_{(x)} \delta u_{(x)}\right]_0^H - \int_0^H \left[K_s u''_{(x)} + f(x)\right] \delta u_{(x)} \int_0^H u_{(x)}\, \delta f(x) dx \tag{59}$$

Equating the terms to zero results in the following equation:

$$K_s u''_{(x)} + f(x) = 0 \tag{60}$$

And border condition:

$$u'_{(0)} = 0 \tag{61}$$

A second order differential equation is obtained. Normalizing the differential equation by the variable $z = x/H$:

$$u''_{(z)} = -\frac{H^2}{K_s} f(z) \tag{62}$$

Assuming a general lateral load (Miranda E. , 1999):

$$f_{(x)} = \frac{W_{max}}{1 - e^{-a}}\left(1 - e^{-a + a\frac{x}{H}}\right) \rightarrow f_{(z)} = \frac{W_{max}}{1 - e^{-a}}\left(1 - e^{-a + az}\right) \tag{63}$$

Replacing it in the differential equation:

$$u''_{(z)} = -\gamma(1 - e^{-a+az}) \tag{64}$$

Where:

$$\gamma = \frac{W_{max}H^2}{K_s(1 - e^{-a})} \tag{65}$$

The expression for $u_{(z)}$ is proposed:

$$u_{(z)} = C_0 + C_1 z - \frac{\gamma}{2}z^2 + \frac{\gamma}{a^2}e^{-a+az} \tag{66}$$

The constants are obtained by evaluating the relevant boundary conditions:

$$\begin{Bmatrix} u_{(1)} = 0 \\ u'_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} C_0 = \gamma\left[\left(\frac{1}{2} - \frac{1}{a^2}\right) + \frac{e^{-a}}{a}\right] \\ C_1 = -\gamma\frac{e^{-a}}{a} \end{Bmatrix} \tag{67}$$

The lateral displacement is obtained by rewriting the expression for $u_{(z)}$:

$$u_{(z)} = \frac{W_{max}H^2}{K_s(1 - e^{-a})}\left[-\frac{1}{2}z^2 - \frac{e^{-a}}{a}z + \left(\frac{1}{2} - \frac{1}{a^2} + \frac{e^{-a}}{a}\right) + \frac{e^{-a+az}}{a^2}\right] \tag{68}$$

The maximum displacement is obtained by evaluating $u_{(z)}$ at 0:

$$u_{(0)} = \frac{W_{max}H^2}{K_s(1 - e^{-a})}\left[\left(\frac{1}{2} - \frac{1}{a^2}\right) + \left(\frac{1}{a} + \frac{1}{a^2}\right)e^{-a}\right] \tag{69}$$

The insterstory drift can be obtained by deriving $u_{(z)}$ once:

$$\Delta_s = \frac{W_{max}H^2}{K_s(1 - e^{-a})}\left(-z - \frac{e^{-a}}{a} + \frac{e^{-a+az}}{a}\right) \tag{70}$$

The global drift is obtained as the quotient between maximum displacement and total height:

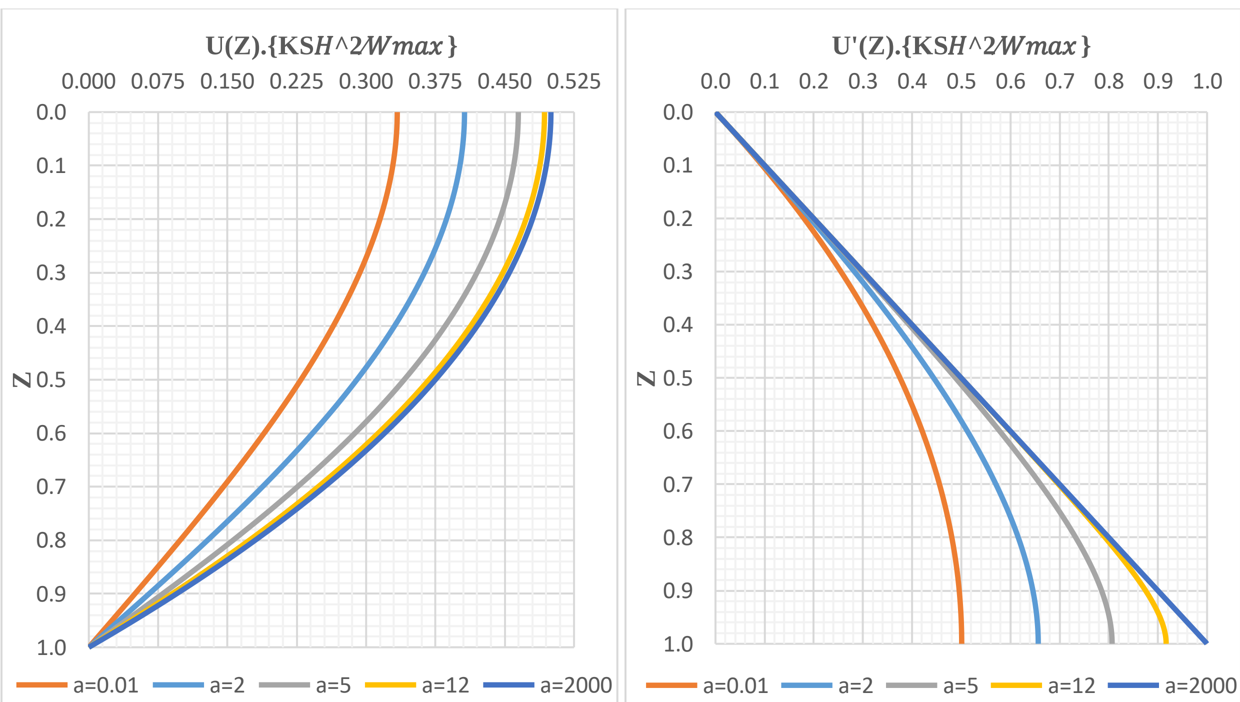
$$\Delta_g = \frac{u_{(0)}}{H} = \frac{W_{max}H}{K_s(1 - e^{-a})}\left[\left(\frac{1}{2} - \frac{1}{a^2}\right) + \left(\frac{1}{a} + \frac{1}{a^2}\right)e^{-a}\right] \tag{71}$$

For the case of a uniformly distributed lateral load $(a \rightarrow \infty)$:

$$\left\{\begin{matrix} u_{(z)} = \dfrac{W_{max} H^2}{2K_s}(1 - z^2) \to u_{(0)} = \dfrac{W_{max} H^2}{2K_s} \\ \Delta_s = -\dfrac{W_{max} H^2}{K_s} z \\ \Delta_g = \dfrac{W_{max} H}{2K_s} \end{matrix}\right\} \tag{72}$$

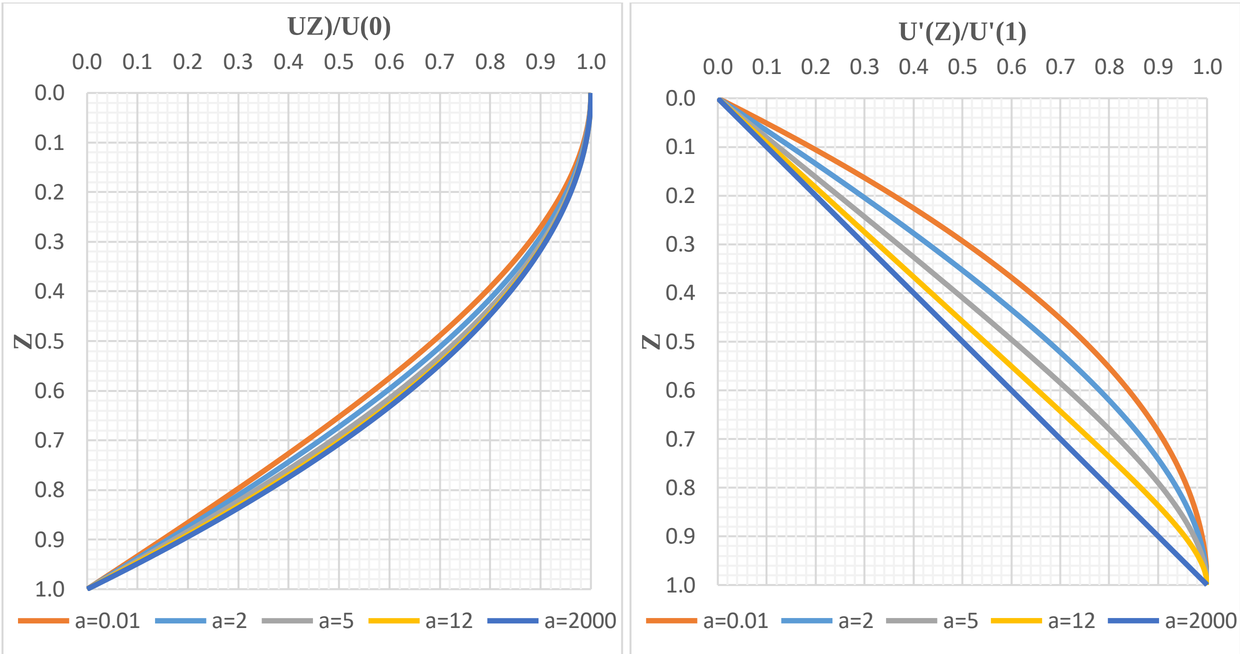
According to the analysis of equations and graphs:

- ✓ The stiffness is inversely proportional to the second power of its height.
- ✓ The lateral displacement profile shows a behavior against lateral loading (concavity to the left).
- ✓ The interstory drift profile shows maximum efficiency at the top of the beam with a tendency to zero at the roof of the building.
- ✓ The normalized lateral displacement profile and the normalized interstory drift profile are dependent on the parameter a.



(a) (b)

Figure 27. Effect of parameter a. a) Lateral displacement and b) Interstory drift.



(a) (b)

Figure 28. Effect of parameter a. a) Normalized lateral displacement and b) Normalized interstory drift.

4.1.2.2 Case 2

- **Calculation of the Transfer Matrix**

According to the second order differential equation and assuming that the external loads act on the floors and not along the floor height, it is possible to write it as follows:

$$u''_{(z)} = 0 \tag{73}$$

The expression for $u_{(z)}$ and $u'_{(z)}$ is proposed:

$$\begin{Bmatrix} u_{(z)} = C_0 + C_1 z \\ u'_{(z)} = C_1 \end{Bmatrix} \tag{74}$$

Writing the equation in matrix form:

$$\begin{Bmatrix} u_i(z_i) \\ u'_i(z_i) \end{Bmatrix} = K_i(z_i) \begin{Bmatrix} C_0 \\ C_1 \end{Bmatrix} \tag{75}$$

Where:

$$K_i(z_i) = \begin{bmatrix} 1 & z_i \\ 0 & 1 \end{bmatrix}_i \tag{76}$$

Evaluating at the base of the i-th floor; i.e., for $z_i = h_i$:

$$\begin{Bmatrix} u_i(h_i) \\ u_i'(h_i) \end{Bmatrix} = K_i(h_i) \begin{Bmatrix} C_0 \\ C_1 \end{Bmatrix} \rightarrow \begin{Bmatrix} C_0 \\ C_1 \end{Bmatrix} = K_i^{-1}(h_i) \begin{Bmatrix} u_i(h_i) \\ u_i'(h_i) \end{Bmatrix} \tag{77}$$

Replacing the vector of coefficients:

$$\begin{Bmatrix} u_i(z_i) \\ u_i'(z_i) \end{Bmatrix} = K_i(z_i) K_i^{-1}(h_i) \begin{Bmatrix} u_i(h_i) \\ u_i'(h_i) \end{Bmatrix} = T_i(z_i) \begin{Bmatrix} u_i(h_i) \\ u_i'(h_i) \end{Bmatrix} \tag{78}$$

Where:

$$T_i(\mathrm{z}) = K_i(z_i) K_i^{-1}(h_i) \tag{79}$$

If we evaluate the forces and displacements at the top of the i-th floor, we have:

$$\begin{Bmatrix} u_i(0) \\ u_i'(0) \end{Bmatrix} = T_i(0) \begin{Bmatrix} u_i(h_i) \\ u_i'(h_i) \end{Bmatrix} \tag{80}$$

This equation shows the relationship of forces and displacements between the top and bottom of the i-th floor.

- **Static Analysis Under Static Point Loads Applied at Floor Level**

The relationship between forces and displacements between two consecutive floors is obtained by taking into account the transfer matrix and the vector of external point forces.

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \end{Bmatrix} = \prod_{k=1}^{n} T_k(0) \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \end{Bmatrix} - \sum_{s=0}^{n} \left[\prod_{k=s+1}^{n} T_k(0) \right] F_s - F_n \tag{81}$$

Expressing it in simplified form:

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \end{Bmatrix} + f \tag{82}$$

Where:

$$\left\{\begin{matrix} \mathrm{t} = \prod_{k=1}^{n} T_k(0) \\ f = -\sum_{s=0}^{n} \left[\prod_{k=s+1}^{n} T_k(0) \right] F_s - F_n \end{matrix}\right\} \tag{83}$$

This equation expresses the relationship between the part forces and displacements of the top and the base of the beam. An important point to note is that the size of the transfer matrix is 6x6 and remains constant for all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ u'_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_{(h_1)} = 0 \\ u'_{(0)} = 0 \end{Bmatrix} \tag{84}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} \\ t_{2,1} & t_{2,2} \end{bmatrix} \begin{Bmatrix} 0 \\ u'_1(h_1) \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{85}$$

Clearing the slope at the base of the model:

$$0 = t_{2,2} u'_1(h_1) + f_2 \rightarrow u'_1(h_1) = -t_{2,2}^{-1} f_2 \tag{86}$$

The lateral displacement at the top of the model is obtained by substituting the displacement at the top of the beam:

$$u_n(0) = -t_{1,2} t_{2,2}^{-1} f_2 + f_1 \tag{87}$$

4.1.3 Timoshenko Beam of Two Field (TB)

Numerous investigations in the literature have used simplified bending or shear beam models to model buildings; however, other studies have shown that not considering shear or bending stiffness simultaneously in the model is inappropriate, as the margin of error exceeds engineering acceptance criteria.

By considering an infinite shear stiffness in the bending beam (EBB) and consequently zero shear deformation, it was assumed that the plane cross-sections perpendicular to the beam axis remain plane and perpendicular to the axis after deformation; however, in the case of structural elements such as shear walls and/or non-slender cores, this effect cannot be neglected and must be taken into account. To overcome this problem the Timoshenko Beam (TB) is developed, which results from an extension of the bending beam (EBB) to allow for the effect of shear deformation.

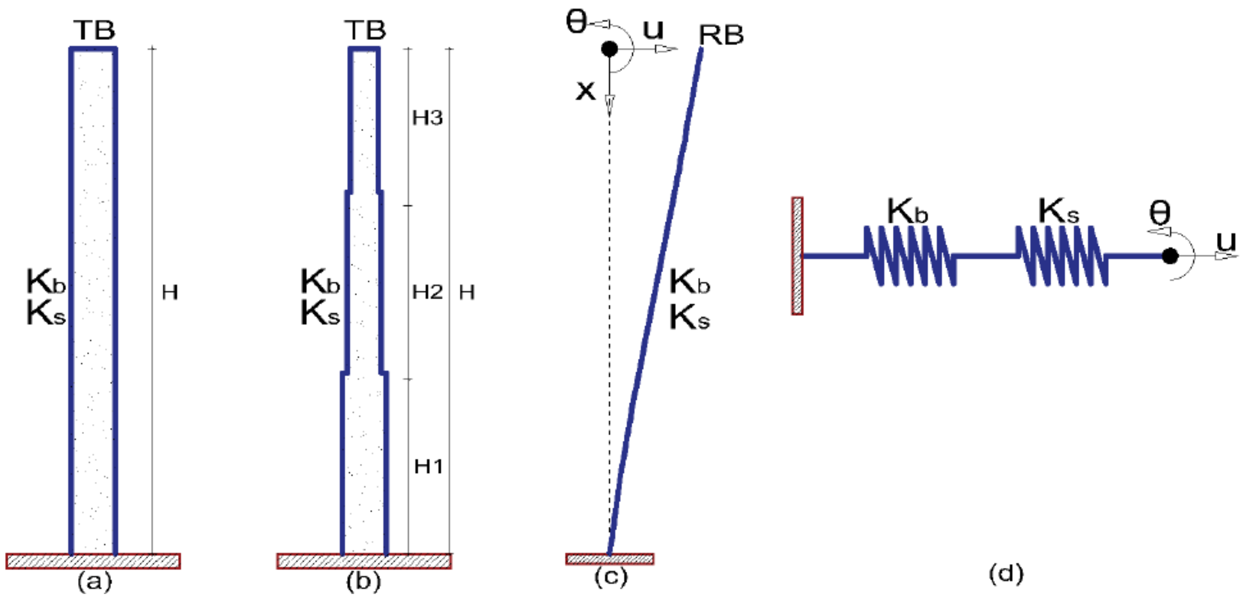


Figure 29. Timoshenko beam of two-field (TB). a) Case 1, b) Case 2 and c) equivalent RB and d) TB stiffness idealization.

The stiffness of the TB beam results from a series coupling of the bending and shear stiffness; that is, the total deformation of the TB beam results from the combination of the bending and shear deformation. The TB beam model takes into account a single transverse motion u and a rotation field θ with stiffnesses K_b and K_s as bending and shear stiffness, respectively.

4.1.3.1 Case 1

The potential energy of the TB model is expressed as:

$$V = \frac{1}{2}\int_0^H \left\{K_b \theta'^{\,2}_{(x)} + K_s\left[u'_{(x)} - \theta_{(x)}\right]^2\right\} dx \tag{88}$$

Where:

$$\left\{K_b = \sum EI_{w,i}\,, K_s = k\sum GA_{w,i}\right\} \tag{89}$$

The work done by the external force is:

$$W = \int_0^H f_{(x)} u_{(x)}\, dx \tag{90}$$

Consequently, the total potential energy of the two-field TB beam subjected to a general lateral load distribution is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H \left\{K_b \theta'^{\,2}_{(x)} + K_s\left[u'_{(x)} - \theta_{(x)}\right]^2\right\} dx - \int_0^H f_{(x)} u_{(x)}\, dx \tag{91}$$

Closed-form solutions of the model on which a transverse load acts are achieved by solving the differential system arising from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\begin{aligned}\delta\mathcal{U} = \int_0^H \{K_b \theta'_{(x)} \delta\theta'_{(x)} + K_s\left[u'_{(x)} - \theta_{(x)}\right]\left[\delta u'_{(x)} - \delta\theta_{(x)}\right]\} dx - \int_0^H f(x)\delta u_{(x)}\, dx \\ - \int_0^H u_{(x)}\, \delta f(x) dx\end{aligned} \tag{92}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\begin{aligned}\delta\mathcal{U} = \left[K_b \theta'_{(x)} \delta\theta_{(x)}\right]_0^H + \{K_s\left[u'_{(x)} - \theta_{(x)}\right]\delta u\}_0^H - \int_0^H \{K_b \theta''_{(x)} + K_s\left[u'_{(x)} - \theta_{(x)}\right]\}\delta\theta_{(x)} \\ - \int_0^H \{K_s\left[u''_{(x)} - \theta'_{(x)}\right] + f(x)\}\delta u_{(x)} - \int_0^H u_{(x)}\, \delta f(x) dx\end{aligned} \tag{93}$$

Equating the terms to zero results in the following equations:

$$\begin{Bmatrix} K_b\theta''_{(x)} + K_s\left[u'_{(x)} - \theta_{(x)}\right] = 0 \\ K_s\left[u''_{(x)} - \theta'_{(x)}\right] + f(x) = 0 \end{Bmatrix} \tag{94}$$

And boundary conditions:

$$\begin{Bmatrix} \theta'_{(0)} = 0 \\ u'_{(0)} - \theta_{(0)} = 0 \end{Bmatrix} \tag{95}$$

Using the method of differential operators for the solution of the system of equations:

$$\begin{Bmatrix} u_{(x)} \\ \theta_{(x)} \end{Bmatrix} = -\begin{bmatrix} K_sD & K_bD^2 - K_s \\ K_sD^2 & -K_sD \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ f_{(x)} \end{Bmatrix} \tag{96}$$

i.e.,

$$\begin{Bmatrix} u''''_{(x)} \\ \theta''''_{(x)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{1}{K_b} f_{(x)} - \dfrac{1}{K_s} f''_{(x)} \\ \dfrac{1}{K_b} f'_{(x)} \end{Bmatrix} \tag{97}$$

A fourth order differential equation is obtained. Normalizing the differential equation by the variable $z = x/H$:

$$u''''_{(z)} = \frac{H^4}{K_b} f_{(z)} - \frac{H^2}{K_s} f''_{(z)} \tag{98}$$

Assuming a general lateral load (Miranda E. , 1999):

$$f_{(x)} = \frac{W_{max}}{1 - e^{-a}}\left(1 - e^{-a+a\frac{x}{H}}\right) \rightarrow f_{(z)} = \frac{W_{max}}{1 - e^{-a}}(1 - e^{-a+az}) \tag{99}$$

Replacing it in the differential equation:

$$u''''_{(z)} = \lambda + \lambda\left(\frac{a^2 - \alpha^2}{\alpha^2}\right)e^{-a+az} \tag{100}$$

Where:

$$\left\{\alpha = H\sqrt{\frac{K_s}{K_b}}\ , \lambda = \frac{W_{max}H^4}{K_b(1-e^{-a})}\right\} \tag{101}$$

The expression for $u_{(z)}$ is proposed:

$$u_{(z)} = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + \frac{\lambda}{24}z^4 + \frac{\lambda(a^2-\alpha^2)}{a^4\alpha^2}e^{-a+az} \tag{102}$$

The constants are obtained by evaluating the relevant boundary conditions:

$$\left\{\begin{array}{c} u_{(1)} = 0 \\ u'_{(1)} = -\dfrac{\lambda}{\alpha^2}\left(1-\dfrac{1}{a}+\dfrac{e^{-a}}{a}\right) \\ u''_{(0)} = -\dfrac{\lambda}{\alpha^2}(1-e^{-a}) \\ u'''_{(0)} = \dfrac{\lambda}{\alpha^2}(ae^{-a}) \end{array}\right\} \rightarrow \left\{\begin{array}{c} C_0 = \lambda\left[\left(\dfrac{1}{8}+\dfrac{1}{a^4}-\dfrac{1}{a^3}\right)+\dfrac{e^{-a}}{a}\left(\dfrac{1}{2a}+\dfrac{1}{3}+\dfrac{1}{\alpha^2}\right)+\dfrac{1}{\alpha^2}\left(\dfrac{1}{2}-\dfrac{1}{a^2}\right)\right] \\ C_1 = \lambda\left[\left(-\dfrac{1}{6}+\dfrac{1}{a^3}\right)-\dfrac{e^{-a}}{a}\left(\dfrac{1}{a}+\dfrac{1}{2}+\dfrac{1}{\alpha^2}\right)\right] \\ C_2 = -\dfrac{\lambda}{2\alpha^2}\left(1-\dfrac{\alpha^2}{a^2}e^{-a}\right) \\ C_3 = \lambda\dfrac{e^{-a}}{6a} \end{array}\right\} \tag{103}$$

The lateral displacement is obtained by rewriting the expression for $u_{(z)}$:

$$\begin{aligned} u_{(z)} = \frac{W_{max}H^4}{K_b(1-e^{-a})}\Bigg\{&\frac{1}{24}z^4 + \frac{e^{-a}}{6a}z^3 + \left(\frac{e^{-a}}{2a^2}-\frac{1}{2\alpha^2}\right)z^2 + \left[\left(-\frac{1}{6}+\frac{1}{a^3}\right)-e^{-a}\left(\frac{1}{a^2}+\frac{1}{2a}+\frac{1}{a\alpha^2}\right)\right]z \\ &+\left[\left(\frac{1}{8}+\frac{1}{a^4}-\frac{1}{a^3}\right)+e^{-a}\left(\frac{1}{2a^2}+\frac{1}{3a}\right)+\frac{1}{\alpha^2}\left(\frac{1}{2}-\frac{1}{a^2}+\frac{e^{-a}}{a}\right)\right]+\frac{(a^2-\alpha^2)}{a^4\alpha^2}e^{-a+az}\Bigg\} \end{aligned}$$

After some simple manipulations, the displacement $u_{(z)}$ can be expressed as the sum of the bending and shear displacement:

$$u_{(z)} = u_{flexion} + u_{shear}$$

Where:

$$\left\{\begin{array}{c} \left\{\begin{aligned} u_{flexion} = \frac{W_{max}H^4}{K_b(1-e^{-a})}\Bigg\{&\frac{1}{24}z^4 + \frac{e^{-a}}{6a}z^3 + \frac{e^{-a}}{2a^2}z^2 + \left[\left(-\frac{1}{6}+\frac{1}{a^3}\right)-e^{-a}\left(\frac{1}{a^2}+\frac{1}{2a}\right)\right]z \\ &+\left[\left(\frac{1}{8}+\frac{1}{a^4}-\frac{1}{a^3}\right)+e^{-a}\left(\frac{1}{2a^2}+\frac{1}{3a}\right)\right]-\frac{1}{a^4}e^{-a+az}\Bigg\} \end{aligned}\right\} \\ u_{shear} = \dfrac{W_{max}H^4}{K_b(1-e^{-a})}\dfrac{1}{\alpha^2}\left[-\dfrac{1}{2}z^2-\dfrac{e^{-a}}{a}z+\left(\dfrac{1}{2}-\dfrac{1}{a^2}+\dfrac{e^{-a}}{a}\right)+\dfrac{e^{-a+az}}{a^2}\right] \end{array}\right\} \tag{104}$$

The maximum displacement is obtained by evaluating $u_{(z)}$ at 0:

$$u_{(0)} = \frac{W_{max}H^4}{K_b(1-e^{-a})}\left\{\left(\frac{1}{8}+\frac{1}{a^4}-\frac{1}{a^3}\right)+e^{-a}\left[\frac{1}{2a^2}+\frac{1}{3a}-\frac{1}{a^4}+\frac{1}{\alpha^2}\left(\frac{1}{a^2}+\frac{1}{a}\right)\right]+\frac{1}{\alpha^2}\left(\frac{1}{2}-\frac{1}{a^2}\right)\right\} \quad (105)$$

The interstory drift can be obtained by deriving $u_{(z)}$ once:

$$\Delta_s = \frac{W_{max}H^4}{K_b(1-e^{-a})}\left\{\frac{1}{6}z^3+\frac{e^{-a}}{2a}z^2+\left(\frac{e^{-a}}{a^2}-\frac{1}{\alpha^2}\right)z+\left[\left(-\frac{1}{6}+\frac{1}{a^3}\right)-e^{-a}\left(\frac{1}{a^2}+\frac{1}{2a}+\frac{1}{a\alpha^2}\right)\right]\right.$$
$$\left.+\frac{(a^2-\alpha^2)}{a^3\alpha^2}e^{-a+az}\right\} \quad (106)$$

The global drift is obtained as the quotient between maximum displacement and total height:

$$\Delta_g = \frac{W_{max}H^3}{K_b(1-e^{-a})}\left\{\left(\frac{1}{8}+\frac{1}{a^4}-\frac{1}{a^3}\right)+e^{-a}\left[\frac{1}{2a^2}+\frac{1}{3a}-\frac{1}{a^4}+\frac{1}{\alpha^2}\left(\frac{1}{a^2}+\frac{1}{a}\right)\right]+\frac{1}{\alpha^2}\left(\frac{1}{2}-\frac{1}{a^2}\right)\right\} \quad (107)$$

For the case of a uniformly distributed lateral load $(a \to \infty)$:

$$\left\{\begin{array}{c} u_{(z)} = \dfrac{W_{max}H^4}{K_b}\left(\dfrac{1}{24}z^4-\dfrac{1}{6}z+\dfrac{1}{8}\right)+\dfrac{W_{max}H^2}{2K_s}(1-z^2) \\ u_{(0)} = \dfrac{W_{max}H^4}{8K_b}+\dfrac{W_{max}H^2}{2K_s} \\ \Delta_s = -\left[\dfrac{W_{max}H^4}{6K_b}(1-z^3)+\dfrac{W_{max}H^2}{K_s}z\right] \\ \Delta_g = \dfrac{W_{max}H^3}{8K_b}+\dfrac{W_{max}H}{2K_s} \end{array}\right\} \quad (108)$$

According to the analysis of equations and graphs:

- ✓ Being dependent on the parameter α, it allows taking into account the shear deformation and modeling a shear wall more accurately. It is known that for slender shear walls the lateral displacement is practically independent of the parameter α because it is deformed only by bending; however, for a non-slender or ordinary shear wall the shear deformation can become significant and important.
- ✓ Displacement, interstory drift and global drift result from the sum of the contribution by bending and shear.

- ✓ The parameter α conditions the lateral displacement profile and the interstory drift profile; that is, the parameter α determines the predominant type of behavior of the beam. For a value of $=0.3$ $(H/L \approx 0.15)$, it shows pure shear behavior; a value of $\alpha = 3$ $(H/L \approx 1.45)$, it shows intermediate behavior between shear and bending; and a value of $\alpha = 30$ $(H/L \approx 14.5)$, it shows pure bending behavior.
- ✓ As the value of parameter α increases the influence of parameter a on the normalized lateral displacement profile becomes less and less, and for a value of $\alpha = 30$ $(H/L \approx 14.5)$, the normalized lateral displacement profile is practically independent of parameter α, as is the case in the flexural beam.
- ✓ The normalized lateral displacement profile and the normalized interstory drift profile are practically identical for all cases and independent of the parameter α.
- ✓ The shear contribution can be safely ignored when α has a value greater than about 8.70, corresponding to a contribution of 5%. For a value of $\alpha = 6$, a 10% contribution is obtained and for a value of $\alpha = 4$ one has a 20% contribution.
- ✓ It is irresponsible to neglect the contribution of the shear when α has a value less than 8.70 because it drastically modifies the behavior of the beam and is on the side of insecurity.
- ✓ The contribution of shear is practically independent of the parameter α.

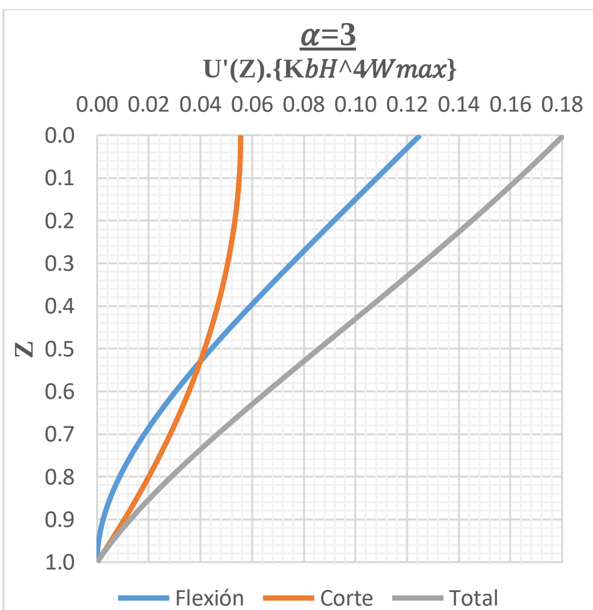


Figure 30. Bending, shear and total displacement for $\alpha = 3$.

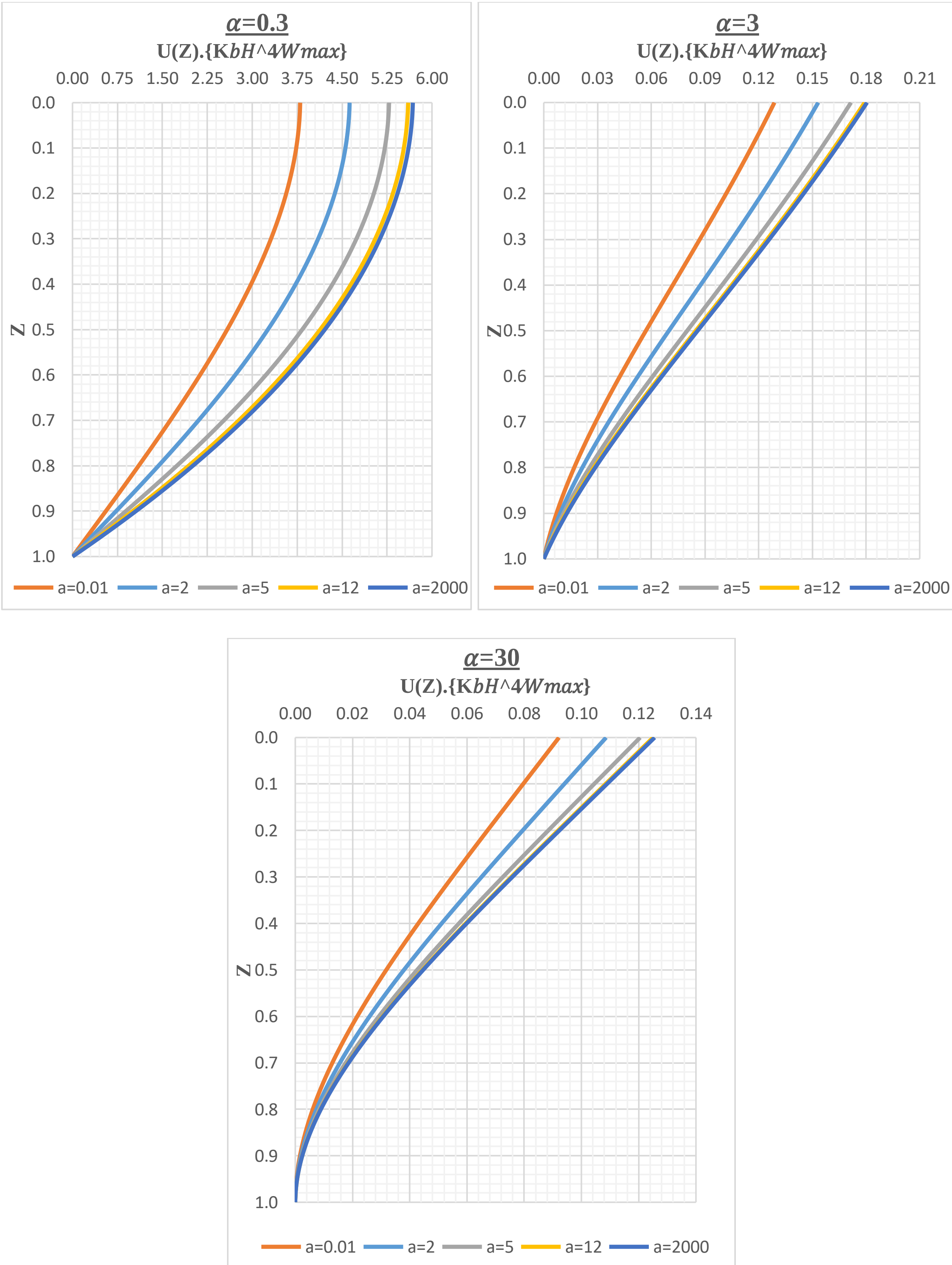


Figure 31. Lateral displacement of the beam and effect of parameter a.

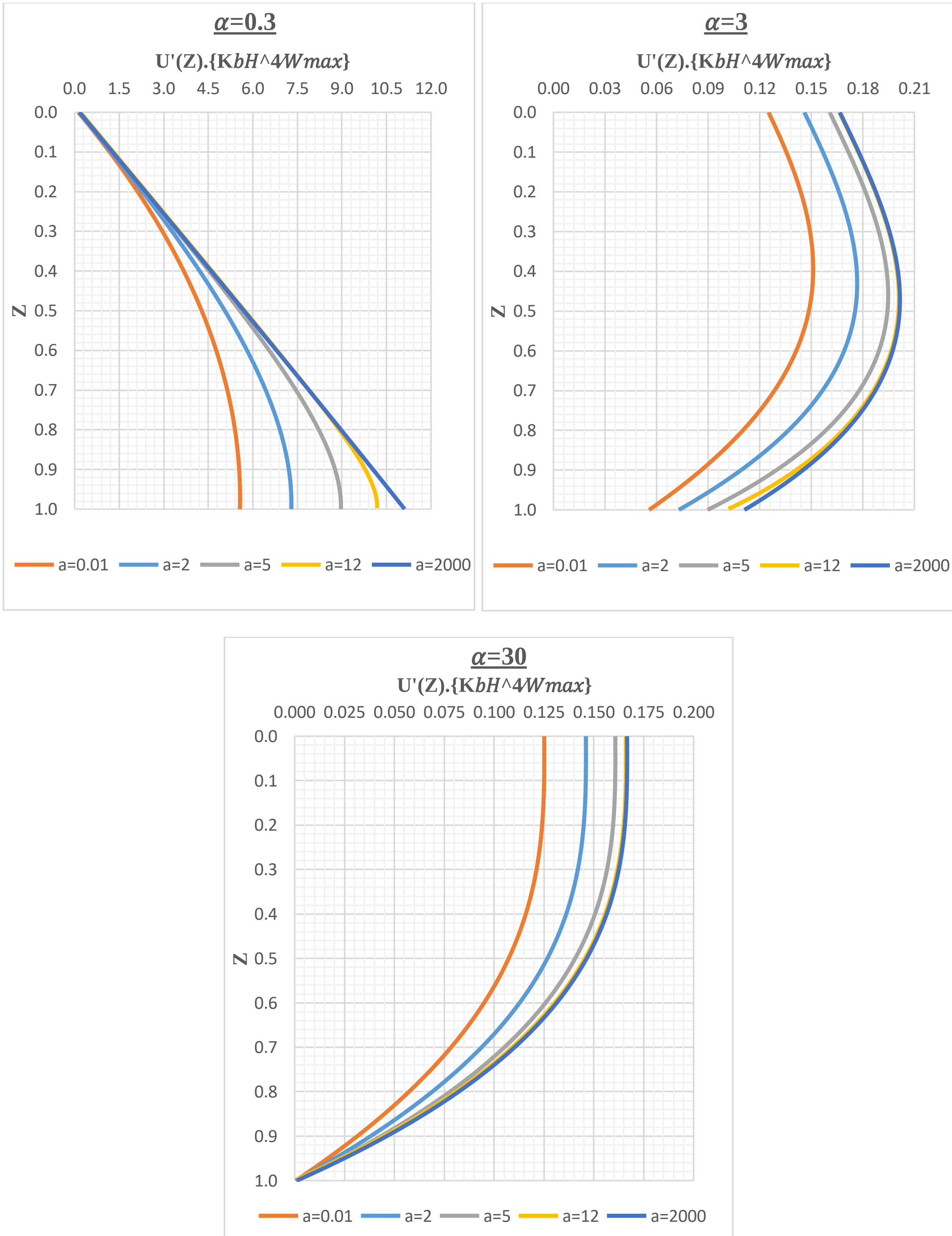


Figure 32. Beam interstory drift and effect of parameter a.

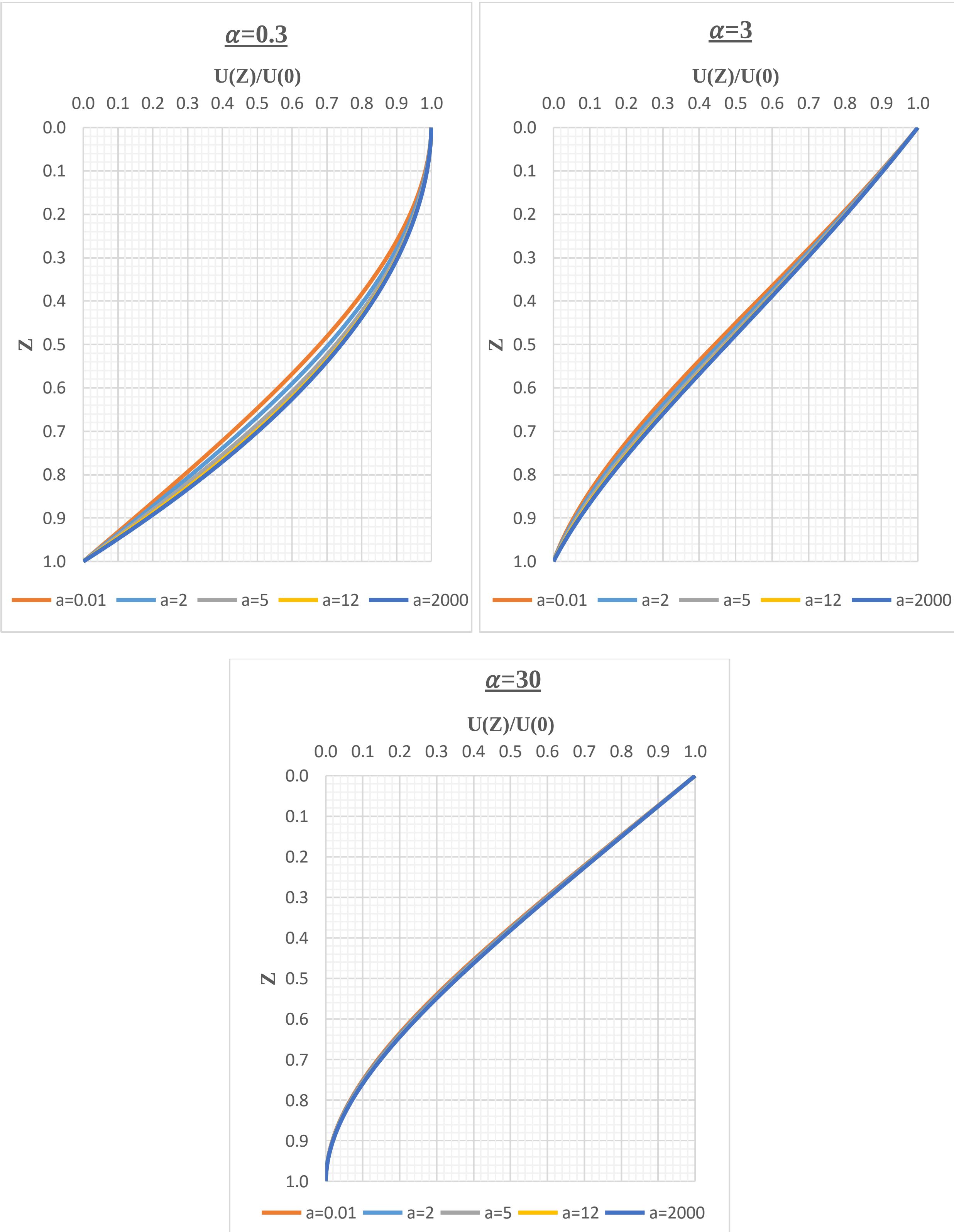


Figure 33. Effect of parameter α on the normalized lateral displacement profile.

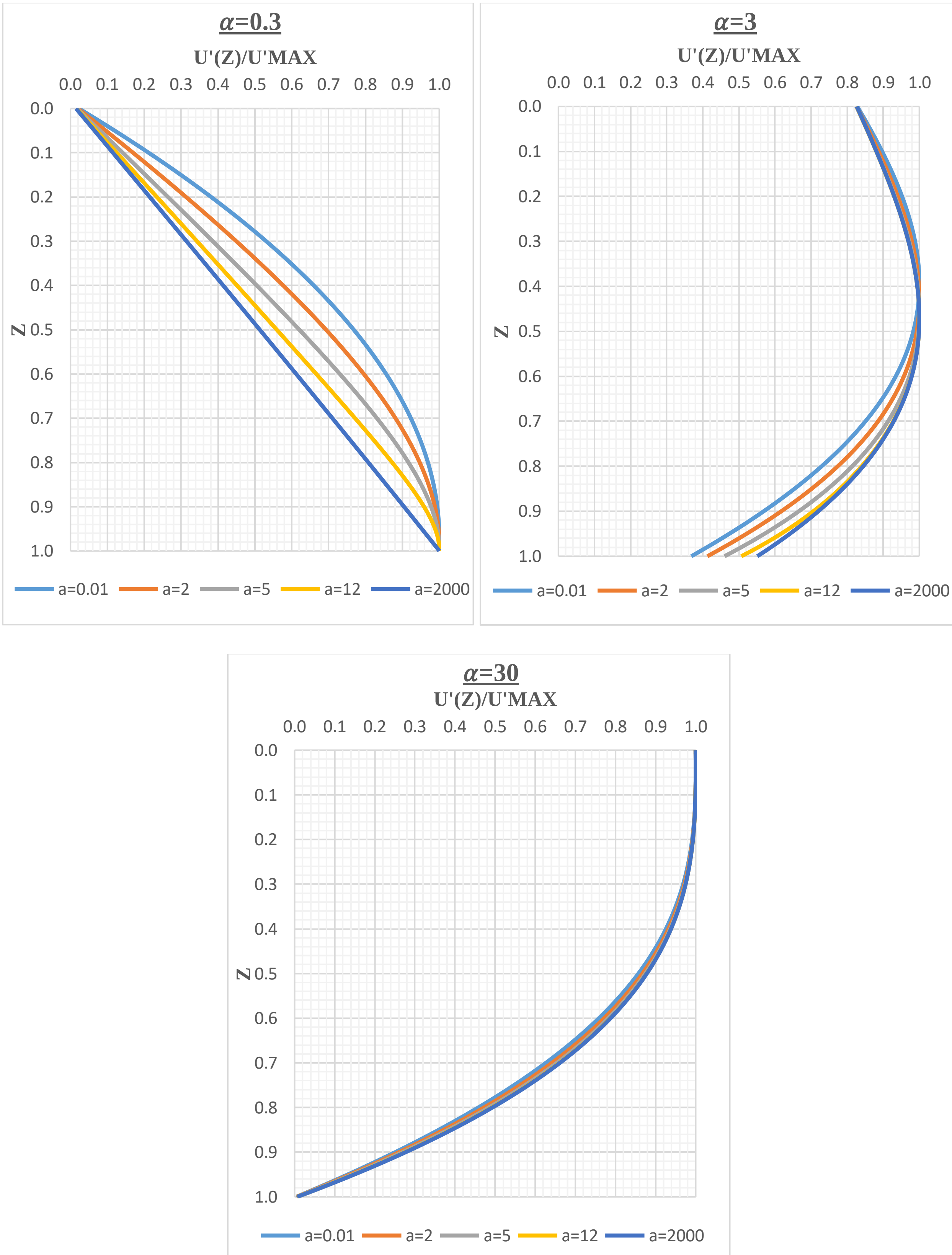


Figure 34. Effect of parameter a on the interstory drift profile.

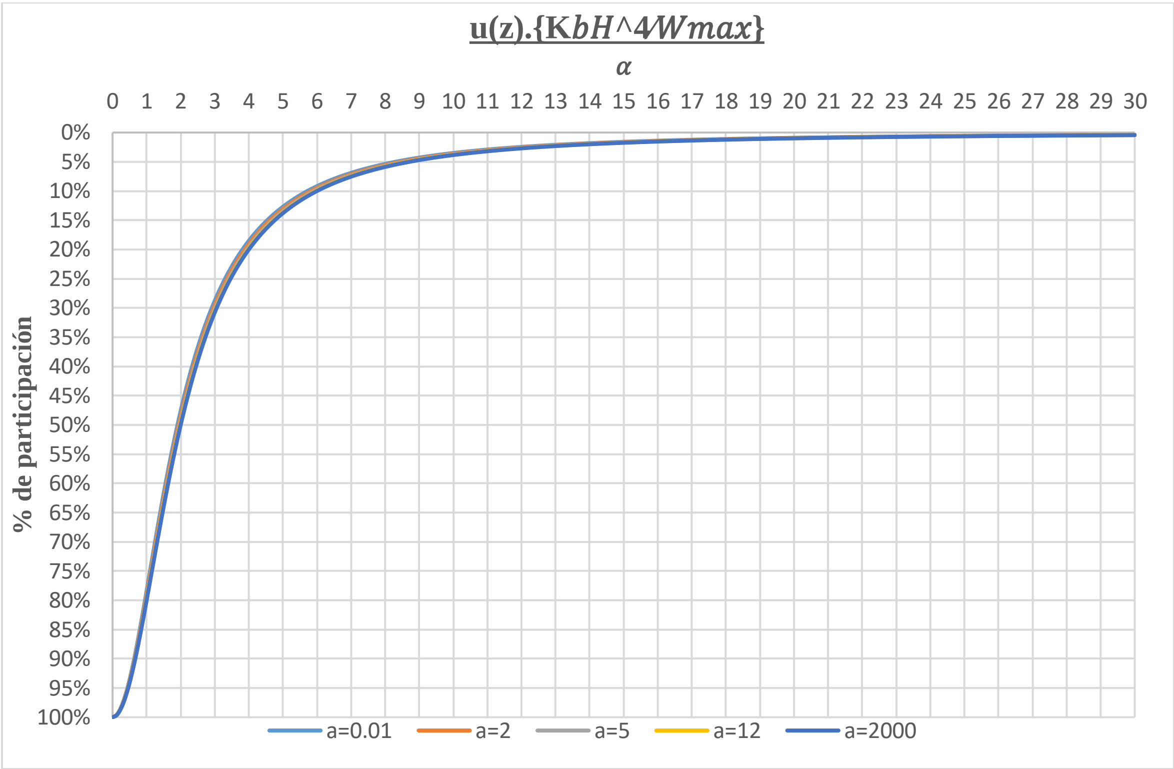


Figure 35. Influence of shear displacement.

4.1.3.2 Case 2

- **Calculation of the Transfer Matrix**

According to the coupled differential equations and assuming that the external loads act on the floors and not along the floor height, it is possible to write it as follows:

$$\begin{Bmatrix} K_b\theta''_{(x)} + K_s\left[u'_{(x)} - \theta_{(x)}\right] = 0 \\ K_s\left[u''_{(x)} - \theta'_{(x)}\right] = 0 \end{Bmatrix} \tag{109}$$

The expression for $u_{(z)}$ and $\theta_{(z)}$ is proposed:

$$\begin{Bmatrix} u_{(z)} = C_0 + C_1 z + C_2 z^2 + C_3 z^3 \\ \theta_{(z)} = C_4 + C_5 z + C_6 z^2 \end{Bmatrix} \tag{110}$$

Expressing the coefficients of the function $\theta_{(z)}$ as a function of the coefficients of $u_{(z)}$:

$$\theta_{(z)} = C_1 + (2z)C_2 + \left(3z^2 + 6\frac{K_b}{K_s}\right)C_3 \tag{111}$$

Internal forces such as bending moment and shear force associated with lateral displacement result:

$$\begin{Bmatrix} M_{(z)} = K_b\theta'_{(z)} = (2K_b)C_2 + (6K_b z)C_3 \\ V_{(z)} = K_s\left[u'_{(z)} - \theta_{(z)}\right] = \left(-\frac{6}{\alpha^{*2}}K_s\right)C_3 \end{Bmatrix} \tag{112}$$

Writing the equations in matrix form:

$$\begin{Bmatrix} u_i(z_i) \\ \theta_i(z_i) \\ M_i(z_i) \\ V_i(z_i) \end{Bmatrix} = K_i(z_i)\begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{Bmatrix} \tag{113}$$

Where:

$$K_i(z_i) = \begin{bmatrix} 1 & z_i & z_i^2 & z_i^3 \\ 0 & 1 & 2z_i & 3z_i^2 + \frac{6}{\alpha^{*2}} \\ 0 & 0 & 2K_b & 6K_b z_i \\ 0 & 0 & 0 & -\frac{6}{\alpha^{*2}}K_s \end{bmatrix}_i \tag{114}$$

- **Static Analysis Under Static Point Loads Applied at Floor Level**

The relationship between forces and displacements between two consecutive floors is obtained by taking into account the transfer matrix and the vector of external point forces.

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_k(0)\begin{Bmatrix} u_1(h_1) \\ \theta_1(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} - \sum_{s=0}^{n}\left[\prod_{k=s+1}^{n} T_k(0)\right]F_s - F_n \tag{115}$$

Expressing it in simplified form:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \mathrm{t}\begin{Bmatrix} u_1(h_1) \\ \theta_1(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} + f \tag{116}$$

Where:

$$\begin{Bmatrix} \mathrm{t} = \prod_{k=1}^{n} T_k(0) \\ f = -\sum_{s=0}^{n} \left[\prod_{k=s+1}^{n} T_k(0) \right] F_s - F_n \end{Bmatrix} \tag{117}$$

This equation expresses the relationship between the part forces and displacements of the top and the base of the beam. An important point to note is that the size of the transfer matrix is 6x6 and remains constant for all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ \theta_{(1)} = 0 \\ \theta'_{(0)} = 0 \\ u'_{(0)} - \theta_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1(h_1) = 0 \\ \theta_1(h_1) = 0 \\ M_n(0) = 0 \\ V_n(0) = 0 \end{Bmatrix} \tag{118}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} \tag{119}$$

By clearing the bending moment and the shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} + \begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix} \rightarrow \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} = -\begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix}^{-1} \begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix} \tag{120}$$

Substituting the internal forces gives the displacement and rotation at the top:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \end{Bmatrix} = -\begin{bmatrix} t_{1,3} & t_{1,4} \\ t_{2,3} & t_{2,4} \end{bmatrix} \begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix}^{-1} \begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{121}$$

4.1.4 Parallel Coupling of Bending Beam and Shear Beam of a Field (CTB) - Translational Behavior

The CTB beam is developed, which considers that the structure consists of a parallel coupling of a bending beam and a shear beam with bending and shear deformation respectively. The beams are assumed to be coupled in parallel by axially stiff members that only transmit horizontal forces and do not deform.

The CTB beam is more efficient than the EBB and SB beam due to the interaction between the displacement profiles. The EBB beam shows a displacement profile that is in favor of the lateral load, with a maximum slope at the bottom; while the SB beam shows a displacement profile that is against the lateral load, with a maximum slope at the top. Joining both beams by means of axially stiff members that only transmit horizontal forces conditions the EBB and SB beam to develop identical lateral displacement; resulting in a bending displacement profile at the bottom constraining displacements at the bottom floors and a shear displacement profile at the top constraining displacements at the top floors. This horizontal interaction effect contributes to increase the lateral stiffness of the structure, making the total stiffness of the structure greater than the sum of the lateral stiffnesses of the EBB and SB beam individually.

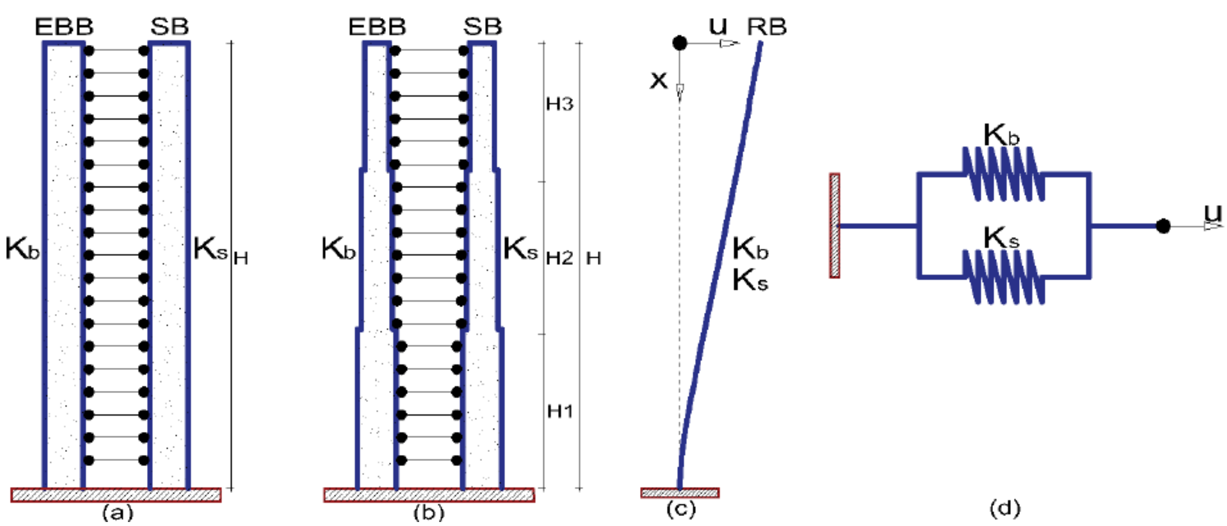


Figure 36. Flexural and shear beam coupling (CTB) of a field. a) Case 1, b) Case 2, c) Equivalent RB and d) Idealization of CTB stiffness.

The CTB beam model takes into account a single transverse motion u with stiffnesses K_b and K_s as local bending and shear stiffnesses, respectively; that is, it ignores the global bending stiffness and consequently does not take into account axial deformations as an additional kinematic field.

This model is suitable for modeling dual frame and shear wall structures; and frames where the effect of global bending is not predominant and therefore can be neglected.

4.1.4.1 Case 1

The potential energy of the CTB model of a field is expressed as:

$$V = \frac{1}{2}\int_0^H \left[K_b {u''_{(x)}}^2 + K_s {u'_{(x)}}^2\right] dx \tag{122}$$

Where:

$$\begin{Bmatrix} K_b = r\sum EI_{w,c} \ , K_s = \left(K_b^{-1} + K_c^{-1}\right)^{-1} \\ K_c = \sum \frac{12EI_{w,c}}{h^2} \ , K_b = \sum \frac{12EI_b}{lh} \ , r = \frac{K_c}{K_c + K_b} , \end{Bmatrix} \tag{123}$$

The work done by the external force is:

$$W = \int_0^H f_{(x)} u_{(x)} \, dx \tag{124}$$

Consequently, the total potential energy of the CTB beam of a classical field subjected to a general lateral load distribution is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H \left[K_b {u''_{(x)}}^2 + K_s {u'_{(x)}}^2\right] dx - \int_0^H f(x) u_{(x)} \, dx \tag{125}$$

Closed-form solutions of the model on which a transverse load acts are achieved by solving the differential system arising from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\delta\mathcal{U} = \int_0^H \left\{K_b u''_{(x)} \delta u''_{(x)} + K_s u'_{(x)} \delta u'_{(x)}\right\} dx - \int_0^H f(x)\delta u_{(x)} \, dx - \int_0^H u_{(x)} \, \delta f(x) dx \tag{126}$$

After integrating by parts and replacing them in the equation, we order the common terms:

$$\delta\mathcal{U} = \left[K_b u''_{(x)} \delta u'_{(x)}\right]_0^H - \left\{\left[K_b u'''_{(x)} - K_s u'_{(x)}\right]\delta u_{(x)}\right\}_0^H + \int_0^H \left[K_b u''''_{(x)} - K_s u''_{(x)} - f(x)\right]\delta u_{(x)} dx - \int_0^H u_{(x)}\, \delta f(x) dx \tag{127}$$

Equating the terms to zero results in the following equation:

$$K_b u''''_{(x)} - K_s u''_{(x)} - f(x) = 0 \tag{128}$$

And boundary conditions:

$$\left\{\begin{matrix} u''_{(0)} = 0 \\ K_b u'''_{(0)} - K_s u'_{(0)} = 0 \end{matrix}\right\} \tag{129}$$

A fourth order differential equation is obtained. Normalizing the differential equation by the variable $z = x/H$:

$$u''''_{(z)} - H^2 \frac{K_s}{K_b} u''_{(z)} = \frac{H^4}{K_b} f(z) \tag{130}$$

Assuming a general lateral load (Miranda E. , 1999):

$$f_{(x)} = \frac{W_{max}}{1 - e^{-a}}\left(1 - e^{-a + a\frac{x}{H}}\right) \rightarrow f_{(z)} = \frac{W_{max}}{1 - e^{-a}}(1 - e^{-a + az}) \tag{131}$$

Replacing it in the differential equation:

$$u''''_{(z)} - \alpha^2 u''_{(z)} = \lambda(1 - e^{-a + az}) \tag{132}$$

Donde:

$$\left\{\alpha = H\sqrt{\frac{K_s}{K_b}}\,, \lambda = \frac{W_{max} H^4}{K_b(1 - e^{-a})}\right\} \tag{133}$$

The expression for $u_{(z)}$ is proposed:

$$u_{(z)} = C_0 + C_1 z + C_2 \cosh(\alpha z) + C_3 \sinh(\alpha z) - \frac{\lambda}{2\alpha^2} z^2 - \frac{\lambda}{a^2(a^2 - \alpha^2)} e^{-a + az} \tag{134}$$

The constants are obtained by evaluating the relevant boundary conditions:

$$\begin{Bmatrix} u_{(1)} = 0 \\ u'_{(1)} = 0 \\ u''_{(0)} = 0 \\ u'''_{(0)} - \alpha^2 u'_{(0)} = 0 \end{Bmatrix} \tag{135}$$

Constants:

$$\begin{Bmatrix} C_0 = \lambda\left[\dfrac{1}{2\alpha^2} + \dfrac{1}{a^2(a^2-\alpha^2)}\right] - (C_1 + C_2\cosh\alpha + C_3\sinh\alpha) \\ C_1 = -\dfrac{\lambda e^{-a}}{a\alpha^2} \\ C_2 = \dfrac{\lambda}{\alpha^2}\left(\dfrac{1}{\alpha^2} + \dfrac{e^{-a}}{a^2-\alpha^2}\right) \\ C_3 = \dfrac{1}{\alpha\cosh\alpha}\left\{\lambda\left[\dfrac{1}{\alpha^2} + \dfrac{1}{a(a^2-\alpha^2)}\right] - (C_1 + C_2\alpha\sinh\alpha)\right\} \end{Bmatrix} \tag{136}$$

The maximum displacement is obtained by evaluating $u_{(z)}$ at 0:

$$u_{(0)} = C_0 + C_2 - \frac{\lambda}{a^2(a^2-\alpha^2)}e^{-a} \tag{137}$$

The instertory drift can be obtained by deriving $u_{(z)}$ once:

$$\Delta_s = C_1 + C_2\alpha\sinh(\alpha z) + C_3\alpha\cosh(\alpha z) - \frac{\lambda}{\alpha^2}z - \frac{\lambda}{a(a^2-\alpha^2)}e^{-a+az} \tag{138}$$

The global drift is obtained as the quotient between maximum displacement and total height:

$$\Delta_g = C_0 + C_2 - \frac{\lambda}{a^2(a^2-\alpha^2)}e^{-a} \tag{139}$$

For the case of a uniformly distributed lateral load $(a \to \infty)$:

$$
\left\{
\begin{array}{c}
u_{(z)} = \frac{W_{max}H^2}{K_s}\left(\frac{1-z^2}{2}\right) + \frac{W_{max}H^4}{K_b}\left\{\frac{\alpha[\sinh(\alpha z) - \sinh\alpha] - 1 + \cosh[\alpha(z-1)]}{\alpha^4\cosh\alpha}\right\} \\
u_{(0)} = \frac{W_{max}H^2}{2K_s} - \frac{W_{max}H^4}{K_b}\left(\frac{\alpha\sinh\alpha + 1 - \cosh\alpha}{\alpha^4\cosh\alpha}\right) \\
\Delta_p = -\frac{W_{max}H^2}{K_s}z + \frac{W_{max}H^4}{K_b}\left\{\frac{\alpha\cosh(\alpha z) + \sinh[\alpha(z-1)]}{\alpha^3\cosh\alpha}\right\} \\
\Delta_g = \frac{W_{max}H}{2K_s} - \frac{W_{max}H^3}{K_b}\left(\frac{\alpha\sinh\alpha + 1 - \cosh\alpha}{\alpha^4\cosh\alpha}\right)
\end{array}
\right\} \tag{140}
$$

We will consider some special cases of analysis:

- When $K_b \to 0$. This situation occurs in few-story coupled pent and/or shear walls where the local bending stiffness is small and can be neglected. Applying limits:

$$
\lim_{K_b \to 0}\left[K_b u''''_{(z)} - H^2 K_s u''_{(z)}\right] = \lim_{K_b \to 0}[wH^4] \to u''_{(z)} = -\frac{wH^2}{K_s} \tag{141}
$$

This equation shows that shear is dominant in the structure and is identical to the differential equation of a shear beam (SB).

- When $K_s \to 0$. This situation occurs in frames and/or coupled shear walls of few stories where the shear stiffness is small and can be neglected. Applying limits:

$$
\lim_{K_s \to 0}\left[K_b u''''_{(z)} - H^2 K_s u''_{(z)}\right] = \lim_{K_s \to 0}[wH^4] \to u''''_{(z)} = \frac{wH^4}{K_b} \tag{142}
$$

This equation shows that shear is dominant in the structure and is identical to the differential equation for a bending beam (EBB).

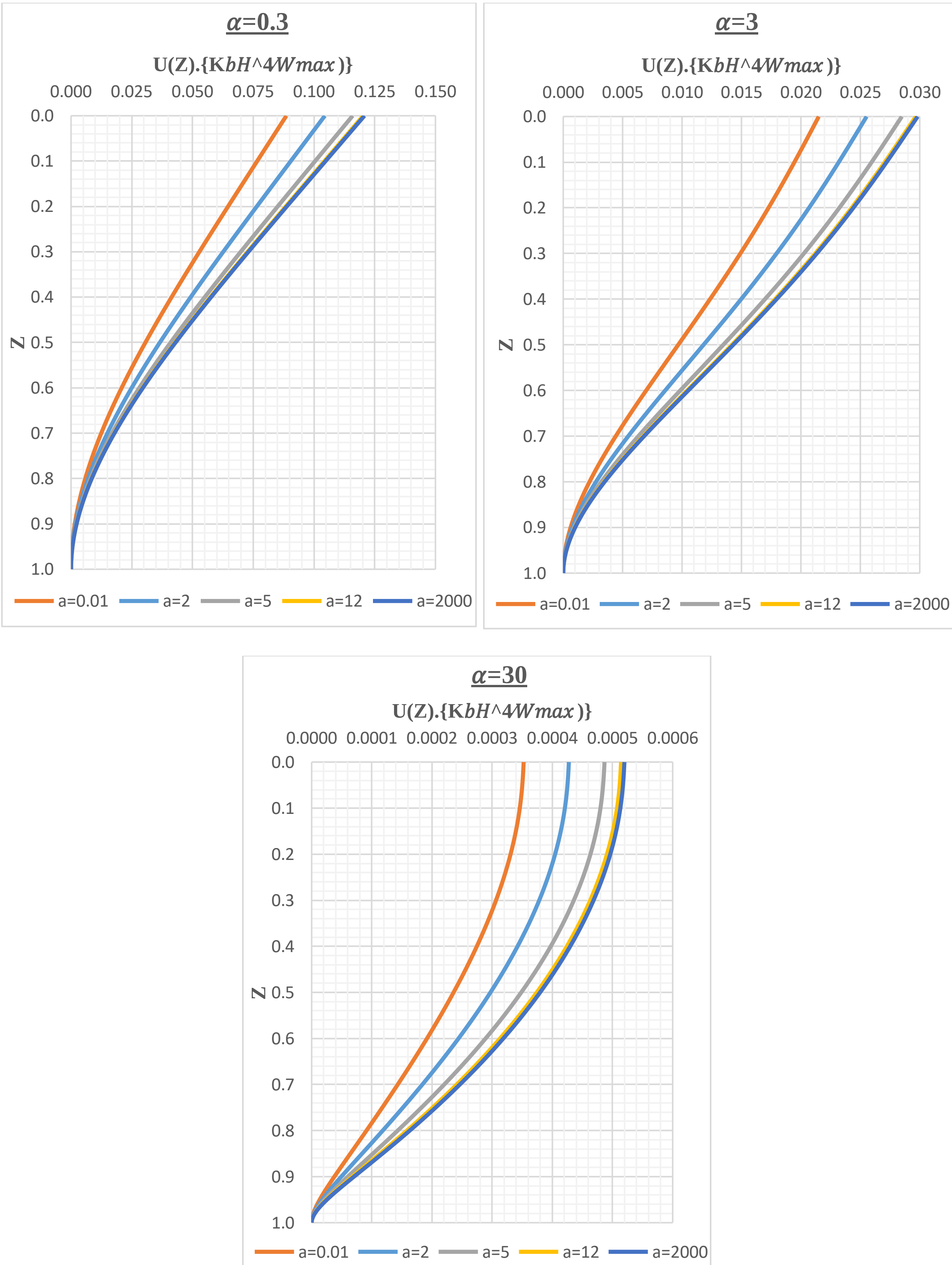


Figure 37. Lateral displacement of the beam and effect of the parameter a.

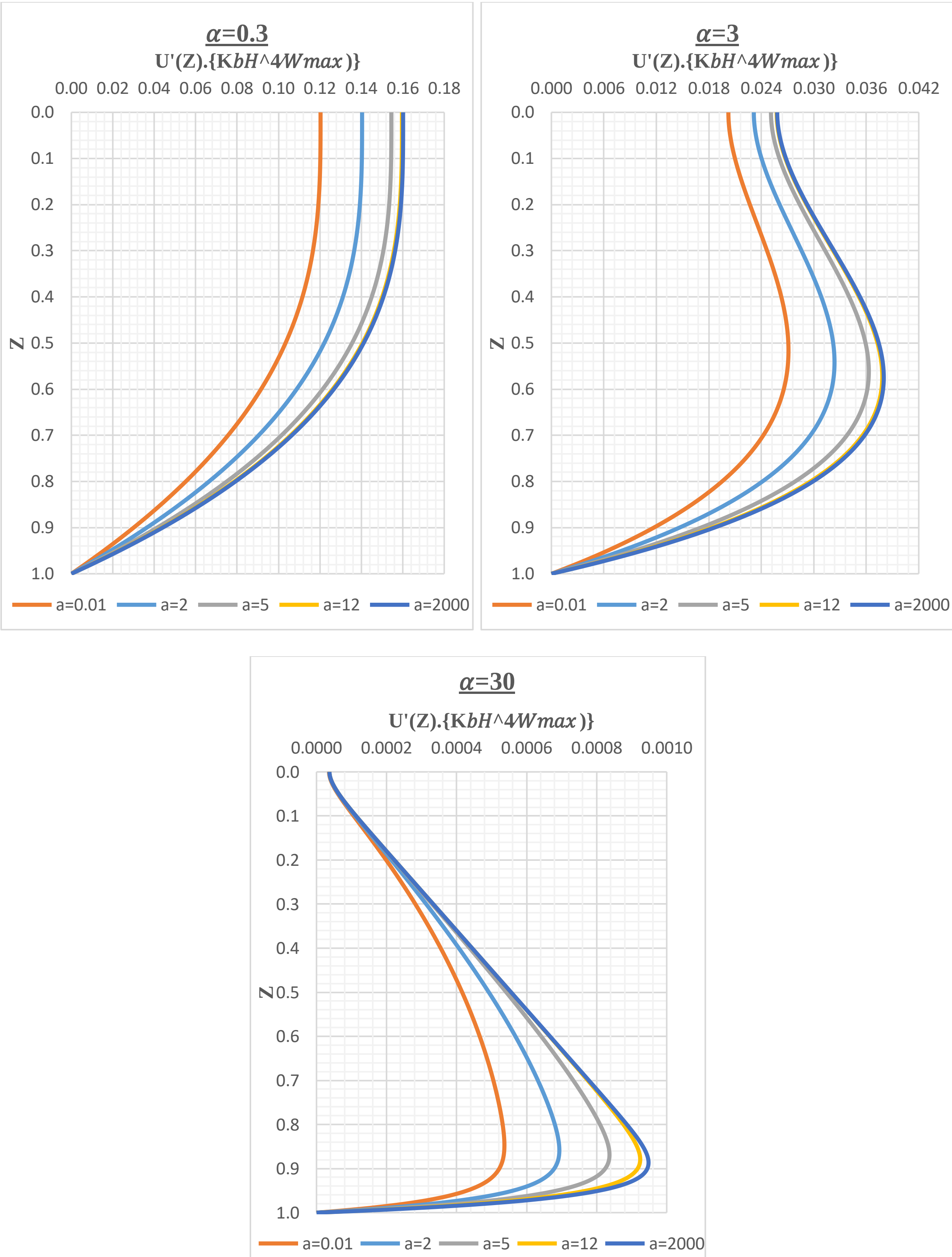


Figure 38. Beam interstory drift and effect of parameter a.

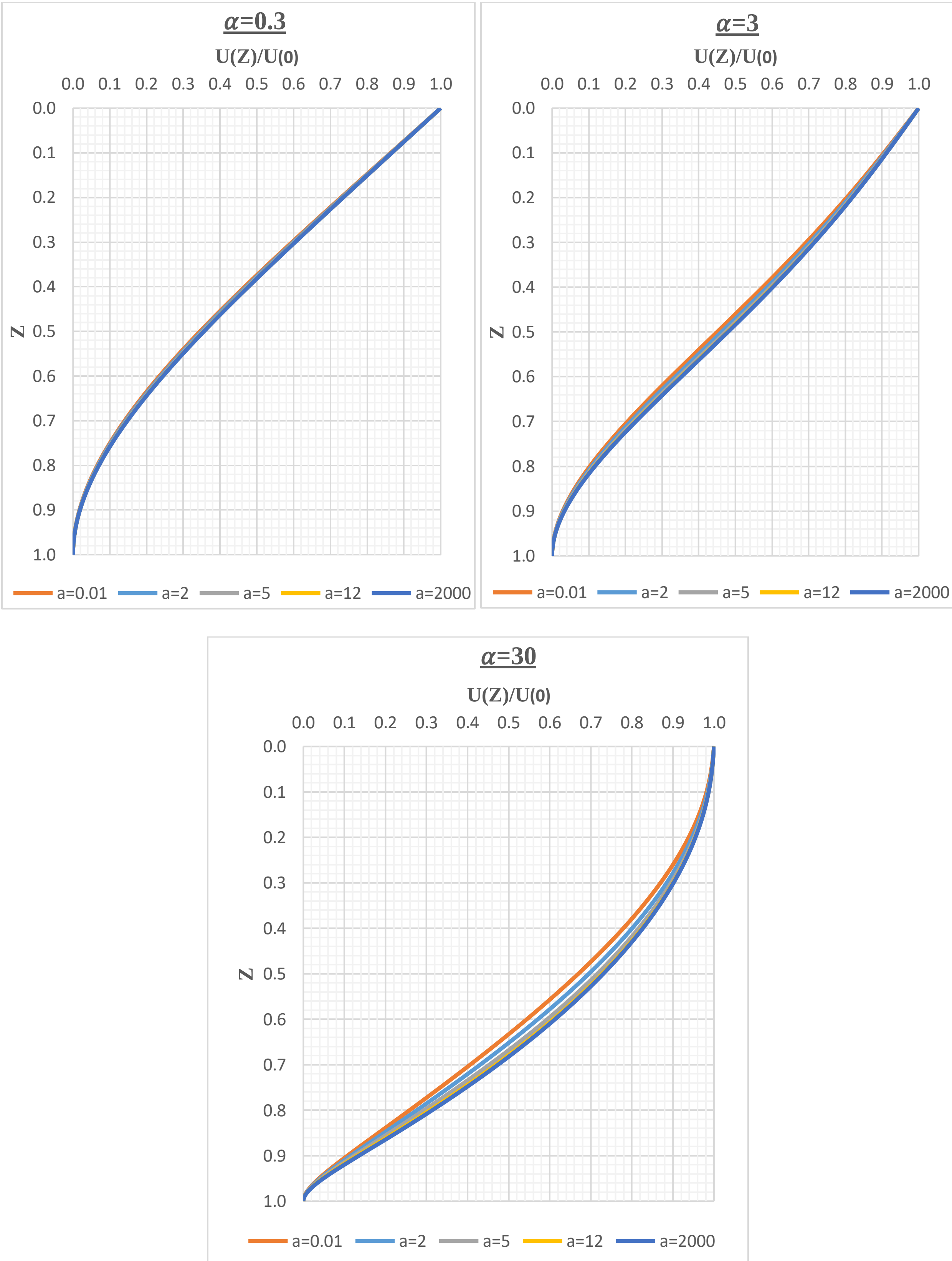


Figure 39. Effect of parameter α on the normalized lateral displacement profile.

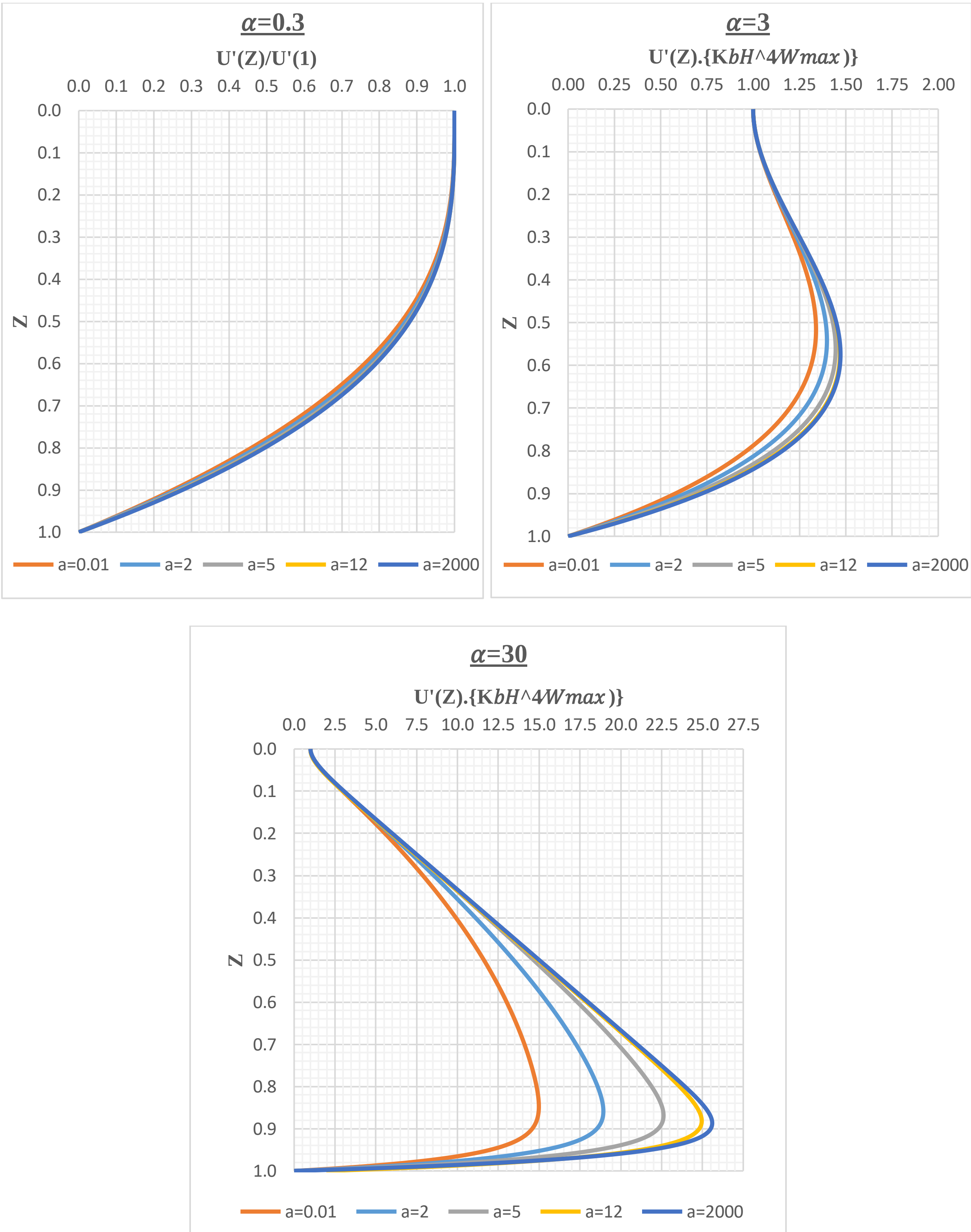


Figure 40. Effect of parameter a on the interstory drift profile.

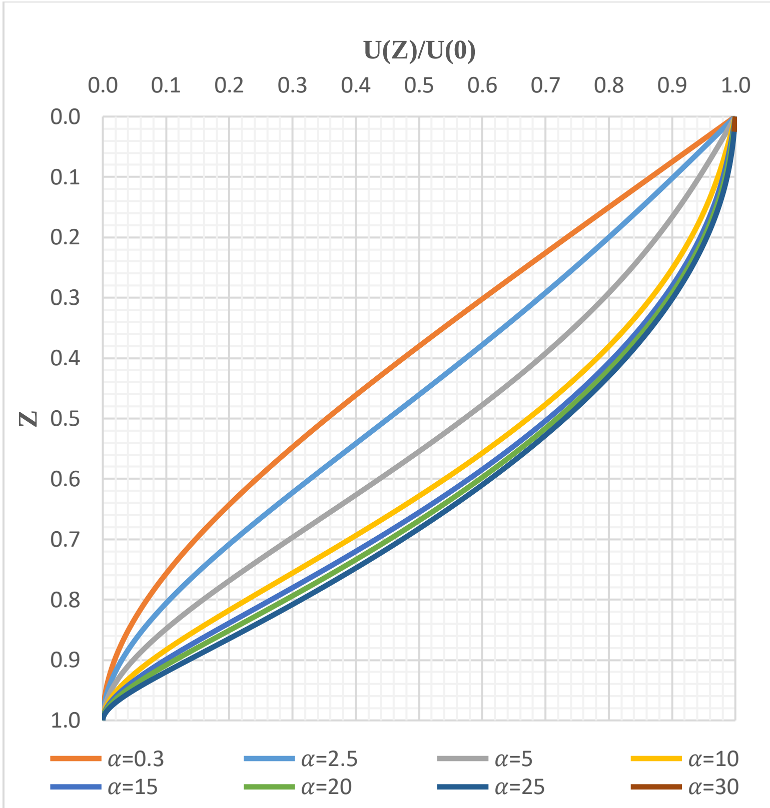


Figure 41. Effect of parameter α on the normalized lateral displacement profile for a uniformly distributed load ($a = 2000$).

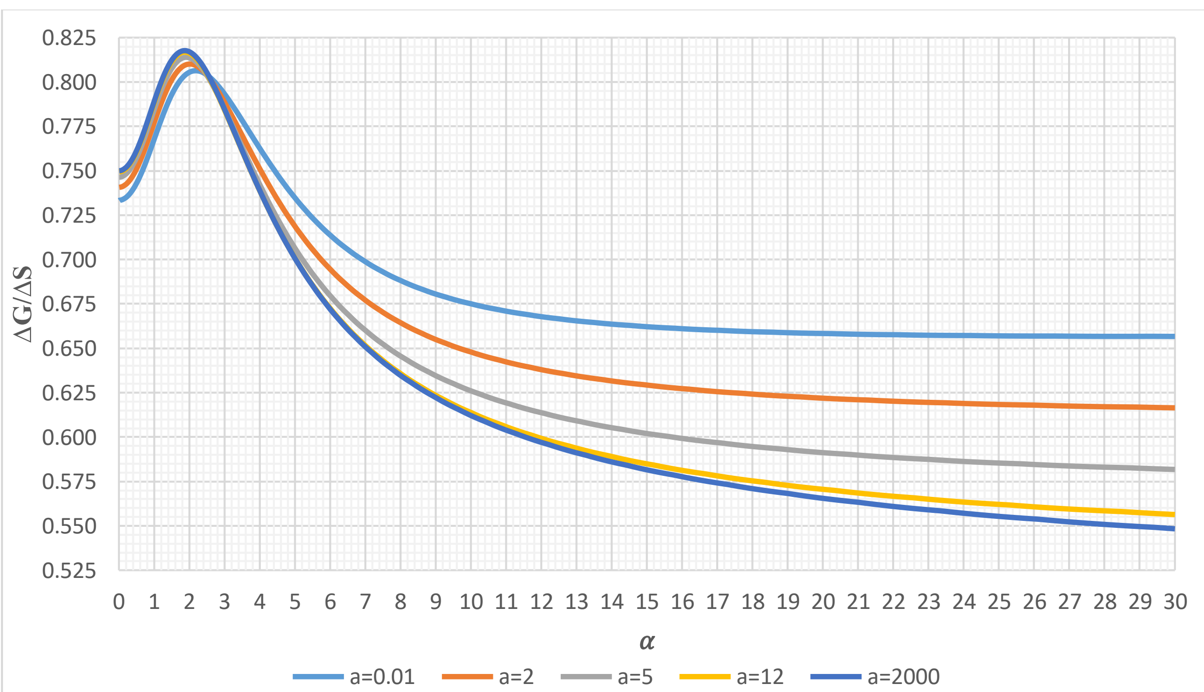


Figure 42. Variation of α vs. drift ratio Δ_g/Δ_s for various cases of a.

According to the analysis of equations and graphs:

- ✓ Displacement, interstory distortion and global distortion result from the sum of the shear contribution and the interaction between bending and shear.
- ✓ The contribution of bending is not used directly as is the effect of shear, because axial deformations have been ignored and the effect of bending is taken into account indirectly in the interaction effect between bending and shear.
- ✓ Since both beams coupled in parallel must move the same, the bending restrains the shear in the lower floors and the shear restrains the bending in the upper floors. As a consequence of this interaction between the two beams, the lateral stiffness increases, concluding that the stiffness of the system is greater than the sum of the bending and shear stiffnesses separately. Furthermore, the degree of interaction between bending and shear is strongly influenced by the parameter α.
- ✓ The parameter α conditions the lateral displacement profile and the interstory distortion profile; that is, the parameter α determines the predominant type of behavior of the beam. For a value of $\alpha = 0.3$, it shows a pure bending behavior; a value of $\alpha = 3$, shows an intermediate behavior between shear and bending; and a value of $\alpha = 30$, shows a behavior tending to pure shear.
- ✓ As the value of parameter α decreases the influence of parameter a on the normalized lateral displacement profile becomes less and less, and for a value of $\alpha = 0.3$, the normalized lateral displacement profile is practically independent of parameter a, as is the case in the flexural beam.
- ✓ The normalized lateral displacement profile and the normalized interstory distortion profile are dependent on the parameter α. This dependence decreases with increasing value of parameter α.
- ✓ It is possible to determine the value of α as a function of $\Delta_g/\Delta_{s,max}$, which for a preliminary analysis can be assumed based on current code provisions. Calculated the value of α, the influence of the interaction between shear and bending can be evaluated.
- ✓ The inflection point coinciding with the point of maximum interstory distortion has a slight influence of parameter a and stabilizes as the value of α increases.

- **Parametric Analysis**

The inflection point in the deflection curve of the portal frame subjected to lateral loads is a key parameter in defining whether shear or bending behavior is dominant. The portion of the equivalent column below the inflection point represents the bending behavior and the upper portion represents the shear behavior. The dominant behavior is defined by the dimensionless parameters α and k.

The inflection point coincides with the maximum drift level dy/dz and is calculated by equating the curvature of the portal deflection to zero. The location of the inflection point is found by iterative analysis, taking into account eq:

$$C_2\alpha^2 \cosh(\alpha z) + C_3\alpha^2 \sinh(\alpha z) + 2C_4 + C_5 a^2 e^{-a+az} = 0 \qquad (143)$$

For the particular case of a uniformly distributed lateral load:

$$-\frac{1}{\alpha^2}z + \frac{\alpha^2 \cosh(\alpha z) - \alpha \sinh(\alpha - \alpha z)}{\alpha^5 \cosh(\alpha)} = 0 \qquad (144)$$

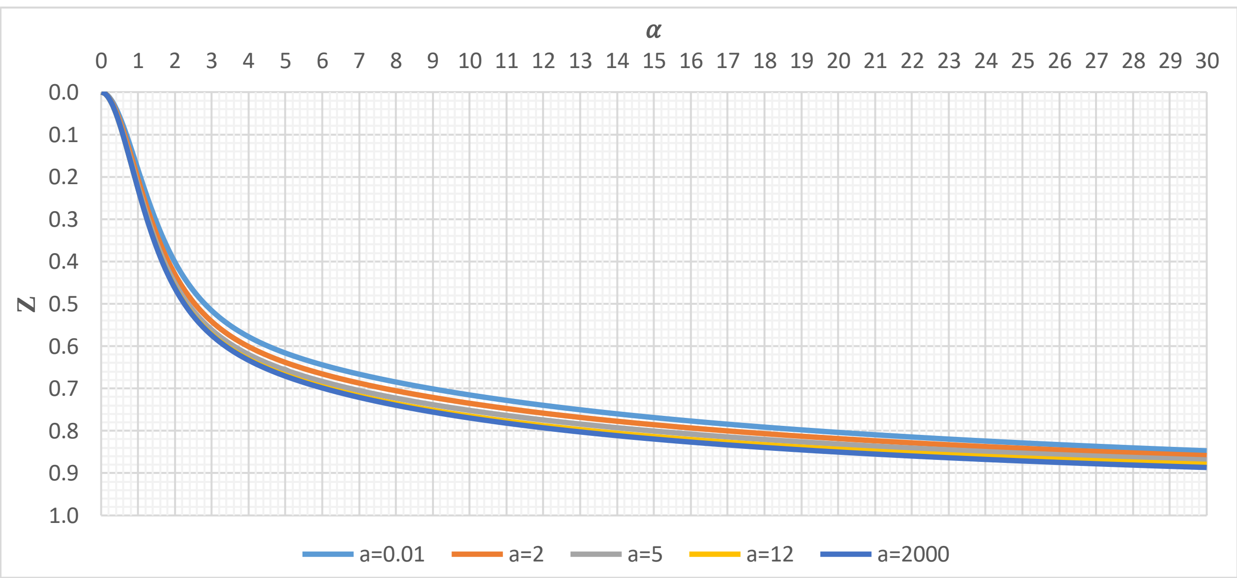


Figure 43. Location of the inflection point.

With respect to the curve in the figure 43, the CTB beam can be divided into four categories according to the value of α:

- Values of $\alpha < 1.5$: Its location is approximately in the upper third of the sandwich beam height and therefore behaves predominantly in bending.

- Values of $1.5 < \alpha < 10$: It is influenced by the shear stiffness. Its location is generally between 0.25 and 0.65 of the height. It is therefore possible that CTB beams in this range represent an apparently balanced behavior, where the interaction between flexural and shear behavior has a high degree of influence.
- Values of $10 < \alpha < 30$: The location of the inflection point practically stabilizes and is no longer influenced by the increase in shear stiffness (increase of α). The CTB beam behaves predominantly in shear.

4.1.4.2 Case 2

- **Calculation of the Transfer Matrix**

According to the differential equation and since it is assumed that the external loads act on the floors and not along the floor height, it is possible to write it as follows:

$$K_b u''''_{(x)} - K_s u''_{(x)} = 0 \tag{145}$$

The expression for $u_{(z)}$ and $u'_{(z)}$ is proposed:

$$\begin{Bmatrix} u_{(z)} = C_0 + C_1 z + C_2 \cosh(\alpha^* z) + C_3 \sinh(\alpha^* z) \\ u'_{(z)} = C_1 + C_2 \alpha^* \sin(\alpha^* z) + C_3 \alpha^* \cosh(\alpha^* z) \end{Bmatrix} \tag{146}$$

Where:

$$\alpha^* = \sqrt{\frac{K_s}{K_b}} \tag{147}$$

Internal forces such as bending moment and shear force associated with lateral displacement result:

$$\begin{Bmatrix} M_{(z)} = K_b u''_{(z)} = \alpha^{*2} \cosh(\alpha^* z)\, K_b C_2 + \alpha^{*2} \sinh(\alpha^* z)\, K_b C_3 \\ V_{(z)} = K_b u'''_{(z)} - K_s u'_{(x)} = (-\alpha^{*2} K_b) C_1 \end{Bmatrix} \tag{148}$$

Writing the equations in matrix form:

$$\begin{Bmatrix} u_i(z_i) \\ u_i'(z_i) \\ M_i(z_i) \\ V_i(z_i) \end{Bmatrix} = K_i(z_i) \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{Bmatrix} \tag{149}$$

Where:

$$K_i(z_i) = \begin{bmatrix} 1 & z_i & \cosh(\alpha^* z) & \sinh(\alpha^* z) \\ 0 & 1 & \alpha^* \sin(\alpha^* z) & \alpha^* \cosh(\alpha^* z) \\ 0 & 0 & \alpha^{*2} \cosh(\alpha^* z)\, K_b & \alpha^{*2} \sinh(\alpha^* z)\, K_b \\ 0 & -\alpha^{*2} K_b & 0 & 0 \end{bmatrix}_i \tag{150}$$

- **Static Analysis Under Static Point Loads Applied at Floor Level**

The relationship between forces and displacements between two consecutive floors is obtained by taking into account the transfer matrix and the vector of external point forces.

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_k(0) \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} - \sum_{s=0}^{n} \left[\prod_{k=s+1}^{n} T_k(0) \right] F_s - F_n \tag{151}$$

Expressing it in simplified form:

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} + f \tag{152}$$

Where:

$$\begin{Bmatrix} \mathrm{t} = \prod_{k=1}^{n} T_k(0) \\ f = -\sum_{s=0}^{n} \left[\prod_{k=s+1}^{n} T_k(0) \right] F_s - F_n \end{Bmatrix} \tag{153}$$

This equation expresses the relationship between the part forces and displacements of the top and the base of the beam. An important point to note is that the size of the transfer matrix is 4x4 and remains constant for all floors.Según las condiciones de contorno definidas en el caso 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ u'_{(1)} = 0 \\ u''_{(0)} = 0 \\ K_b u'''_{(0)} - K_s u'_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1(h_1) = 0 \\ u'_1(h_1) = 0 \\ M_n(0) = 0 \\ V_n(0) = 0 \end{Bmatrix} \tag{154}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} \tag{155}$$

By clearing the bending moment and the shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} + \begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix} \rightarrow \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} = -\begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix}^{-1} \begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix} \tag{156}$$

Substituting the internal forces gives the displacement and slope at the top:

$$\begin{Bmatrix} u_n(0) \\ u'_n(0) \end{Bmatrix} = -\begin{bmatrix} t_{1,3} & t_{1,4} \\ t_{2,3} & t_{2,4} \end{bmatrix} \begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix}^{-1} \begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{157}$$

4.1.5 Parallel Coupling of Bending Beam and Shear Beam of a Field (CTB) - Torsional Behavior

Torsional behavior is suitable for modeling cores, because once a certain number of floors are reached the cores become necessary due to the fact that they create three-dimensional units such as elevators. In the two main directions they act as shear walls and have significant torsional resistance which can constitute an important resistance with respect to the overall torsion of the building.

Cores can be considered as thin-walled open cross-section elements with a particular uncommon behavior. In the presence of bending moments and axial loading the behavior is similar to that of a solid beam, but in the presence of torsional moments the relative axial displacement of the beam complicates its behavior. The main characteristic of thin-walled beams is that they can undergo longitudinal extension as a result of torsion; consequently, longitudinal normal stresses are created that are proportional to the deformations, leading to an internal equilibrium of the longitudinal forces in each cross-section. These stresses, which arise as a result of relative section deformation and which are not examined in Saint Venant's pure torsion theory, can reach very large values in thin-walled beams (Vlasov, 1961).

- **Saint Venant's Theory (Uniform Torsion)**

In the case of uniform torsion, the value of rotation and longitudinal displacement of the beam is constant throughout the element. This occurs if the constant torsional moment is applied in opposite directions at the ends of the free beam so that it does not distort. The torsional stiffness affecting the rotation of the beam is defined as GJ and can be calculated for open and closed sections:

a) For cores of open sections, the stress distribution is very similar to that of thin rectangular sections. Therefore, the shear stresses are parallel to the section walls and change linearly along the thickness. To calculate J, the Bredt-Batho formula is used:

$$J = \frac{1}{3}\sum_{i=1}^{m} h_i v_i^3 \tag{158}$$

b) For cores with closed sections, there are significant increases in torsional stiffness with respect to the open section and the shear stress distribution is constant. The following formula is used to calculate J:

$$J = \frac{4A_0^2}{\sum_{i=1}^{m} \frac{h_i}{v_i}} \tag{159}$$

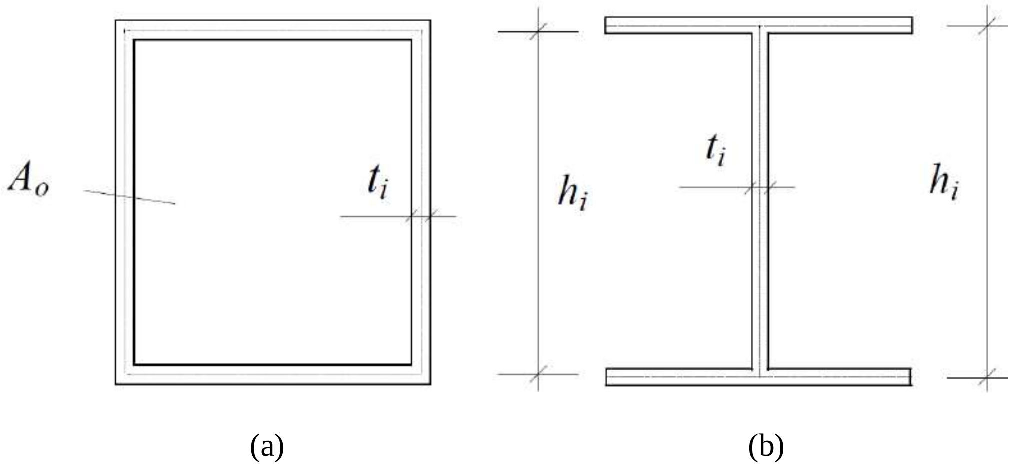


Figure 44. (a) Closed section structural core, and (b) Open section structural core (Zalka, 2020).

- **Vlasov's Theory (Non-Uniform Torsion)**

In the case of non-uniform torsion, according to Vlasov's theory the rotation of the beam is variable throughout the element. When sections are free to deform, a beam responds in uniform torsion, on the contrary, if the deformation is restricted, due to the complex distribution of longitudinal forces, the shear forces in the cross-section can be related to two different modes of torsional behavior. One due to uniform torsion, and the other due to deformation torsion. Therefore, its torsional stiffness originates from two sources: Saint Venant's pure torsional stiffness (GJ) and deformation torsional stiffness (EI_w).

For pure Saint Venant torsional stiffness, the torsional constant (GJ) is defined in closed form, but for deformation stiffness the calculation of the deformation constant (I_w) is much more complicated. There is no generally valid procedure for the calculation of the deformation constant, but closed-form solutions exist for several known cross-sections.

4.1.5.1 Case 1

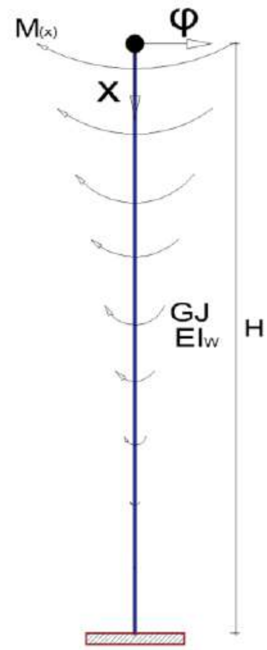


Figure 45. Core subjected to a uniformly distributed torsional moment.

The torsional moment is equal to the sum of uniform torsion and non-uniform torsion:

$$EI_w\varphi''''_{(x)} - GJ\varphi''_{(x)} = M_{(x)} \tag{160}$$

A case of special interest to the structural engineer is cores that are partially enclosed by slabs or floor beams, such as elevators. The effect of connecting elements can always be safely ignored, but their contribution is usually significant and in some cases needs to be taken into account for economic reasons. Connecting elements prevent the core section from deforming and increase its torsional stiffness. Investigations by Vlasov (1961) show that the phenomenon is taken into account by modifying the torsional differential equation.

$$EI_w\varphi''''_{(x)} - GJ^*\varphi''_{(x)} = M_{(x)} \tag{161}$$

Where:

$$\left\{\begin{matrix} J^* = J + \bar{J} \\ \bar{J} = \dfrac{4A_0^2}{\dfrac{l^3 sG}{12EI_b} + \dfrac{1.2ls}{A_b}}, A_b = t_b d, I_b = \dfrac{t_b d^3}{12} \end{matrix}\right\} \tag{162}$$

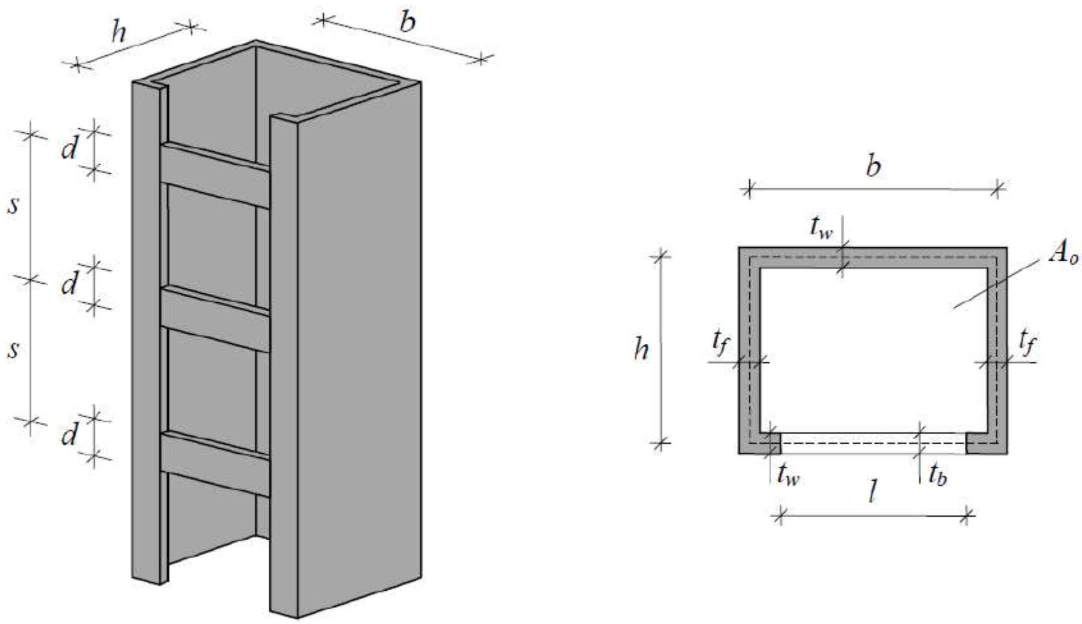


Figure 46. Core partially enclosed by slabs and/or beams (Zalka, 2020).

However, numerical investigations show that when the depth of the connecting beam (d) is relatively large, the effect of the connecting beams tends to be overestimated and may result in a value for the torsional stiffness that is greater than that of a completely closed core, which is clearly impossible.

Taking this observation into account, (Zalka, 2020) proposes to use the following equation as a conservative approximation:

$$\bar{J} = \frac{4h^2b^2}{\frac{2b-1}{t_w} + \frac{l}{t_w^*} + \frac{2h}{t_f}} \tag{163}$$

Where:

$$t_w^* = \frac{d}{s} t_b \tag{164}$$

The above observation is shown in the graph for values of $b = h = 5m, s = 3m, L = 1.80m, t_w = t_f = 0.25m, t_b = 0.20m, E = 23000\frac{MN}{m2}$ y $G = 9580\frac{MN}{m2}$. As can be seen, the formula proposed by Vlasov overestimates the torsional stiffness for relatively large values of d⁄s. In structural engineering practice, overestimating the torsional stiffness can give a false sense of security to the engineer. However, Vlasov's formula seems to be more accurate than the formula proposed by Dr. Zalka at an initial stage ($d/s < 0.3$), but then Vlasov's formula starts to overestimate the torsional stiffness moderately and dangerously.

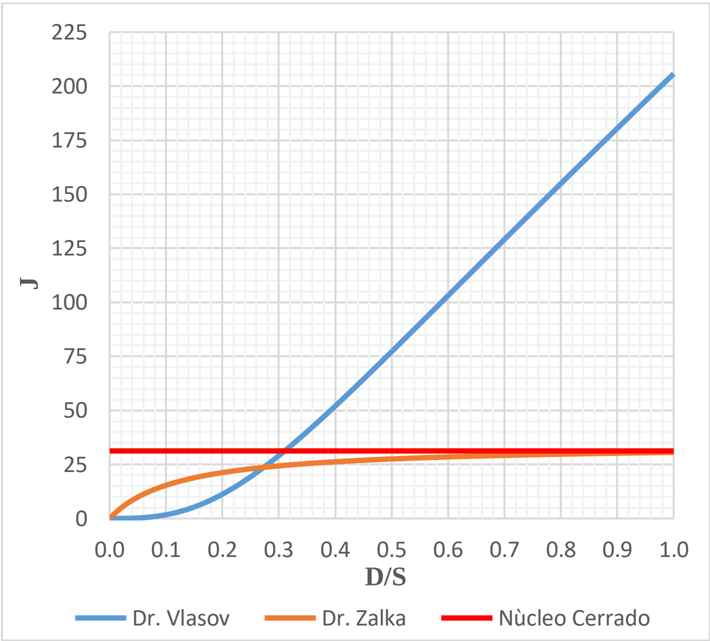


Figure 47. Comparison of the torsional parameter J.

Assuming a general torsional load (Miranda E. , 1999):

$$M_{(x)} = \frac{M_{max}}{1 - e^{-a}}\left(1 - e^{-a+a\frac{x}{H}}\right) \rightarrow M_{(z)} = \frac{M_{max}}{1 - e^{-a}}(1 - e^{-a+az}) \tag{165}$$

Replacing it in the differential equation:

$$\varphi''''_{(z)} - \beta^2 \varphi''_{(z)} = \lambda(1 - e^{-a+az}) \tag{166}$$

Where:

$$\beta = H\sqrt{\frac{GJ^*}{EI_w}}, \lambda = \frac{M_{max}H^4}{EI_w(1 - e^{-a})} \tag{167}$$

If the torsional stiffnesses EI_w and GJ^* are replaced by their lateral equivalents K_b and K_s respectively, it follows that the translational and torsional CTB equations are identical; that is, the same conclusions obtained for the CTB beam subjected to a general lateral load are applicable for the CTB beam subjected to a general torsional load.

$$\varphi_{(z)} = C_0 + C_1 z + C_2 \cosh(\beta z) + C_3 \sinh(\beta z) - \frac{\lambda}{2\beta^2} z^2 - \frac{\lambda}{a^2(a^2-\beta^2)} e^{-a+az} \tag{168}$$

Where:

$$\left\{\begin{array}{c} C_0 = \lambda\left[\frac{1}{2\beta^2} + \frac{1}{a^2(a^2-\beta^2)}\right] - (C_1 + C_2\cosh\beta + C_3\sinh\beta) \\ C_1 = -\lambda\left(\frac{e^{-a}}{a\beta^2}\right) \\ C_2 = \frac{\lambda}{\beta^2}\left(\frac{1}{\beta^2} + \frac{e^{-a}}{a^2-\beta^2}\right) \\ C_3 = \frac{1}{\beta\cosh\beta}\left\{\lambda\left[\frac{1}{\beta^2} + \frac{1}{a(a^2-\beta^2)}\right] - (C_1 + C_2\beta\sinh\beta)\right\} \end{array}\right\} \tag{169}$$

For the case of a uniformly distributed torsional load ($a \to \infty$), the expression for $\varphi_{(z)}$ results:

$$\varphi_{(z)} = \frac{M_{max}H^2}{GJ^*}\left(\frac{1-z^2}{2}\right) + \frac{M_{max}H^4}{EI_w}\left\{\frac{\beta[\sinh(\beta z) - \sinh\beta] - 1 + \sinh(\beta - \beta z)}{\beta^4\cosh\beta}\right\} \tag{170}$$

This deflection expression clearly shows how the bending and shear contributors interact, producing an interaction between them.

$$\left\{\begin{array}{c} \varphi_{(shear)} = \frac{M_{max}H^2}{GJ^*}\left(\frac{1-z^2}{2}\right) \\ \varphi_{(interaction)} = \frac{M_{max}H^4}{EI_w}\left\{\frac{\beta[\sinh(\beta z) - \sinh\beta] - 1 + \sinh(\beta - \beta z)}{\beta^4\cosh\beta}\right\} \end{array}\right\} \tag{171}$$

Evaluating the maximum deflection when z = 0:

$$\varphi_{(z)} = \frac{M_{max}H^2}{2GJ^*} + \frac{M_{max}H^4}{EI_w}\left[\frac{(1-\beta)\sinh\beta - 1}{\beta^4\cosh\beta}\right] \tag{172}$$

4.1.5.2 Case 2

- **Calculation of the Transfer Matrix**

According to the differential equation and assuming that the external loads act on the floors and not along the floor height, it is possible to write it as follows:

$$EI_w\varphi''''_{(x)} - GJ^*\varphi''_{(x)} = 0 \tag{173}$$

The expression for $\varphi_{(z)}$ and $\varphi'_{(z)}$ is proposed:

$$\begin{Bmatrix} \varphi_{(z)} = C_0 + C_1 z + C_2\cosh(\alpha^*_\varphi z) + C_3\sinh(\alpha^*_\varphi z) \\ \varphi'_{(z)} = C_1 + C_2\alpha^*_\varphi\sin(\alpha^*_\varphi z) + C_3\alpha^*_\varphi\cosh(\alpha^*_\varphi z) \end{Bmatrix} \tag{174}$$

Where:

$$\alpha^*_\varphi = \sqrt{\frac{GJ^*}{EI_w}} \tag{175}$$

Internal forces such as bending moment and shear force associated with lateral displacement result:

$$\begin{Bmatrix} M_{(z)} = EI_w\varphi''_{(z)} = \alpha^{*2}_\varphi\cosh(\alpha^*_\varphi z)\,EI_w C_2 + \alpha^{*2}_\varphi\sinh(\alpha^*_\varphi z)\,EI_w C_3 \\ V_{(z)} = EI_w\varphi'''_{(z)} - GJ^*\varphi'_{(x)} = (-\alpha^{*2}_\varphi EI_w)C_1 \end{Bmatrix} \tag{176}$$

Writing the equations in matrix form:

$$\begin{Bmatrix} \varphi_i(z_i) \\ \varphi'_i(z_i) \\ M_i(z_i) \\ V_i(z_i) \end{Bmatrix} = K_i(z_i)\begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{Bmatrix} \tag{177}$$

Where:

$$K_i(z_i) = \begin{bmatrix} 1 & z_i & \cosh(\alpha^*_\varphi z) & \sinh(\alpha^*_\varphi z) \\ 0 & 1 & \alpha^*_\varphi\sin(\alpha^*_\varphi z) & \alpha^*\cosh(\alpha^*_\varphi z) \\ 0 & 0 & \alpha^{*2}_\varphi\cosh(\alpha^*_\varphi z)\,EI_w & \alpha^{*2}_\varphi\sinh(\alpha^*_\varphi z)\,EI_w \\ 0 & -\alpha^{*2}_\varphi EI_w & 0 & 0 \end{bmatrix}_i \tag{178}$$

- **Static Analysis Under Static Point Loads Applied at Floor Level**

The relationship between forces and displacements between two consecutive floors is obtained by taking into account the transfer matrix and the vector of external point forces.

$$\begin{Bmatrix} \varphi_n(0) \\ \varphi'_n(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_k(0) \begin{Bmatrix} \varphi_1(h_1) \\ \varphi'_1(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} - \sum_{s=0}^{n} \left[\prod_{k=s+1}^{n} T_k(0) \right] F_s - F_n \tag{179}$$

Expressing it in simplified form:

$$\begin{Bmatrix} \varphi_n(0) \\ \varphi'_n(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} \varphi_1(h_1) \\ \varphi'_1(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} + f \tag{180}$$

Where:

$$\begin{Bmatrix} \mathrm{t} = \prod_{k=1}^{n} T_k(0) \\ f = -\sum_{s=0}^{n} \left[\prod_{k=s+1}^{n} T_k(0) \right] F_s - F_n \end{Bmatrix} \tag{181}$$

This equation expresses the relationship between the part forces and displacements of the top and the base of the beam. An important point to note is that the size of the transfer matrix is 4x4 and remains constant for all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} \varphi_{(1)} = 0 \\ \varphi'_{(1)} = 0 \\ \varphi''_{(0)} = 0 \\ EI_w \varphi'''_{(0)} - GJ^* \varphi'_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} \varphi_1(h_1) = 0 \\ \varphi'_1(h_1) = 0 \\ M_n(0) = 0 \\ V_n(0) = 0 \end{Bmatrix} \tag{182}$$

Replacing:

$$\begin{Bmatrix} \varphi_n(0) \\ \varphi'_n(0) \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} \tag{183}$$

By clearing the bending moment and the shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} + \begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix} \rightarrow \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} = -\begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix}^{-1} \begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix} \tag{184}$$

Substituting the internal force gives the rotation and its slope at the top of the beam:

$$\begin{Bmatrix} \varphi_n(0) \\ \varphi_n'(0) \end{Bmatrix} = -\begin{bmatrix} t_{1,3} & t_{1,4} \\ t_{2,3} & t_{2,4} \end{bmatrix} \begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix}^{-1} \begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{185}$$

4.1.6 Sandwich Beam of Two-Field (SWB)

The sandwich beam (SWB) is developed, which considers that the structure consists of a parallel coupling of a Timoshenko beam (TB) and a bending beam (EBB) with deformation by global bending and shear; and local bending respectively. The beams are assumed to be coupled in parallel by axially stiff members that only transmit horizontal forces and do not deform.

This model is suitable for modeling coupled portal frames and shear walls. The SWB beam model takes into account two kinematic fields: a transverse motion (u) and a rotational motion (θ); with stiffnesses K_{b1}, K_s and K_{b2} as global bending, shear and local stiffnesses, respectively.

The great acceptance that the literature has applied to the sandwich beam (SWB) to analyze structures such as portal frames, coupled shear walls and even buildings in global form, is due to its ability to correctly describe the three fundamental deformations of any structure; that is: shear deformation, global bending deformation, and local bending deformation.

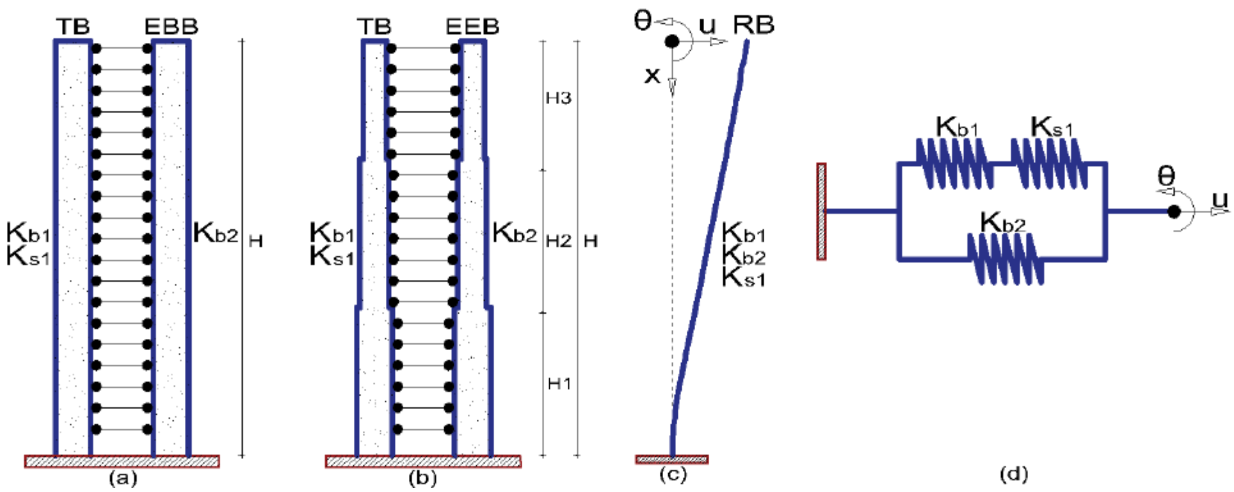


Figure 48. Sandwich beam of two-field (SWB). a) Case 1, b) Case 2, c) equivalent RB and d) Idealization of SWB stiffness.

4.1.6.1 Case 1

The potential energy of the two-field SWB model is expressed as follows:

$$V = \frac{1}{2}\int_0^H \left\{K_{b1}{\theta'_{(x)}}^2 + K_{s1}\left[\theta_{(x)} - u'_{(x)}\right]^2\right\}dx + \frac{1}{2}\int_0^H K_{b2}{u''_{(x)}}^2\,dx \tag{186}$$

The characteristic stiffnesses are evaluated according to the structural element:

- Coupled shear wall:

$$\begin{cases} K_{b1} = \sum_{i=1}^{w} EA_{w,i} c_i^2, K_{b2} = r \sum_{i=1}^{w} EI_{w,i}, K_{s1} = \left(K_b^{-1} + K_w^{-1}\right)^{-1} \\ K_b = \sum_{i=1}^{b} \frac{6EI_{b,i}[(l^* + S_1)^2 + (l^* + S_2)^2]}{l^{*3} h \left(1 + 12 \frac{kEI_{b,i}}{l^{*2} GA_{b,i}}\right)}, K_w = \sum_{i=1}^{w} \frac{12EI_{w,i}}{h^2}, r = \frac{K_c}{K_c + K_b} \end{cases} \tag{187}$$

- Frame:

$$\begin{cases} K_{b1} = \sum_{i=1}^{c} EA_{c,i} c_i^2, K_{b2} = r \sum_{i=1}^{c} EI_{c,i}, K_{s1} = \left(K_b^{-1} + K_c^{-1}\right)^{-1} \\ K_b = \sum_{i=1}^{b} \frac{12EI_{b,i}}{lh}, K_c = \sum_{i=1}^{c} \frac{12EI_{c,i}}{h^2}, r = \frac{K_c}{K_c + K_b} \end{cases} \tag{188}$$

The work done by the external force is:

$$W = \int_0^H f_{(x)} u_{(x)}\, dx \tag{189}$$

Consequently, the total potential energy of the two-field SWB beam subjected to a general lateral load distribution is expressed as:

$$\mathcal{U} = \frac{1}{2} \int_0^H \left\{ K_{b1} {\theta'_{(x)}}^2 + K_{s1} \left[\theta_{(x)} - u'_{(x)}\right]^2 + K_{b2} {u''_{(x)}}^2 \right\} dx - \int_0^H f(x) u_{(x)}\, dx \tag{190}$$

Closed-form solutions of the model on which a transverse load acts are achieved by solving the differential system arising from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\begin{aligned} \delta\mathcal{U} = \int_0^H & \{K_{b1} \theta'_{(x)} \delta\theta'_{(x)} + K_{s1} \left[\theta_{(x)} - u'_{(x)}\right] \delta\theta_{(x)} - K_{s1} \left[\theta_{(x)} - u'_{(x)}\right] \delta u'_{(x)} \\ & + K_{b2} u''_{(x)} \delta u''_{(x)}\} dx - \int_0^H f(x) \delta u_{(x)}\, dx - \int_0^H u_{(x)}\, \delta f(x) dx \end{aligned} \tag{191}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\delta\mathcal{U} = \left[K_{b1}\theta'_{(x)}\delta\theta_{(x)}\right]_0^H - \left\{K_{s1}\left[\theta_{(x)} - u'_{(x)}\right] + K_{b2}u'''_{(x)}\right\}\delta u_{(x)}\Big|_0^H + \left[K_{b2}u''_{(x)}\delta u'_{(x)}\right]_0^H$$
$$+ \int_0^H \left\{K_{s1}\left[\theta_{(x)} - u'_{(x)}\right] - K_{b1}\theta''_{(x)}\right\}\delta\theta_{(x)}dx$$
$$+ \int_0^H \left\{K_{s1}\left[\theta'_{(x)} - u''_{(x)}\right] + K_{b2}u''''_{(x)} - f_{(x)}\right\}\delta u_{(x)}dx - \int_0^H u_{(x)}\,\delta f(x)dx \tag{192}$$

Equating the terms to zero results in the following equations:

$$\begin{Bmatrix} K_{s1}\left[\theta_{(x)} - u'_{(x)}\right] - K_{b1}\theta''_{(x)} = 0 \\ K_{s1}\left[\theta'_{(x)} - u''_{(x)}\right] + K_{b2}u''''_{(x)} - f_{(x)} = 0 \end{Bmatrix} \tag{193}$$

And boundary conditions:

$$\begin{Bmatrix} \theta'_{(0)} = 0 \\ u''_{(0)} = 0 \\ K_{s1}\left[\theta_{(0)} - u'_{(0)}\right] + K_{b2}u'''_{(0)} = 0 \end{Bmatrix} \tag{194}$$

Using the method of coefficients for the solution of the system of equations:

$$\begin{Bmatrix} u_{(x)} \\ \theta_{(x)} \end{Bmatrix} = \begin{bmatrix} -K_{s1}D & K_{s1} - K_{b1}D^2 \\ K_{s1}D^2 - K_{b2}D^4 & K_{s1}D \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ f_{(x)} \end{Bmatrix} \tag{195}$$

I.e.,

$$\begin{Bmatrix} u''''''_{(x)} - \left(\dfrac{K_{s1}}{K_{b1}} + \dfrac{K_{s1}}{K_{b2}}\right)u''''_{(x)} \\ \theta''''''_{(x)} - \left(\dfrac{K_{s1}}{K_{b1}} + \dfrac{K_{s1}}{K_{b2}}\right)\theta''''_{(x)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{1}{K_{b2}}\left[f''_{(x)} - \dfrac{K_{s1}}{K_{b1}}f_{(x)}\right] \\ -\dfrac{K_{s1}}{K_{b1}K_{b2}}f'_{(x)} \end{Bmatrix} \tag{196}$$

A sixth order differential equation is obtained. Normalizing the differential equation by the variable $z = x/H$:

$$u''''''_{(z)} - \left[\left(\frac{K_{s1}}{K_{b1}} + \frac{K_{s1}}{K_{b2}}\right)H^2\right]u''''_{(z)} = \frac{H^4}{K_{b2}}\left[f''_{(z)} - H^2\frac{K_{s1}}{K_{b1}}f_{(z)}\right] \tag{197}$$

Defining three parameters:

$$\left\{\alpha = H\sqrt{\frac{K_{s1}}{K_{b2}}}, \kappa = \sqrt{1+\frac{K_{b2}}{K_{b1}}}, \lambda = \frac{W_{max}H^4}{K_{b2}(1-e^{-a})}\right\} \tag{198}$$

Replacing the first two parameters:

$$u_{(z)}'''''' - (\alpha\kappa)^2 u_{(z)}'''' = \frac{H^4}{K_{b2}}\left[f_{(z)}'' - \alpha^2(\kappa^2-1)f_{(z)}\right] \tag{199}$$

Assuming a general lateral load (Miranda E. , 1999):

$$f_{(x)} = \frac{W_{max}}{1-e^{-a}}\left(1-e^{-a+a\frac{x}{H}}\right) \to f_{(z)} = \frac{W_{max}}{1-e^{-a}}(1-e^{-a+az}) \tag{200}$$

Replacing the lateral load and the third parameter:

$$u_{(z)}'''''' - (\alpha\kappa)^2 u_{(z)}'''' = -\lambda\alpha^2(\kappa^2-1) + \lambda[\alpha^2(\kappa^2-1)-a^2]e^{-a+az} \tag{201}$$

The expression for $u_{(z)}$ is proposed:

$$\begin{aligned} u_{(z)} = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + C_4\cosh(\alpha\kappa z) + C_5\sinh(\alpha\kappa z) + \frac{\lambda(\kappa^2-1)}{24\kappa^2}z^4 \\ + \frac{\lambda[\alpha^2(\kappa^2-1)-a^2]}{a^4[a^2-(\alpha\kappa)^2]}e^{-a+az} \end{aligned} \tag{202}$$

The constants are obtained by evaluating the relevant boundary conditions (the origin of x is at the base of the model):

$$\left\{\begin{matrix} u_{(1)} = 0, u_{(1)}' = 0 \\ u_{(0)}'' = 0 \\ u_{(1)}''' = \lambda\left(1-\frac{1}{a}+\frac{e^{-a}}{a}\right) \\ u_{(0)}'''' = \lambda(1-e^{-a}) \\ u_{(1)}''''' = \lambda\left[\alpha^2\left(1-\frac{1}{a}+\frac{e^{-a}}{a}\right)-a\right] \end{matrix}\right\} \tag{203}$$

Constants:

$$
\left\{
\begin{array}{c}
\left\{
\begin{array}{c}
C_0 = -(C_1 + C_2 + C_3) - C_4 \operatorname{sech}(\alpha\kappa) - \dfrac{\lambda\alpha^2}{(\alpha\kappa)^5}\tanh(\alpha\kappa)\left[\left(1 - \dfrac{1}{a} + \dfrac{e^{-a}}{a}\right) + \dfrac{a}{a^2 - (\alpha\kappa)^2}\right] \\
-\lambda\left\{\dfrac{1}{24}\left(1 - \dfrac{1}{k^2}\right) + \dfrac{\alpha^2(\kappa^2 - 1) - a^2}{a^4[a^2 - (\alpha\kappa)^2]}\right\}
\end{array}
\right. \\
C_1 = -(2C_2 + 3C_3) - \dfrac{\lambda\alpha^2}{(\alpha\kappa)^4}\left[\left(1 - \dfrac{1}{a} + \dfrac{e^{-a}}{a}\right) + \dfrac{a}{a^2 - (\alpha\kappa)^2}\right] - \lambda\left\{\dfrac{1}{6}\left(1 - \dfrac{1}{k^2}\right) + \dfrac{\alpha^2(\kappa^2 - 1) - a^2}{a^3[a^2 - (\alpha\kappa)^2]}\right\} \\
C_2 = -C_4\dfrac{(\alpha\kappa)^2}{2} + \dfrac{\lambda e^{-a}}{2a^2}\left[1 + \dfrac{\alpha^2}{a^2 - (\alpha\kappa)^2}\right] \\
C_3 = \dfrac{\lambda e^{-a}}{6a}\left(1 - \dfrac{1}{k^2}\right) \\
C_4 = \dfrac{\lambda}{(\alpha\kappa)^4}\left[\dfrac{1}{k^2} + \dfrac{e^{-a}\alpha^2}{a^2 - (\alpha\kappa)^2}\right] \\
C_5 = -C_4\tanh(\alpha\kappa) + \dfrac{\lambda\alpha^2}{(\alpha\kappa)^5}\operatorname{sech}(\alpha\kappa)\left[\left(1 - \dfrac{1}{a} + \dfrac{e^{-a}}{a}\right) + \dfrac{a}{a^2 - (\alpha\kappa)^2}\right]
\end{array}
\right.
$$

(204)

The maximum displacement is obtained by evaluating $u_{(z)}$ at 0:

$$u_{(0)} = C_0 + C_4 + \frac{\lambda[\alpha^2(\kappa^2 - 1) - a^2]}{a^4[a^2 - (\alpha\kappa)^2]}e^{-a} \tag{205}$$

The interstory drift can be obtained by deriving $u_{(z)}$ once:

$$\begin{aligned}\Delta_s = {} & C_1 + 2C_2 z + 3C_3 z^2 + C_4(\alpha\kappa)\sinh(\alpha\kappa z) + C_5(\alpha\kappa)\cosh(\alpha\kappa z) + \frac{\lambda(\kappa^2 - 1)}{6\kappa^2}z^3 \\ & + \frac{\lambda[\alpha^2(\kappa^2 - 1) - a^2]}{a^3[a^2 - (\alpha\kappa)^2]}e^{-a+az}\end{aligned} \tag{206}$$

The global drift is obtained as the quotient between maximum displacement and total height:

$$\Delta_g = \left[C_0 + C_4 + \frac{\lambda[\alpha^2(\kappa^2 - 1) - a^2]}{a^4[a^2 - (\alpha\kappa)^2]}e^{-a}\right] \tag{207}$$

For the case of a uniformly distributed lateral load ($a \to \infty$), the constants result:

$$\begin{cases} C_0 = \lambda\left\{-\dfrac{1}{\alpha^3\kappa^5\cosh(\alpha\kappa)}\left[\dfrac{1}{\alpha\kappa}+\sinh(\alpha\kappa)\right]+\dfrac{1}{2\alpha^2\kappa^4}+\dfrac{1}{8}\left(1-\dfrac{1}{\kappa^2}\right)\right\} \\ C_1 = -\lambda\dfrac{1}{6}\left(1-\dfrac{1}{\kappa^2}\right) \\ C_2 = -\lambda\dfrac{1}{2\alpha^2\kappa^4} \\ C_3 = 0 \\ C_4 = \lambda\dfrac{1}{\alpha^4\kappa^6} \\ C_5 = \lambda\dfrac{1}{\alpha^3\kappa^5\cosh(\alpha\kappa)}\left[1-\dfrac{\sinh(\alpha\kappa)}{(\alpha\kappa)}\right] \end{cases} \tag{208}$$

Replacing constants:

$$\begin{cases} \begin{Bmatrix} u_{(z)} = \lambda\left(\dfrac{\kappa^2-1}{\kappa^2}\right)\left(\dfrac{1}{24}z^4-\dfrac{1}{6}z+\dfrac{1}{8}\right)+\dfrac{1}{2\kappa^4\alpha^2}\lambda(1-z^2) \\ -\dfrac{1}{\kappa^2}\lambda\left\{\dfrac{1-\cosh(\alpha\kappa z-\alpha\kappa)-(\alpha\kappa)[\sinh(\alpha\kappa z)-\sinh(\alpha\kappa)]}{(\alpha\kappa)^4\cosh(\alpha\kappa)}\right\} \end{Bmatrix} \\ u_{(0)} = \lambda\left(\dfrac{\kappa^2-1}{8\kappa^2}\right)+\dfrac{1}{2\kappa^4\alpha^2}\lambda-\dfrac{1}{\kappa^2}\lambda\left[\dfrac{1-\cosh(\alpha\kappa)+(\alpha\kappa)\sinh(\alpha\kappa)}{(\alpha\kappa)^4\cosh(\alpha\kappa)}\right] \\ \Delta_p = \lambda\left(\dfrac{\kappa^2-1}{6\kappa^2}\right)(z^3-1)-\dfrac{1}{\kappa^4\alpha^2}\lambda z+\dfrac{1}{\kappa^2}\lambda\left[\dfrac{\sinh(\alpha\kappa z-\alpha\kappa)+(\alpha\kappa)\cosh(\alpha\kappa z)}{(\alpha\kappa)^3\cosh(\alpha\kappa)}\right] \\ \Delta_g = \dfrac{1}{H}\left\{\lambda\left(\dfrac{\kappa^2-1}{8\kappa^2}\right)+\dfrac{1}{2\kappa^4\alpha^2}\lambda-\dfrac{1}{\kappa^2}\lambda\left[\dfrac{1-\cosh(\alpha\kappa)+(\alpha\kappa)\sinh(\alpha\kappa)}{(\alpha\kappa)^4\cosh(\alpha\kappa)}\right]\right\} \end{cases} \tag{209}$$

After some simple manipulations, the control parameters result:

$$\begin{cases} \begin{Bmatrix} u_{(z)} = \dfrac{W_{max}H^4}{K_{b1}+K_{b2}}\left(\dfrac{1}{24}z^4-\dfrac{1}{6}z+\dfrac{1}{8}\right)+\dfrac{1}{2k^4}\dfrac{W_{max}H^2}{K_{s1}}(1-z^2) \\ -\dfrac{1}{\kappa^2}\dfrac{W_{max}H^4}{K_{b2}}\left\{\dfrac{1-\cosh(\alpha\kappa z-\alpha\kappa)-(\alpha\kappa)[\sinh(\alpha\kappa z)-\sinh(\alpha\kappa)]}{(\alpha\kappa)^4\cosh(\alpha\kappa)}\right\} \end{Bmatrix} \\ u_{(0)} = \dfrac{W_{max}H^4}{8(K_{b1}+K_{b2})}+\dfrac{1}{2k^4}\dfrac{W_{max}H^2}{K_{s1}}-\dfrac{1}{\kappa^2}\dfrac{W_{max}H^4}{K_{b2}}\left[\dfrac{1-\cosh(\alpha\kappa)+(\alpha\kappa)\sinh(\alpha\kappa)}{(\alpha\kappa)^4\cosh(\alpha\kappa)}\right] \\ \Delta_p = \dfrac{W_{max}H^4}{6(K_{b1}+K_{b2})}(z^3-1)-\dfrac{1}{k^4}\dfrac{W_{max}H^2}{K_{s1}}z+\dfrac{1}{\kappa^2}\dfrac{W_{max}H^4}{K_{b2}}\left\{\dfrac{\sinh(\alpha\kappa z-\alpha\kappa)+(\alpha\kappa)[\cosh(\alpha\kappa z)]}{(\alpha\kappa)^3\cosh(\alpha\kappa)}\right\} \\ \Delta_g = \dfrac{W_{max}H^3}{8(K_{b1}+K_{b2})}+\dfrac{1}{2k^4}\dfrac{W_{max}H}{K_{s1}}-\dfrac{1}{\kappa^2}\dfrac{W_{max}H^3}{K_{b2}}\left[\dfrac{1-\cosh(\alpha\kappa)+(\alpha\kappa)\sinh(\alpha\kappa)}{(\alpha\kappa)^4\cosh(\alpha\kappa)}\right] \end{cases} \tag{210}$$

It is important to note that in the shear deflection equation, the $1/\kappa^4$ term generally tends to unity. It is noted that the lateral deflection equation obtained is identical to the equation proposed by

Nollet (1979) and Zalka (2020). This expression clearly shows how the bending and shear contributors interact; producing an interaction between them.

$$\begin{Bmatrix} u_{(flexión)} = \dfrac{W_{max}H^4}{K_{b1}+K_{b2}}\left(\dfrac{1}{24}z^4 - \dfrac{1}{6}z + \dfrac{1}{8}\right) \\ u_{(shear)} = \dfrac{1}{2k^4}\dfrac{W_{max}H^2}{K_{s1}}(1-z^2) \\ u_{(interaction)} = -\dfrac{1}{\kappa^2}\dfrac{W_{max}H^4}{K_{b2}}\left\{\dfrac{1-\cosh(\alpha\kappa z - \alpha\kappa) - (\alpha\kappa)[\sinh(\alpha\kappa z) - \sinh(\alpha\kappa)]}{(\alpha\kappa)^4\cosh(\alpha\kappa)}\right\} \end{Bmatrix} \quad (211)$$

And for the maximum deflection:

$$\begin{Bmatrix} u_{(flexión)} = \dfrac{W_{max}H^4}{8(K_{b1}+K_{b2})} \\ u_{(shear)} = \dfrac{1}{k^4}\dfrac{W_{max}H^2}{2K_{s1}} \\ u_{(interaction)} = -\dfrac{1}{\kappa^2}\dfrac{W_{max}H^4}{K_{b2}}\left[\dfrac{1-Cosh(\alpha\kappa) + (\alpha\kappa)\sinh(\alpha\kappa)}{(\alpha\kappa)^4\cosh(\alpha\kappa)}\right] \end{Bmatrix} \quad (212)$$

We will consider some special cases of analysis:

a) When $\frac{K_{b2}}{K_{b1}} \to 0$ $(K_{b1} \to \infty,\ K_{b2} \to 0)$. This situation occurs in multi-span few-story coupled portal and/or shear walls, where the global bending is larger in magnitude compared to the local bending. Evaluating the limit of κ:

$$\kappa = \lim_{\frac{K_{b2}}{K_{b1}}\to 0} \kappa = \lim_{\frac{K_{b2}}{K_{b1}}\to 0} \sqrt{1+\frac{K_{b2}}{K_{b1}}} = \sqrt{1+0} = 1 \leftrightarrow \kappa = 1 \quad (213)$$

After integrating twice, evaluate two boundary conditions and apply limits:

$$u''_{(z)} = -\frac{wH^2}{K_s} \quad (214)$$

This equation shows that shear is dominant in the structure and is identical to the differential equation of a shear beam (SB):

$$u_{(z)} = \frac{W_{max}H^2}{2K_{s1}}(1-z^2) \to u_{(0)} = \frac{W_{max}H^2}{2K_{s1}} \quad (215)$$

b) When $h_v \to 0$. This situation occurs in portal frames and/or coupled shear walls where the connecting beams have little camber and consequently little bending stiffness. In this case the function of the connecting beams is primarily to transmit the horizontal loads and force the columns and/or shear walls to work together. Evaluating the limit of K_s and α:

$$K_s = \lim_{K_v \to 0} \frac{K_v K_c}{K_c + K_v} = \lim_{K_v \to 0} K_v \frac{K_c}{K_c + K_v} = 0$$
$$\alpha = \lim_{h_v \to 0} \alpha = \lim_{K_v \to 0} \alpha = \lim_{K_v \to 0} H \sqrt{\frac{K_s}{K_{b2}}} = 0 \qquad (216)$$

After integrating twice and evaluating two boundary conditions:

$$u''''_{(z)} = \frac{wH^4}{K_{b2}} \qquad (217)$$

This equation shows that local bending is dominant in the structure and is identical to the differential equation for a bending beam (EBB):

$$u_{(z)} = \frac{W_{max}H^4}{K_{b2}}\left(\frac{1}{24}z^4 - \frac{1}{6}z + \frac{1}{8}\right) \to u_{(0)} = \frac{W_{max}H^4}{8K_{b2}} \qquad (218)$$

c) When $h_v \to \infty$. This situation turns out to be a theoretical case with little practical application and occurs in coupled portal frames and/or shear walls where the connecting beams have a very large cant, as a consequence the shear stiffness increases and the local bending stiffness decreases drastically because the shear reduction factor tends to zero. In this case the function of the connecting beams is to try to fully couple the columns and/or shear walls so that they work as a single unit. Evaluating the limit of r and K_s:

$$r = \lim_{K_v \to \infty} r = \lim_{K_v \to \infty} \frac{K_c}{K_c + K_v} = \lim_{K_v \to \infty} \frac{1}{1 + \frac{K_v}{K_c}} = 0 \leftrightarrow r = 0 \leftrightarrow K_{b2} = 0$$
$$K_s = \lim_{K_v \to \infty} \frac{1}{\frac{1}{K_v} + \frac{1}{K_c}} = \lim_{K_v \to \infty} \frac{1}{0 + \frac{1}{K_c}} = K_c \leftrightarrow K_s = K_c \leftrightarrow K_s = \infty \qquad (219)$$

After integrating twice, evaluate two boundary conditions and apply limits:

$$u''_{(z)} = \frac{wH^4 z^2}{2(K_{b1} + K_{b2})} \tag{220}$$

It is obtained:

$$u_{(z)} = \frac{W_{max}H^4}{K_{b1} + K_{b2}}\left(\frac{1}{24}z^4 - \frac{1}{6}z + \frac{1}{8}\right) \rightarrow u_{(0)} = \frac{W_{max}H^4}{8(K_{b1} + K_{b2})} \tag{221}$$

This equation shows that in the structure the total bending (global + local) is dominant and is identical to the differential equation of a bending beam (EBB).

d) When $K_{b1} \rightarrow \infty$. This situation occurs in coupled portal and/or shear walls where $A_i \rightarrow \infty$; i.e., where the effect of axial deformations is neglected. Evaluating the limit of κ:

$$\kappa = \lim_{K_{b1}\rightarrow\infty} \kappa = \lim_{K_{b1}\rightarrow\infty} \sqrt{1 + \frac{K_{b2}}{K_{b1}}} = \sqrt{1 + 0} = 1 \leftrightarrow \kappa = 1 \tag{222}$$

After integrating twice, evaluate two boundary conditions and apply limits:

$$u''''_{(z)} - \alpha^2 u''_{(z)} = \frac{wH^4}{K_{b2}} \tag{223}$$

This equation shows that in the structure the global bending is neglected, which leads to the SWB beam behaving like a CTB beam of a field taking into account only the effect of local bending and shear.

$$u_{(z)} = \frac{W_{max}H^2}{K_s}\left(\frac{1 - z^2}{2}\right) + \frac{W_{max}H^4}{K_b}\left\{\frac{\alpha[\sinh(\alpha z) - \sinh\alpha] - 1 + \cosh[\alpha(z - 1)]}{\alpha^4 \cosh\alpha}\right\}$$
$$u_{(0)} = \frac{W_{max}H^2}{2K_s} - \frac{W_{max}H^4}{K_b}\left[\frac{1 + \alpha\sinh\alpha - \cosh\alpha}{\alpha^4 \cosh\alpha}\right] \tag{224}$$

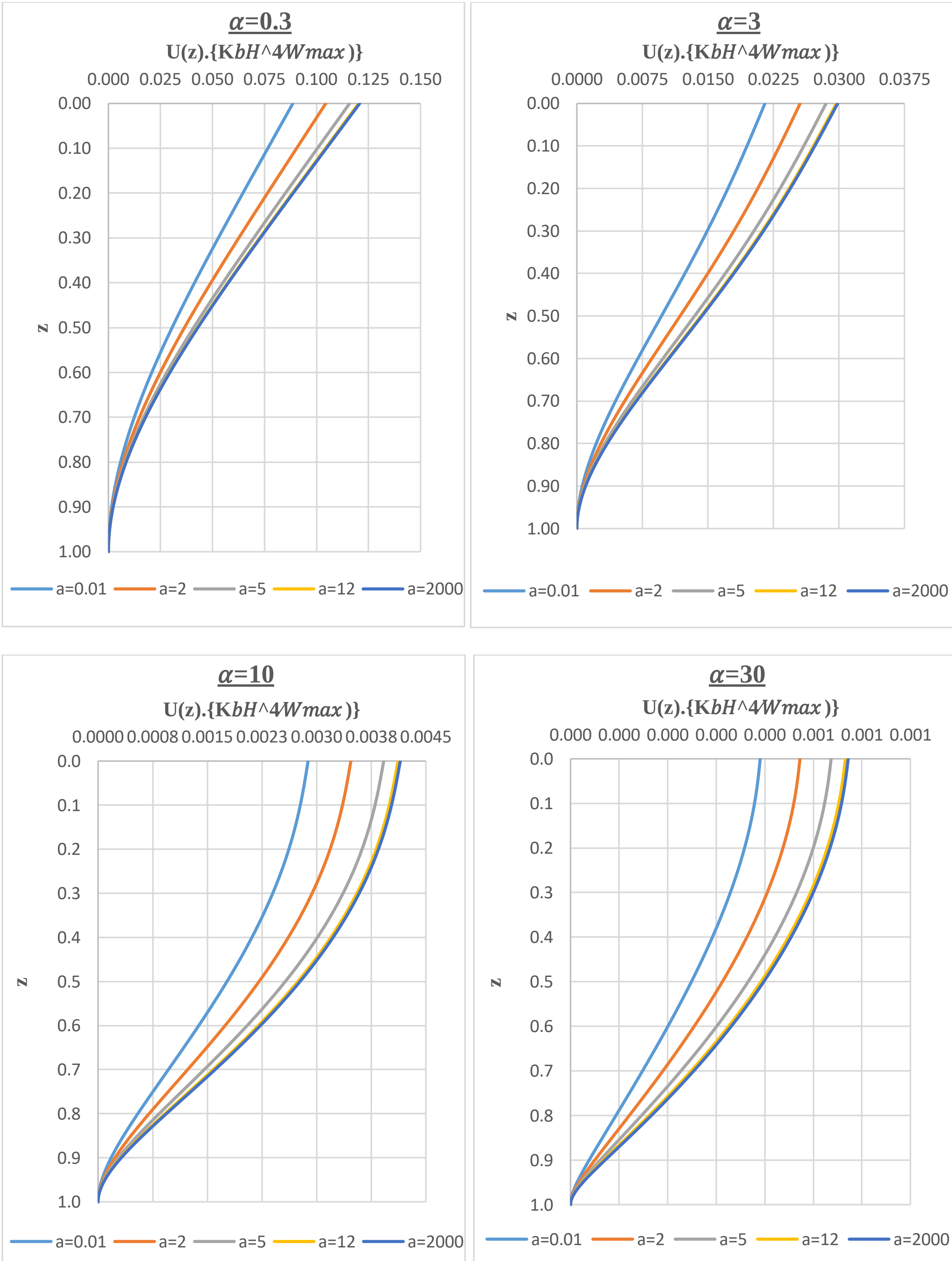


Figure 49. Lateral displacement and effect of parameter a for $k = 1.00021$.

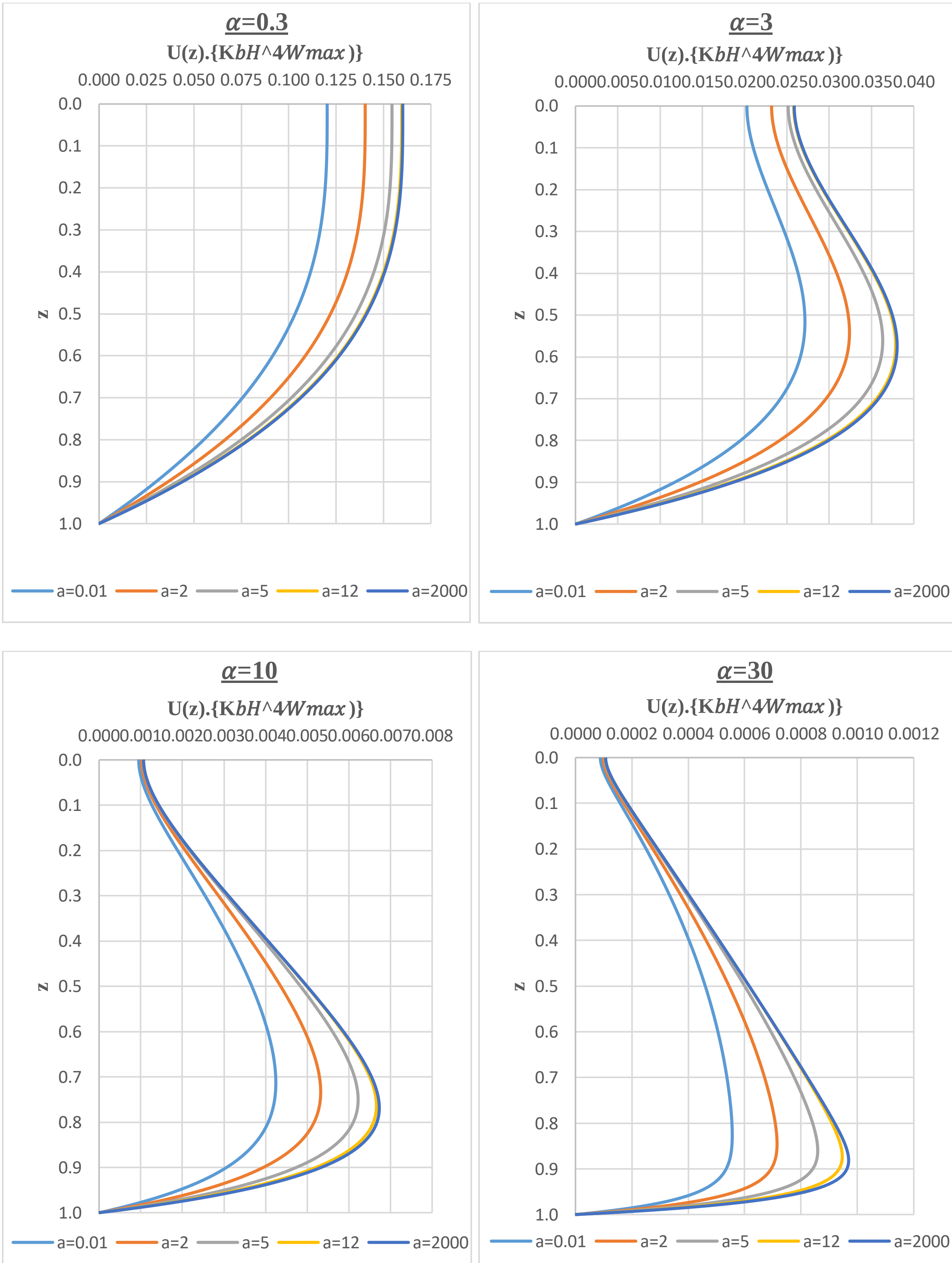


Figure 50. Interstory drift and effect of parameter a for $k = 1.00021$.

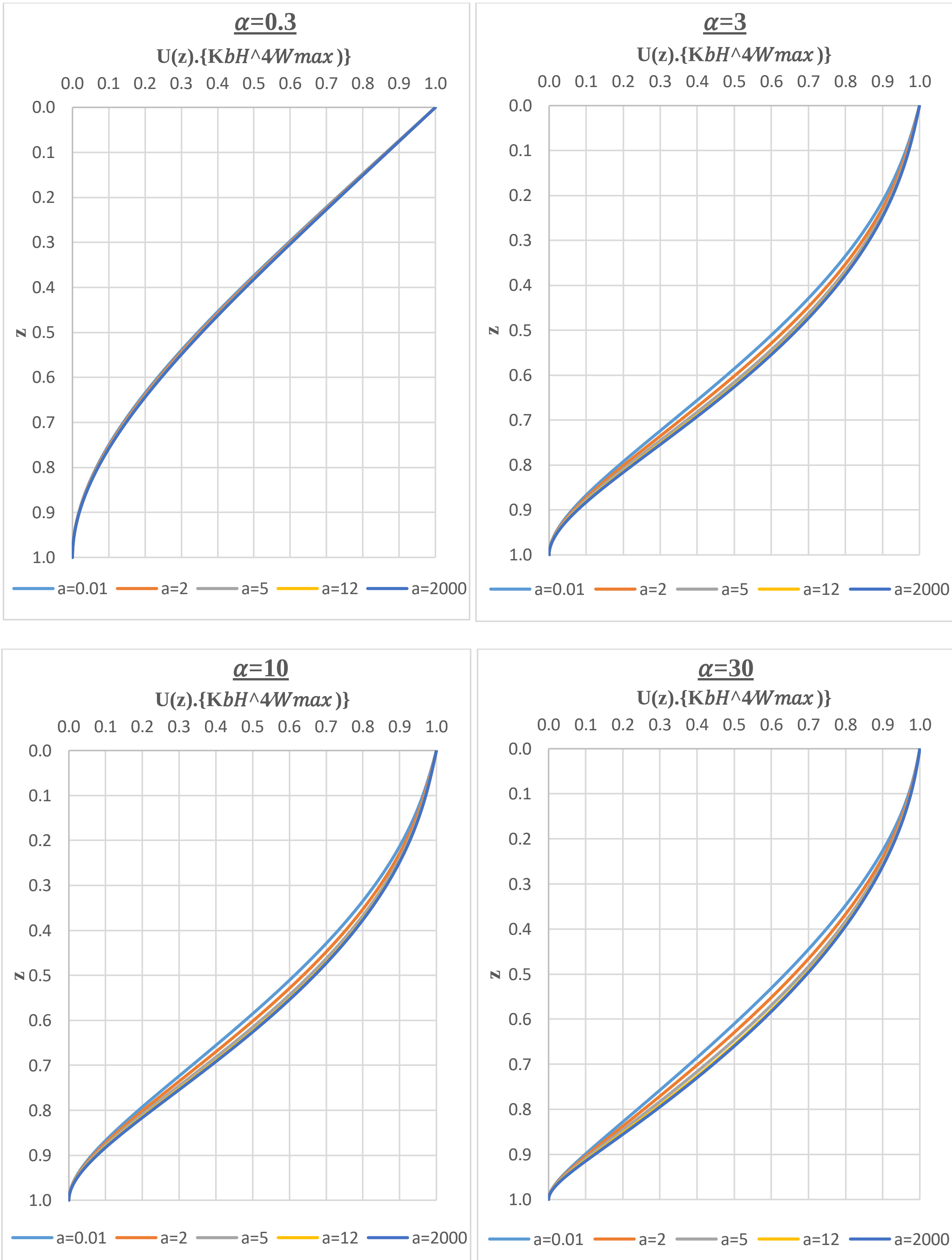


Figure 51. Normalized lateral displacement and effect of parameter a for $k = 1.00021$.

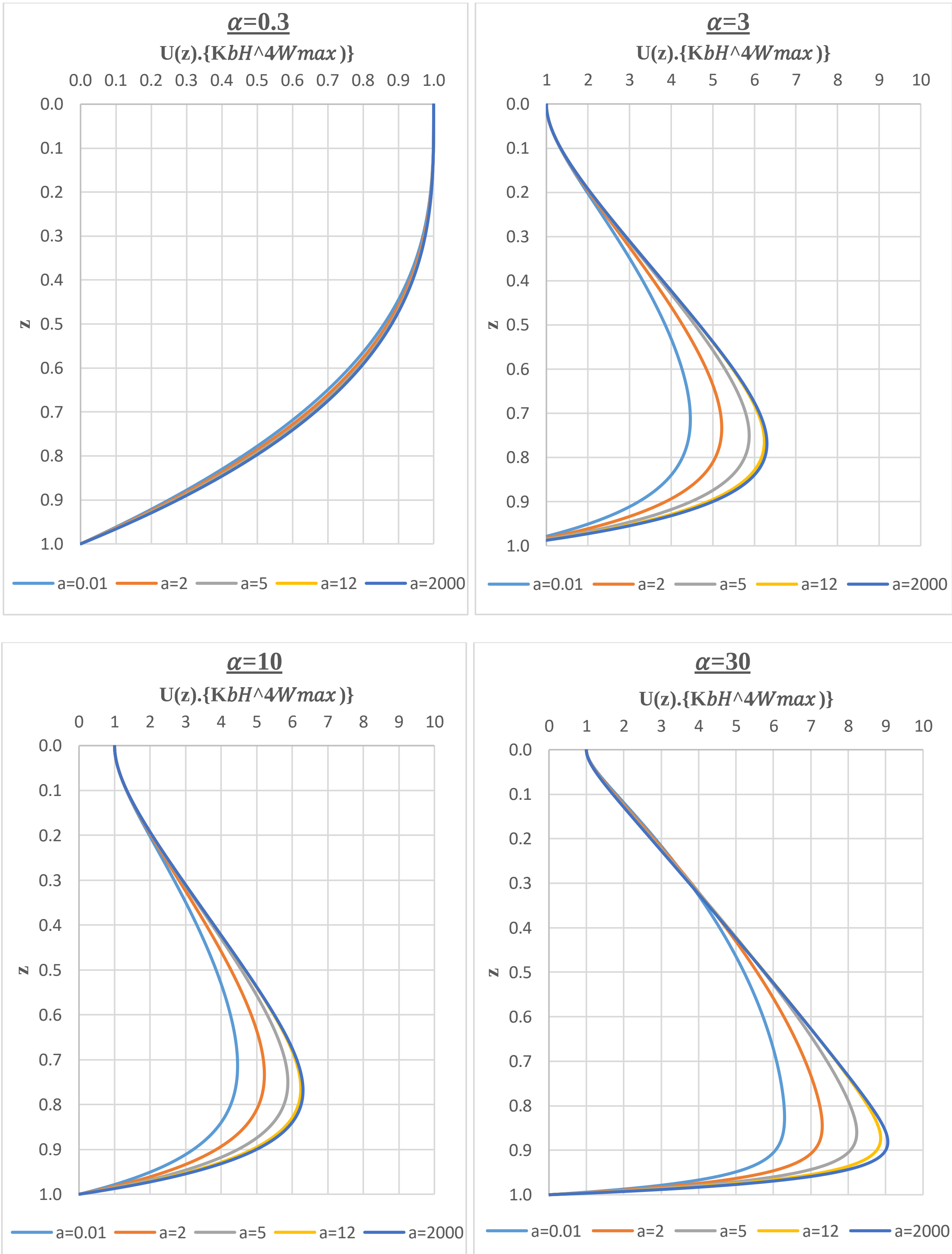


Figure 52. Normalized interstory drift and effect of parameter a for $k = 1.00021$.

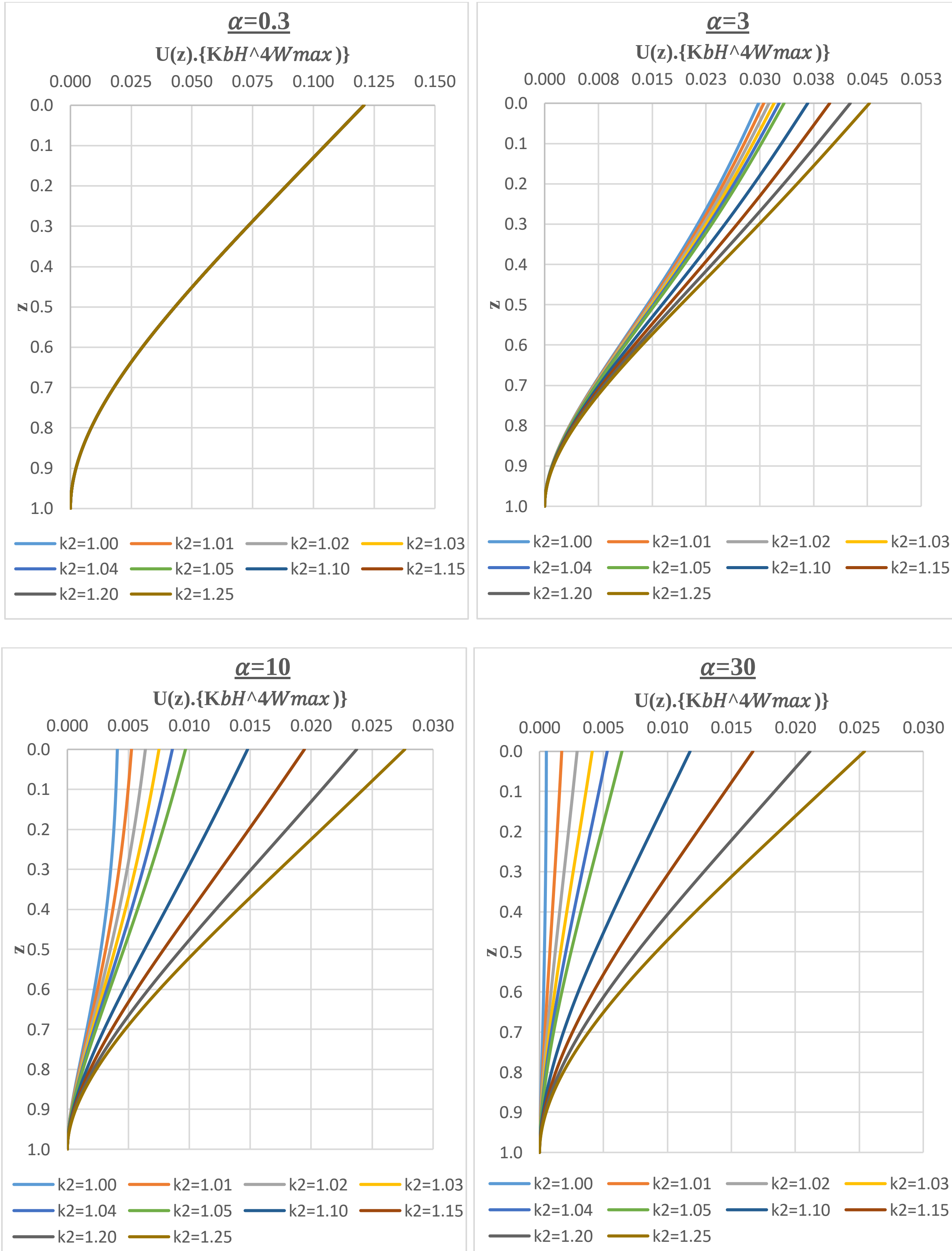


Figure 53. Lateral displacement and effect of parameter κ for $a = 2000$.

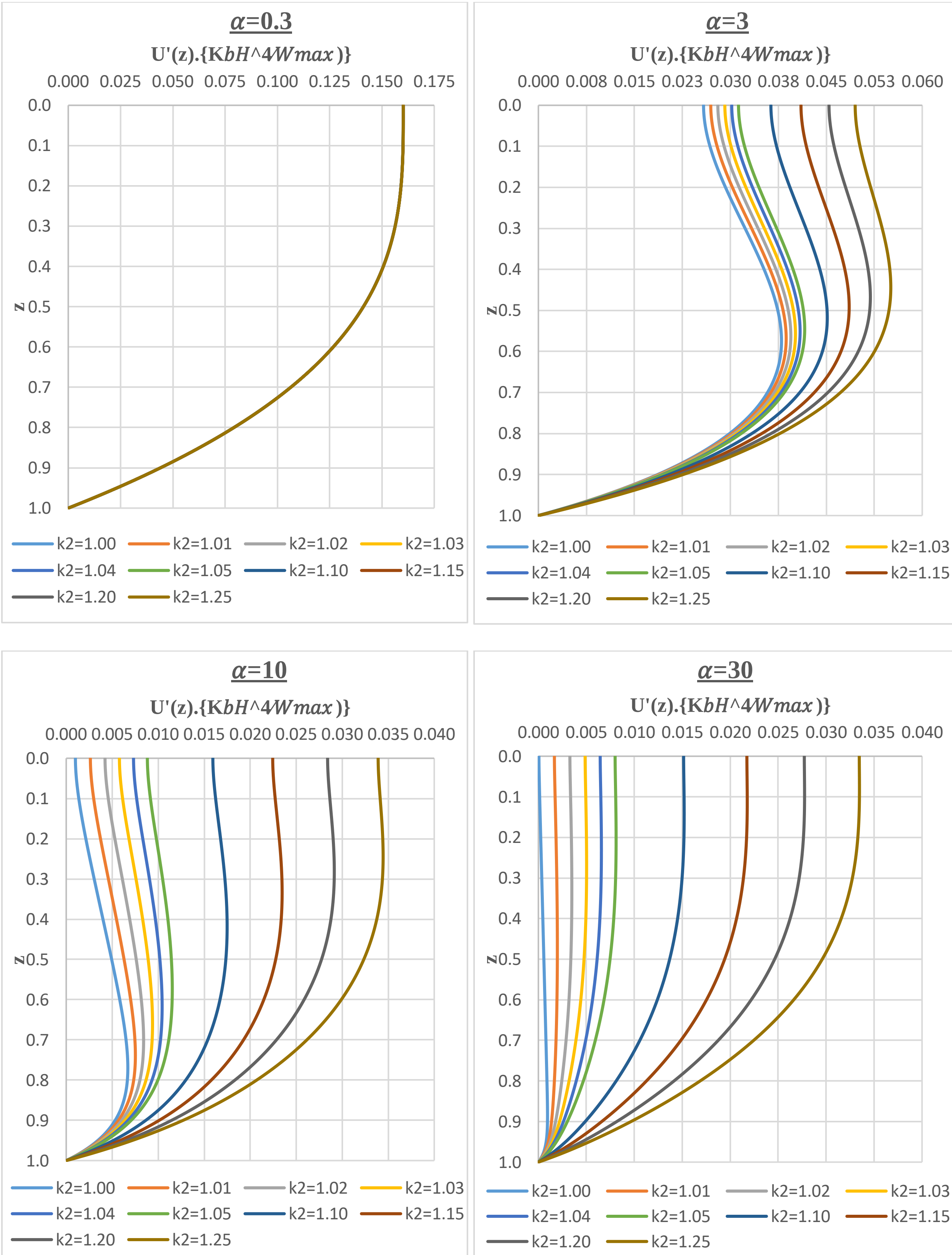


Figure 54. Interstory drift and effect of parameter κ for $\alpha = 2000$.

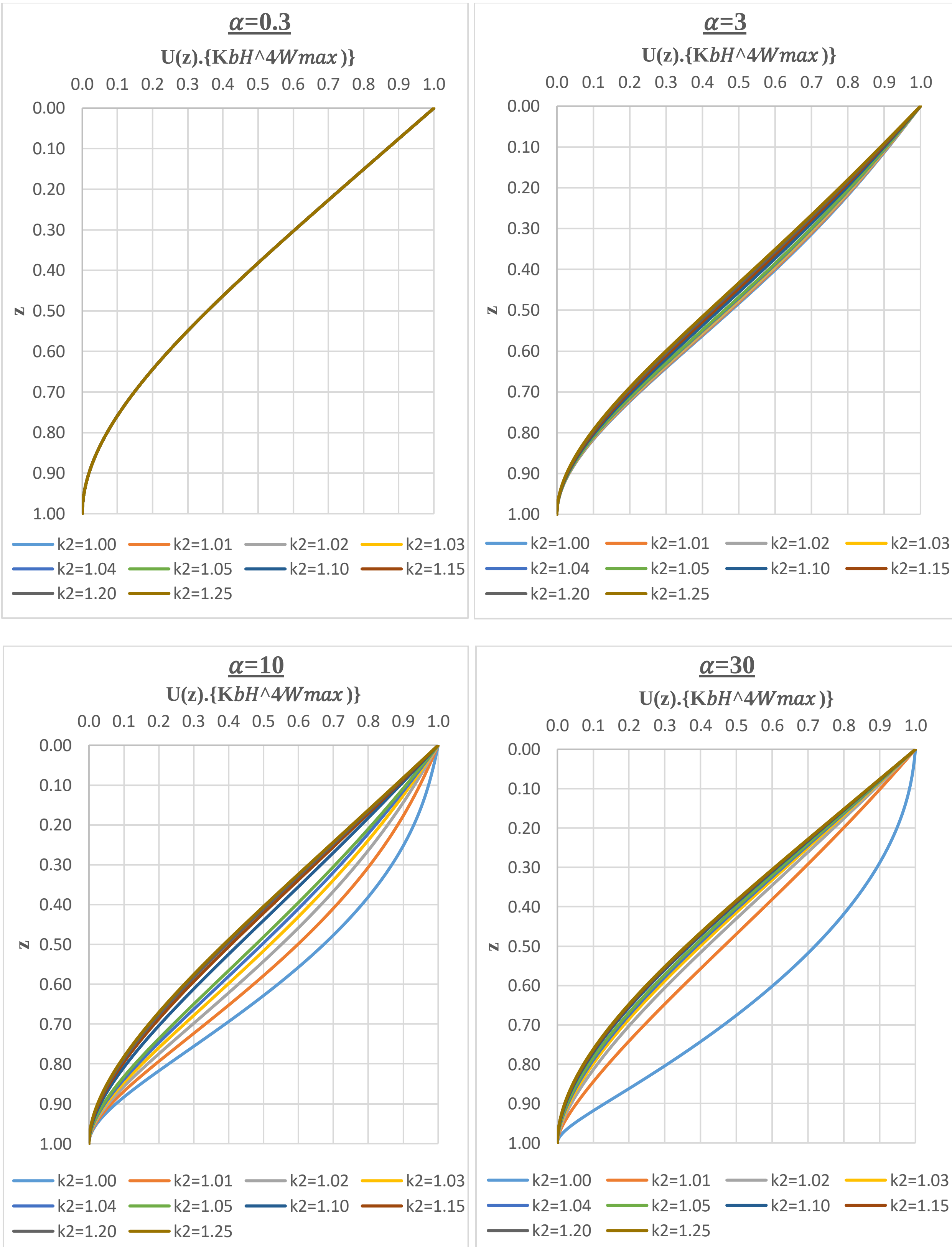


Figure 55. Normalized lateral displacement and effect of parameter κ for $a = 2000$.

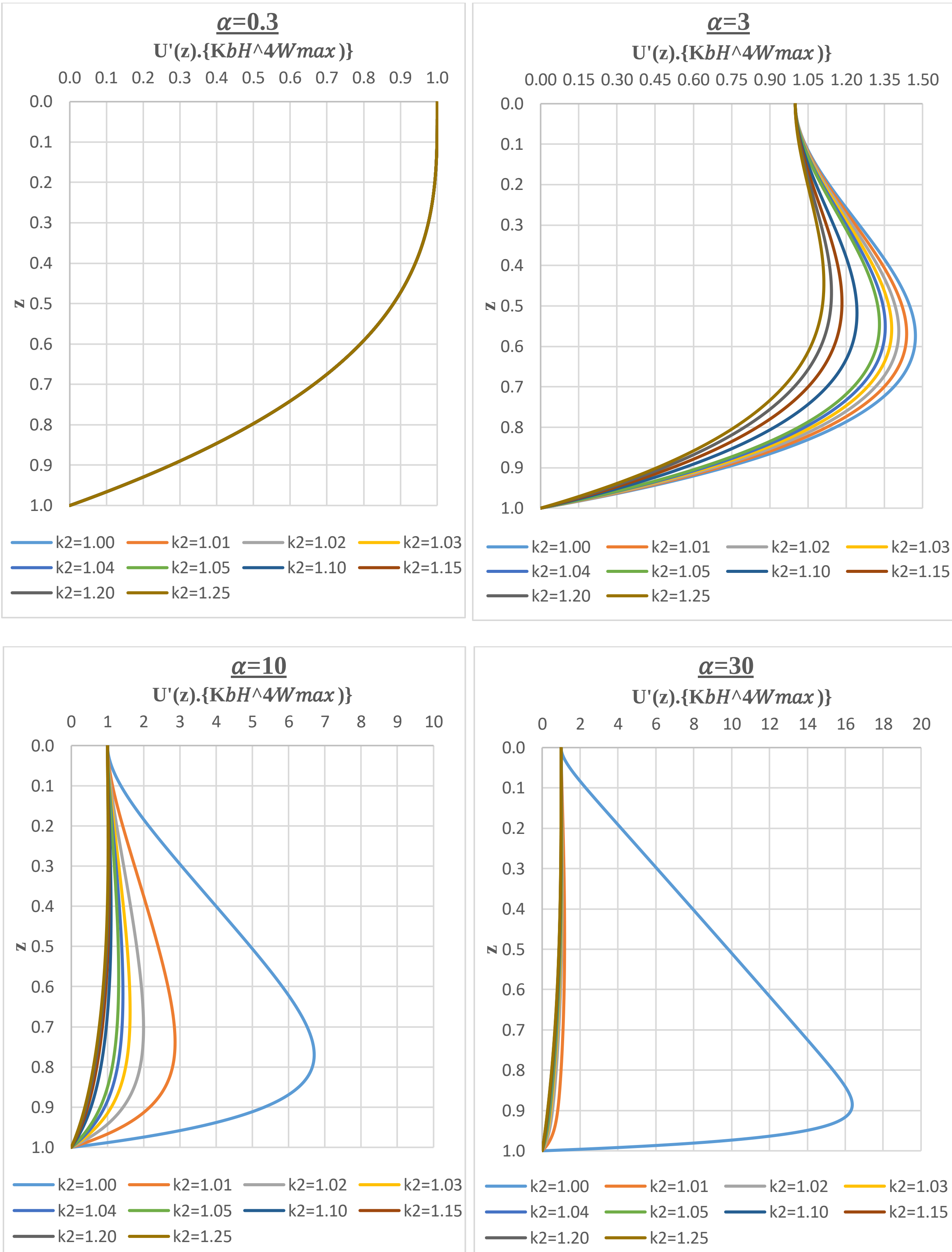


Figure 56. Normalized interstory drift and effect of parameter κ for $\alpha = 2000$.

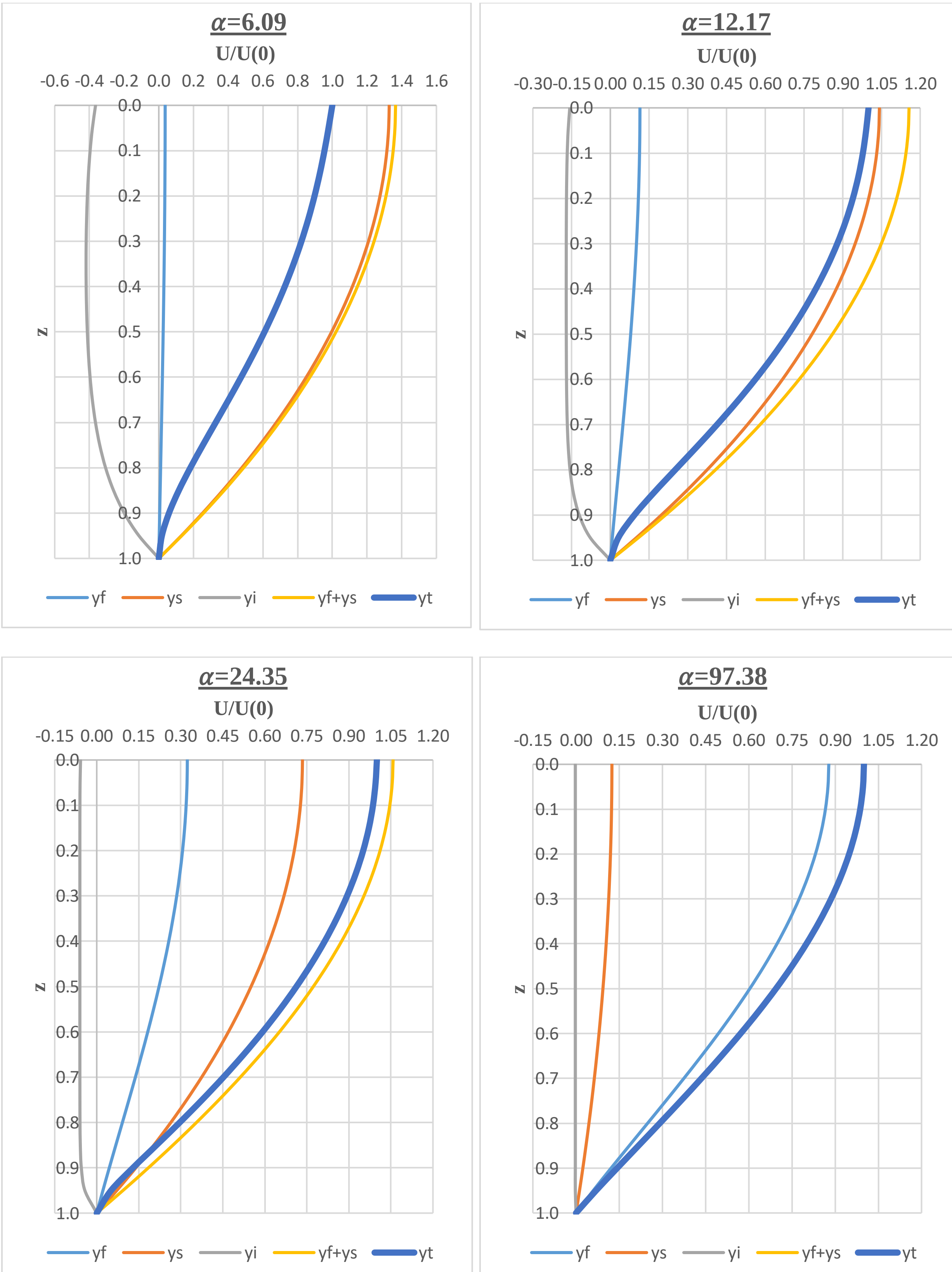


Figure 57. Displacement types (bending, shear and interaction) for $k = 1.00148$.

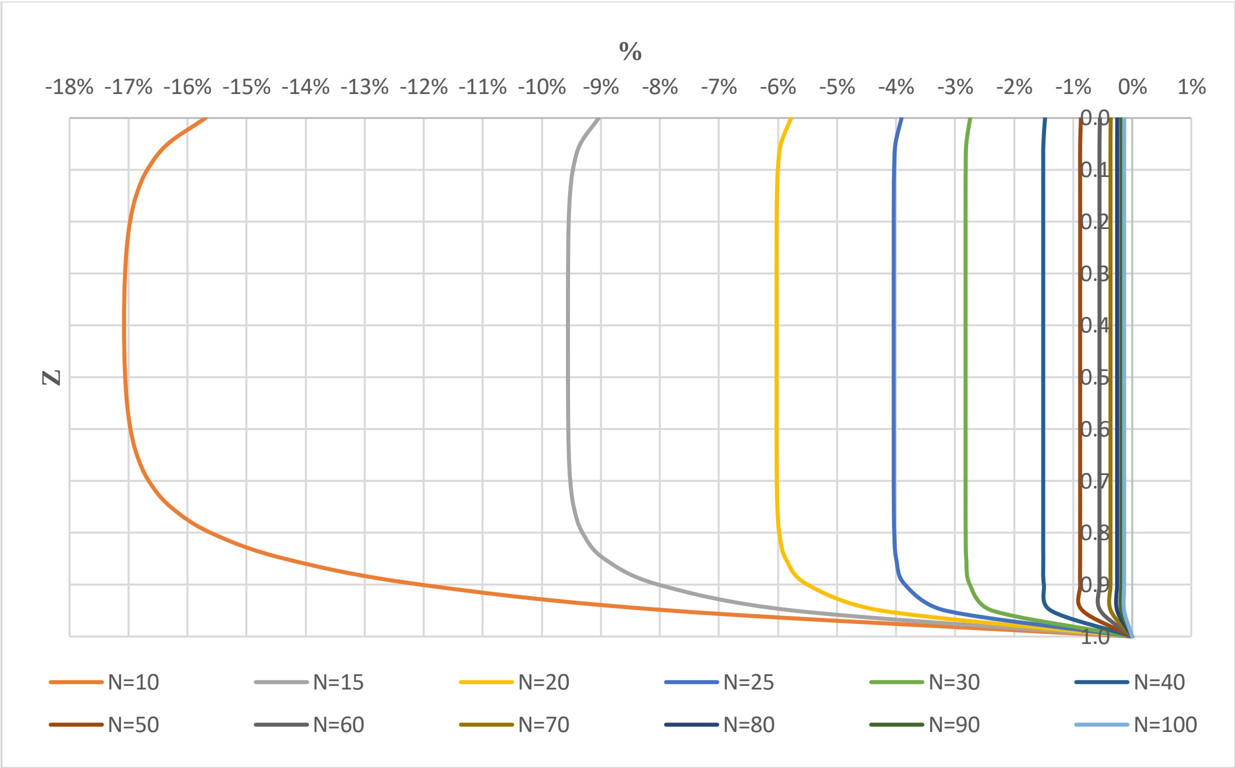


Figure 58. Percentage share of interaction deflection with respect to total displacement.

According to the analysis of equations and graphs:

- ✓ Displacement, interstory drift and global drift result from the sum of the contribution from bending (local bending + global bending), from shear and from the interaction between bending and shear.
- ✓ The lateral stiffness increases due to the interaction effect, concluding that the stiffness of the system is greater than the sum of the individual stiffnesses from bending and shear. The interaction effect is beneficial because it reduces the total displacement of the structure.
- ✓ The parameter α conditions the lateral displacement profile and the interstory drift profile; that is, the parameter α determines the predominant type of behavior in the beam. For a value of $\alpha = 0.3$, it shows a pure bending behavior; a value of $\alpha = 3$, shows a behavior intermediate between shear and bending; and a value of $\alpha = 30$, shows a behavior tending to pure shear.
- ✓ As the value of parameter α decreases the influence of parameter a on the normalized lateral displacement profile becomes less and less, and for a value of $\alpha = 0.3$, the

normalized lateral displacement profile is practically independent of parameter α, as is the case in the flexural beam.

- ✓ The normalized lateral displacement profile and the normalized interstory drift profile are dependent on the parameter α. This dependence decreases with decreasing value of parameter α.
- ✓ The lateral displacement profile and the interstory drift profile are dependent on the parameter κ. This dependence decreases with decreasing value of parameter α.
- ✓ For small values of α (few floors), with respect to the total displacement the contribution of interaction is important, the contribution of bending is negligible and the contribution of shear dominates the behavior. As the value of α increases; the interaction becomes increasingly insignificant, the effect of bending becomes increasingly important, and the effect of shear is drastically reduced.
- ✓ It would seem that ignoring the interaction is acceptable since it considerably reduces the calculations and as α increases it becomes increasingly negligible, but it should be noted that it is only acceptable for intermediate to high values of α, for low-story structures ignoring the interaction is very conservative.
- ✓ As the value of α increases, the profile of the interaction is approximately constant and its contribution decreases considerably.
- ✓ The effect of the interaction is dependent on the type of dominant behavior. When the dominant behavior is practically pure shear the interaction is considerable, as bending begins to dominate the behavior the interaction decreases to a practically negligible value for pure bending values.

- **Parametric Analysis**

The inflection point in the deflection curve of the portal frame subjected to lateral loads is a key parameter in defining whether shear or bending behavior is dominant. The portion of the equivalent column below the inflection point represents the bending behavior and the upper portion represents the shear behavior. The dominant behavior is defined by the dimensionless parameters α and k^2.

The inflection point, which coincides with the maximum drift level $dy\,/\,dz$, is calculated by equating the curvature of the portal deflection to zero. The location of the inflection point is found by iterative analysis, taking into account equation:

$$2C_2 + 6C_3 z + C_4(\alpha\kappa)^2\cosh(\alpha\kappa z) + C_5(\alpha\kappa)^2\sinh(\alpha\kappa z) + \frac{\lambda(\kappa^2-1)}{2\kappa^2}z^2 + \frac{\lambda[a^2(\kappa^2-1)-a^2]}{a^2[a^2-(\alpha\kappa)^2]}e^{-a+az} = 0 \tag{225}$$

For the particular case of a uniformly distributed lateral load:

$$\frac{\kappa^2-1}{2\kappa^2}z^2 - \frac{1}{2\kappa^4\alpha^2}(1-z^2) + \frac{1}{\kappa^2}\frac{\cosh(\alpha\kappa z-\alpha\kappa)+(\alpha\kappa)\sinh(\alpha\kappa z)}{(\alpha\kappa)^2\cosh(\alpha\kappa)} = 0 \tag{226}$$

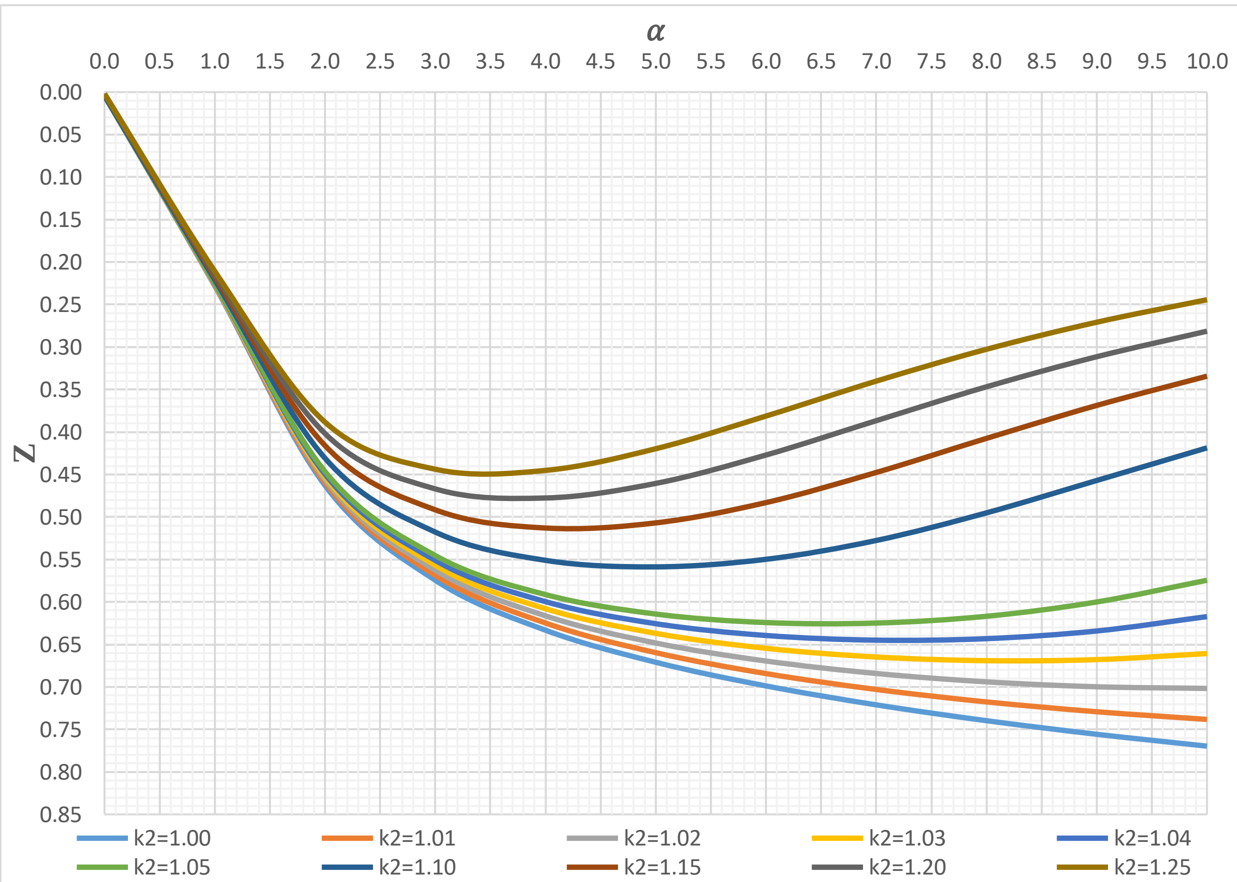


Figure 59. Location of the inflection point for $\alpha H < 10$.

With respect to the curve in Figure 59, the behavior of the sandwich beam can be divided into two categories according to the value of α:

a) Values of $\alpha < 1.5$: The inflection point is practically independent of the parameterκ^2, i.e., the axial deformations of the vertical elements do not significantly influence the calculation of the inflection point. Its location is approximately in the upper third of the height of the sandwich beam and therefore it behaves predominantly in bending.

b) Values of $1.5 < \alpha < 10$: It is influenced by the shear stiffness and axial deformations of the vertical elements. For the range from 1.5 to 4, the increase of shear stiffness (increase of α) decreases the inflection point increasing the shear influence, on the contrary, for the range from 4 to 10, the increase of shear stiffness and k^2 increases the inflection point if there is axial deformation, increasing the bending influence of the sandwich beam. Their location is generally between 0.25 and 0.70 of the height. It is therefore possible that sandwich beams in this range represent an apparently balanced behavior, where the interaction between flexural and shear behavior has a high degree of influence.

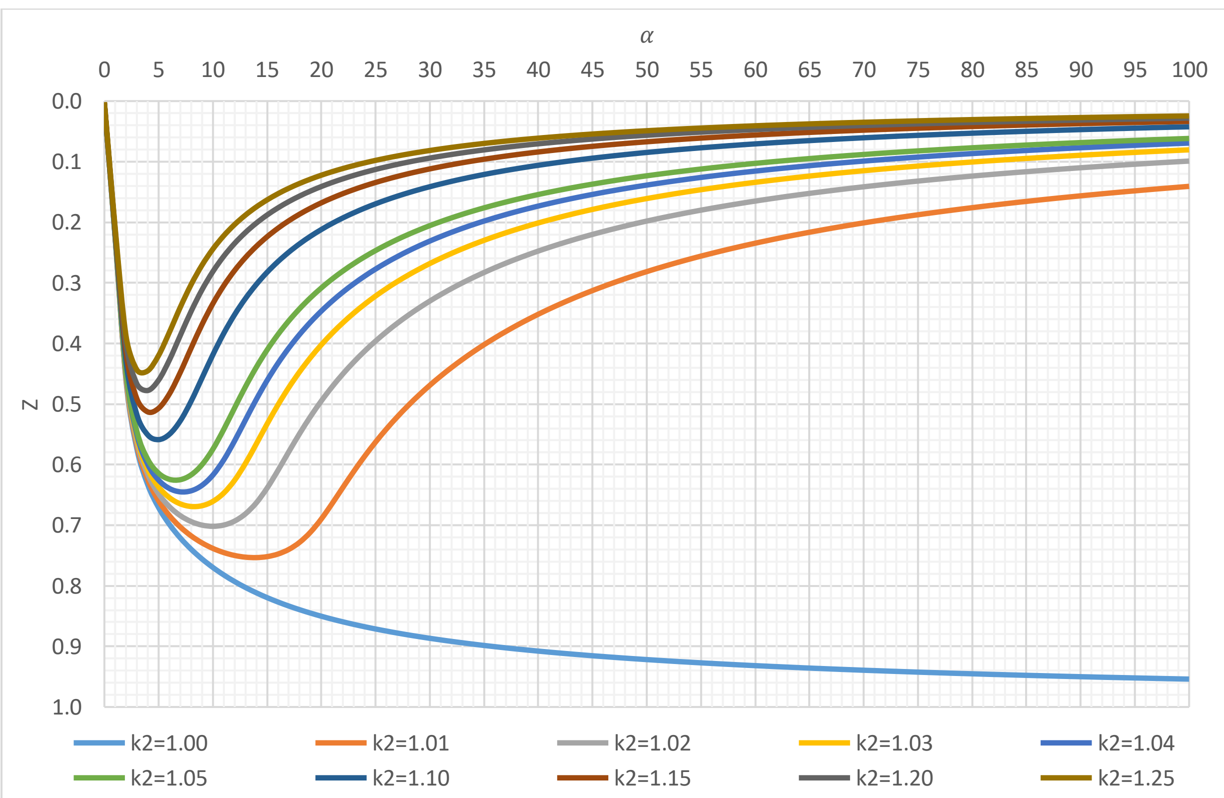


Figure 60. Location of the inflection point for $10 < \alpha H < 100$.

With respect to the curves in Figures 59 and 60, the behavior of the sandwich beam can be divided into four categories according to the value of α:

a) Values of $\alpha < 1.5$: The inflection point is practically independent of the parameter k^2, i.e., the axial deformations of the vertical elements do not significantly influence the calculation of the inflection point. Its location is approximately in the upper third of the height of the sandwich beam and therefore it behaves predominantly in bending.

b) Values of $1.5 < \alpha < 10$: It is influenced by the shear stiffness and axial deformations of the vertical elements. For the range from 1.5 to 4, the increase of shear stiffness (increase of α) decreases the inflection point increasing the shear influence, on the contrary, for the range from 4 to 10, the increase of shear stiffness and k^2 increases the inflection point if there is axial deformation, increasing the flexural influence of the sandwich beam. Their location is generally between 0.25 and 0.70 of the height. It is therefore possible that sandwich beams in this range represent an apparently balanced behavior, where the interaction between flexural and shear behavior has a high degree of influence.

c) Values of $10 < \alpha < 40$: The location of the inflection point increases with α and k^2. When k^2=0, the sandwich beam behaves predominantly in shear, but as k^2increases, the sandwich beam behaves predominantly in flexure.

d) Values of $\alpha > 40$: The location of the inflection point stabilizes and is practically no longer influenced by the increase in shear stiffness (increase of αH). The sandwich beam behaves predominantly in bending because the increase of k^2 (increased axial deflections of the columns) further raises the inflection point, from 0.80 for k^2=1.005 to approximately 0.98 ≈ 1 for $k^2 = 1.25$.

4.1.6.2 Case 2

- **Calculation of the Transfer Matrix**

According to the coupled differential equations and assuming that the external loads act on the floors and not along the floor height, it is possible to write it as follows:

$$\begin{Bmatrix} K_{s1}\left[\theta_{(z)} - u'_{(z)}\right] - K_{b1}\theta''_{(z)} = 0 \\ K_{s1}\left[\theta'_{(z)} - u''_{(z)}\right] + K_{b2}u''''_{(z)} = 0 \end{Bmatrix} \tag{227}$$

The expression for $u_{(z)}$ and $\theta_{(z)}$ is proposed:

$$\begin{Bmatrix} u_{(z)} = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + C_4 \cosh(\alpha^* \kappa z) + C_5 \sinh(\alpha^* \kappa z) \\ \theta_{(z)} = C_6 + C_7 z + C_8 z^2 + C_9 \cosh(\alpha^* \kappa z) + C_{10} \sinh(\alpha^* \kappa z) \end{Bmatrix} \tag{228}$$

Where:

$$\left\{ \alpha^* = \sqrt{\frac{K_{s1}}{K_{b2}}}, \kappa = \sqrt{1 + \frac{K_{b2}}{K_{b1}}} \right\} \tag{229}$$

Expressing the coefficients of the function $\theta_{(z)}$ as a function of the coefficients of $u_{(z)}$:

$$\begin{aligned} \theta_{(z)} = C_1 + (2z)C_2 + \left[3z^2 + \frac{6}{\alpha^{*2}(\kappa^2 - 1)}\right] C_3 - [\alpha^* \kappa(\kappa^2 - 1) \sinh(\alpha^* \kappa z)]C_4 \\ - [\alpha^* \kappa(\kappa^2 - 1) \cosh(\alpha^* \kappa z)]C_5 \end{aligned} \tag{230}$$

Internal forces such as bending moment and shear force associated with lateral displacement result:

$$\left\{ \begin{array}{c} \begin{Bmatrix} M_{\mathrm{l}(z)} = K_{b1}\theta'_{(z)} = (2K_{b1})C_2 + (6K_{b1}z)C_3 - [(\alpha^*\kappa)^2(\kappa^2 - 1) \cosh(\alpha^* \kappa z)\, K_{b1}]C_4 \\ -[(\alpha^*\kappa)^2(\kappa^2 - 1) \sinh(\alpha^* \kappa z)\, K_{b1}]C_5 \end{Bmatrix} \\ \begin{Bmatrix} M_{\mathrm{r}(z)} = K_{b2}u''_{(x)} = (2K_{b2})C_2 + (6K_{b2}z)C_3 + [(\alpha^*\kappa)^2 \cosh(\alpha^* \kappa z)\, K_{b2}]C_4 \\ +[(\alpha^*\kappa)^2 \sinh(\alpha^* \kappa z) K_{b2}]C_5 \end{Bmatrix} \\ V_{(z)} = K_{s1}\left[\theta_{(x)} - u'_{(x)}\right] + K_{b2}u'''_{(x)} = (6K_{b1} + 6K_{b2})C_3 \end{array} \right\} \tag{231}$$

Writing the equations in matrix form:

$$\begin{Bmatrix} u_i(z_i) \\ u_i'(z_i) \\ \theta_i(z_i) \\ M_{\mathrm{l}}(z_i) \\ M_{\mathrm{r}}(z_i) \\ V_i(z_i) \end{Bmatrix} = K_i(z_i) \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{Bmatrix} \tag{232}$$

Where:

$$K_i(z_i) = \begin{bmatrix} 1 & z & z^2 & z^3 & \cosh(\alpha^*\kappa z) & \sinh(\alpha^*\kappa z) \\ 0 & 1 & 2z & 3z^2 & (\alpha^*\kappa)\sinh(\alpha^*\kappa z) & (\alpha^*\kappa)\cosh(\alpha^*\kappa z) \\ 0 & 1 & 2z & 3z^2 + \dfrac{6}{\alpha^{*2}(\kappa^2-1)} & -\alpha^*\kappa(\kappa^2-1)\sinh(\alpha^*\kappa z) & -\alpha^*\kappa(\kappa^2-1)\cosh(\alpha^*\kappa z) \\ 0 & 0 & 2K_{b1} & 6K_{b1}z & -(\alpha^*\kappa)^2(\kappa^2-1)\cosh(\alpha^*\kappa z)\,K_{b1} & -(\alpha^*\kappa)^2(\kappa^2-1)\sinh(\alpha^*\kappa z)\,K_{b1} \\ 0 & 0 & 2K_{b2} & 6K_{b2}z & (\alpha^*\kappa)^2\cosh(\alpha^*\kappa z)\,K_{b2} & (\alpha^*\kappa)^2\sinh(\alpha^*\kappa z)K_{b2} \\ 0 & 0 & 0 & 6K_{b1}+6K_{b2} & 0 & 0 \end{bmatrix} \tag{233}$$

- **Static Analysis Under Static Point Loads Applied at Floor Level**

The relationship between forces and displacements between two consecutive floors is obtained by taking into account the transfer matrix and the vector of external point forces.

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ \theta_n(0) \\ M_{\mathrm{ln}}(0) \\ M_{\mathrm{rn}}(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_k(0) \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ \theta_1(h_1) \\ M_{\mathrm{l}1}(h_1) \\ M_{\mathrm{r}1}(h_1) \\ V_1(h_1) \end{Bmatrix} - \sum_{s=0}^{n}\left[\prod_{k=s+1}^{n} T_k(0)\right] F_s - F_n \tag{234}$$

Expressing it in simplified form:

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ \theta_n(0) \\ M_{\mathrm{ln}}(0) \\ M_{\mathrm{rn}}(0) \\ V_n(0) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ \theta_1(h_1) \\ M_{\mathrm{l}1}(h_1) \\ M_{\mathrm{r}1}(h_1) \\ V_1(h_1) \end{Bmatrix} + f \tag{235}$$

Where:

$$\begin{Bmatrix} \mathrm{t} = \displaystyle\prod_{k=1}^{n} T_k(0) \\ f = -\displaystyle\sum_{s=0}^{n}\left[\prod_{k=s+1}^{n} T_k(0)\right] F_s - F_n \end{Bmatrix} \tag{236}$$

This equation expresses the relationship between the part forces and displacements of the top and the base of the beam. An important point to note is that the size of the transfer matrix is 6x6 and remains constant for all floors.Según las condiciones de contorno definidas en el caso 1:

$$\left\{\begin{matrix} u_{(1)}=0 \\ u'_{(1)}=0 \\ \theta_{(1)}=0 \\ \theta'_{(0)}=0 \\ u''_{(0)}=0 \\ K_{s1}\left[\theta_{(0)}-u'_{(0)}\right]+K_{b2}u'''_{(0)}=0 \end{matrix}\right\} \rightarrow \left\{\begin{matrix} u_1(h_1)=0 \\ u'_1(h_1)=0 \\ \theta_1(h_1)=0 \\ M_{ln}(0)=0 \\ M_{rn}(0)=0 \\ V_n(0)=0 \end{matrix}\right\} \tag{237}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ u'_n(0) \\ \theta_n(0) \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & t_{1,5} & t_{1,6} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} & t_{2,5} & t_{2,6} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} & t_{3,5} & t_{3,6} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,1} & t_{5,2} & t_{5,3} & t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,1} & t_{6,2} & t_{6,3} & t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ M_{l1}(h_1) \\ M_{r1}(h_1) \\ V_1(h_1) \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{Bmatrix} \tag{238}$$

By clearing the bending moment and the shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} M_{l1}(h_1) \\ M_{r1}(h_1) \\ V_1(h_1) \end{Bmatrix} + \begin{Bmatrix} f_4 \\ f_5 \\ f_6 \end{Bmatrix} \tag{239}$$

Clearing:

$$\begin{Bmatrix} M_{11}(h_1) \\ M_{21}(h_1) \\ V_1(h_1) \end{Bmatrix} = -\begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix}^{-1} \begin{Bmatrix} f_4 \\ f_5 \\ f_6 \end{Bmatrix} \tag{240}$$

Substituting the internal forces we obtain the displacement, its derivative and the rotation at the top of the beam:

$$\begin{Bmatrix} u_n(0) \\ u'_n(0) \\ \theta_n(0) \end{Bmatrix} = -\begin{bmatrix} t_{1,4} & t_{1,5} & t_{1,6} \\ t_{2,4} & t_{2,5} & t_{2,6} \\ t_{3,4} & t_{3,5} & t_{3,6} \end{bmatrix} \begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix}^{-1} \begin{Bmatrix} f_4 \\ f_5 \\ f_6 \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} \tag{241}$$

4.1.7 Generalized Sandwich Beam of Three-Field (GSB1)

The Generalized Sandwich Beam (GSB1) is presented, which considers that the structure consists of a parallel coupling of two Timoshenko Beams (TB) joined by means of axially rigid members that only transmit horizontal forces and do not deform. This beam (GSB), results from the generalization of the Sandwich Beam (SWB) by including an additional rotational kinematic field, where the shear deformation in stiffer elements such as shear walls will be taken into account. Taking into account this new kinematic field and the introduction of shear stiffness, this beam (GSB1) can be used as a general replacement beam for all structural elements such as shear walls, coupled shear walls and portal frames.

Bozdoğan (2010) using the GSB1 beam adapted the transfer matrix method to static, dynamic and stability analysis of tall buildings using the continuous method. Moghadasi (2015) using the GSB1 beam and the one-dimensional finite element method offered solutions for static analysis, undamped free vibration analysis and classical damped analysis of tall buildings.

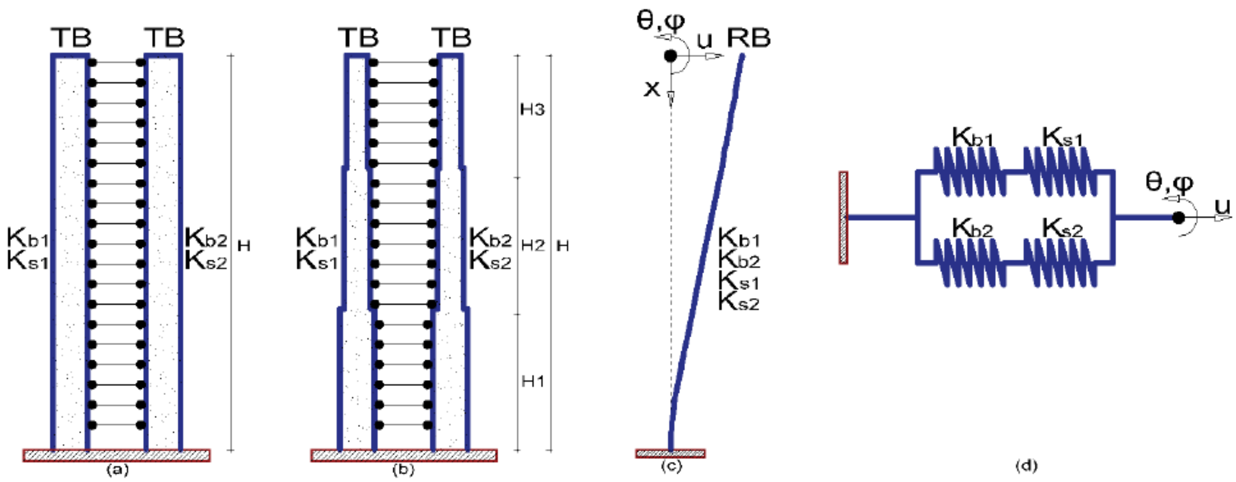


Figure 61. Generalized Sandwich Beam (GSB1) of three fields. a) Case 1, b) Case 2, c) Equivalent RB and d) Idealization of GSB1 stiffness.

The model has three kinematic fields, a single transverse motion u and a different rotation field (θ and ψ) in each timoshenko beam (TB) neglecting the axial extensibility of the TBs. In the model; K_{b1} and K_{s1} are the bending and shear stiffnesses in the left TB, while K_{b2} and K_{s2} are the bending and shear stiffnesses in the right TB. It should be noted that the value of K_{s1} may be negligible (for practical cases) with respect to K_{s2}; because the columns turn out to be thin enough.

4.1.7.1 Case 1

The potential energy of the three-field GSB1 model is expressed as:

$$V = \frac{1}{2}\int_0^H \left\{K_{b1}{\psi'}^2 + K_{s1}\left[u'_{(x)} - \psi_{(x)}\right]^2 + K_{b2}{\theta'_{(x)}}^2 + K_{s2}\left[u'_{(x)} - \theta_{(x)}\right]^2\right\} dx \tag{242}$$

- Coupled shear wall:

$$\begin{Bmatrix} K_{b1} = \sum_{i=1}^{w} EA_{wi}c_i^2\,, K_{b2} = r\sum_{i=1}^{w} EI_{wi}\,, K_{s1} = \left(K_b^{-1} + K_w^{-1}\right)^{-1}, K_{s2} = \sum_{i=1}^{w} GA_{w,i} \\ K_b = \sum_{i=1}^{b} \frac{6EI_{b,i}[(l^* + S_1)^2 + (l^* + S_2)^2]}{l^{*3}h\left(1 + 12\,\frac{kEI_{b,i}}{l^{*2}GA_{b,i}}\right)}\,, K_w = \sum_{i=1}^{w}\frac{12EI_w}{h^2}, r = \frac{K_c}{K_c + K_b} \end{Bmatrix} \tag{243}$$

- Frame:

$$\begin{Bmatrix} K_{b1} = \sum_{i=1}^{c} EA_{c,i}c_i^2\,, K_{b2} = \sum_{i=1}^{c} rEI_{c,i}\,, K_{s1} = \left(K_b^{-1} + K_c^{-1}\right)^{-1}, K_{s2} = \sum_{i=1}^{c} GA_{c,i} \\ K_{s2} = \sum_{i=1}^{c} GA_{ci}\,, K_b = \sum_{i=1}^{b}\frac{12EI_{b,i}}{lh}, K_c = \sum_{i=1}^{c}\frac{12EI_{c,i}}{h^2}, r = \frac{K_c}{K_c + K_b} \end{Bmatrix} \tag{244}$$

- Dual (frame + shear wall):

$$\begin{Bmatrix} K_{b1} = \sum_{i=1}^{c} EA_{c,i}c_i^2\,, K_{b2} = \sum_{i=1}^{c} rEI_{c,i} + \sum_{i=1}^{w} rEI_{w,i}\,, K_{s1} = (K_b^{-1} + K_c^{-1})^{-1}, K_{s2} = \sum_{i=1}^{w} GA_{w,i} \\ K_b = \sum_{i=1}^{b}\frac{12EI_{b,i}}{lh}\ ;\ K_c = \sum_{i=1}^{c}\frac{12EI_{c,i}}{h^2}\ ;\ r = \frac{K_c}{K_c + K_b} \end{Bmatrix} \tag{245}$$

The work done by the external force is:

$$W = \int_0^H f_{(x)}u_{(x)}\,dx \tag{246}$$

Consequently, the total potential energy of the three-field GSB1 beam subjected to a general lateral load distribution is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H \left\{K_{b1}{\psi'_{(x)}}^2 + K_{s1}\left[u'_{(x)} - \psi_{(x)}\right]^2 + K_{b2}{\theta'_{(x)}}^2 + K_{s2}\left[u'_{(x)} - \theta_{(x)}\right]^2\right\} dx - \int_0^H f_{(x)} u_{(x)}\, dx \qquad (247)$$

Closed-form solutions of the model on which a transverse load acts are achieved by solving the differential system arising from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\begin{aligned}\delta\mathcal{U} = \int_0^H \{&K_{b1}\psi'_{(x)}\delta\psi'_{(x)} + K_{s1}\left[u'_{(x)} - \psi_{(x)}\right]\left[\delta u'_{(x)} - \delta\psi_{(x)}\right] + K_{b2}\theta'_{(x)}\delta\theta'_{(x)} \\ &+ K_{s2}\left[u'_{(x)} - \theta_{(x)}\right]\left[\delta u'_{(x)} - \delta\theta_{(x)}\right]\}dx - \int_0^H f(x)\delta u_{(x)}\, dx \\ &- \int_0^H u_{(x)}\, \delta f(x) dx\end{aligned} \qquad (248)$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\begin{aligned}\delta\mathcal{U} = &\left[K_{b1}\psi'_{(x)}\delta\psi_{(x)}\right]_0^H + \left\{K_{s1}\left[u'_{(x)} - \psi_{(x)}\right] + K_{s2}\left[u'_{(x)} - \theta_{(x)}\right]\right\}\delta {u_{(x)}}_0^H + \left[K_{b2}\theta'_{(x)}\delta\theta_{(x)}\right]_0^H \\ &- \int_0^H \left\{K_{b1}\psi''_{(x)} + K_{s1}\left[u'_{(x)} - \psi_{(x)}\right]\right\}\delta\theta_{(x)} \\ &- \int_0^H \left[K_{s1}\left(u''_{(x)} - \psi'_{(x)}\right) + K_{s2}\left(u''_{(x)} - \theta'_{(x)}\right) + f(x)\right]\delta u_{(x)} \\ &- \int_0^H \left\{K_{b2}\theta''_{(x)} + K_{s2}\left[u'_{(x)} - \theta_{(x)}\right]\right\}\delta\psi_{(x)} - \int_0^H u_{(x)}\, \delta f(x) dx\end{aligned} \qquad (249)$$

Equating the terms to zero results in the following equations:

$$\left\{\begin{matrix} K_{b1}\psi''_{(x)} + K_{s1}\left[u'_{(x)} - \psi_{(x)}\right] = 0 \\ K_{b2}\theta''_{(x)} + K_{s2}\left[u'_{(x)} - \theta_{(x)}\right] = 0 \\ K_{s1}\left(u''_{(x)} - \psi'_{(x)}\right) + K_{s2}\left(u''_{(x)} - \theta'_{(x)}\right) + f(x) = 0 \end{matrix}\right\} \qquad (250)$$

And boundary conditions:

$$\left\{\begin{matrix} \psi'_{(0)} = 0 \\ \theta'_{(0)} = 0 \\ (K_{s1} + K_{s2})u'_{(0)} - K_{s1}\psi_{(0)} - K_{s2}\theta_{(0)} = 0 \end{matrix}\right\} \qquad (251)$$

Using the method of differential operators for the solution of the system of equations:

$$\begin{Bmatrix} u_{(x)} \\ \theta_{(x)} \\ \psi_{(x)} \end{Bmatrix} = -\begin{bmatrix} K_{s1}D & 0 & K_{b1}D^2 - K_{s1} \\ K_{s2}D & K_{b2}D^2 - K_{s2} & 0 \\ (K_{s1}+K_{s2})D^2 & -K_{s2}D & -K_{s1}D \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 0 \\ f_{(x)} \end{Bmatrix} \tag{252}$$

i.e.,

$$\begin{Bmatrix} u''''''_{(x)} - \dfrac{K_{s1}K_{s2}(K_{b1}+K_{b2})}{K_{b1}K_{b2}(K_{s1}+K_{s2})}u''''_{(x)} \\ \theta''''''_{(x)} - \dfrac{K_{s1}K_{s2}(K_{b1}+K_{b2})}{K_{b1}K_{b2}(K_{s1}+K_{s2})}\theta''''_{(x)} \\ \psi''''''_{(x)} - \dfrac{K_{s1}K_{s2}(K_{b1}+K_{b2})}{K_{b1}K_{b2}(K_{s1}+K_{s2})}\psi''''_{(x)} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{1}{(K_{s1}+K_{s2})}f''''_{(x)} + \dfrac{K_{b1}K_{s2}+K_{s1}K_{b2}}{K_{b1}K_{b2}(K_{s1}+K_{s2})}f''_{(x)} - \dfrac{K_{s1}K_{s2}}{K_{b1}K_{b2}(K_{s1}+K_{s2})}f_{(x)} \\ \dfrac{K_{s1}}{K_{b1}(K_{s1}+K_{s2})}f'''_{(x)} - \dfrac{K_{s1}K_{s2}}{K_{b1}K_{b2}(K_{s1}+K_{s2})}f'_{(x)} \\ \dfrac{K_{s1}}{K_{b1}(K_{s1}+K_{s2})}f'''_{(x)} - \dfrac{K_{s1}K_{s2}}{K_{b1}K_{b2}(K_{s1}+K_{s2})}f'_{(x)} \end{Bmatrix} \tag{253}$$

A sixth order differential equation is obtained. Normalizing the differential equation by the variable $z = x/H$:

$$u''''''_{(z)} - \frac{K_{s1}K_{s2}(K_{b1}+K_{b2})}{K_{b1}K_{b2}(K_{s1}+K_{s2})}H^2u''''_{(z)} = -\frac{H^2}{(K_{s1}+K_{s2})}f''''_{(z)} + \frac{K_{b1}K_{s2}+K_{s1}K_{b2}}{K_{b1}K_{b2}(K_{s1}+K_{s2})}H^4f''_{(z)} - \frac{K_{s1}K_{s2}}{K_{b1}K_{b2}(K_{s1}+K_{s2})}H^6f_{(z)} \tag{254}$$

Defining five parameters:

$$\begin{Bmatrix} \alpha = H\sqrt{\dfrac{K_{s1}K_{s2}}{K_{b2}(K_{s1}+K_{s2})}}, \kappa = \sqrt{1+\dfrac{K_{b2}}{K_{b1}}}, \eta_\varphi = H\sqrt{\dfrac{K_{s1}}{K_{b1}}}, \eta_\theta = H\sqrt{\dfrac{K_{s2}}{K_{b2}}} \\ \lambda = \dfrac{W_{max}H^2}{(K_{s1}+K_{s2})(1-e^{-a})} \end{Bmatrix} \tag{255}$$

Replacing the first four parameters:

$$u''''''_{(z)} - (\alpha\kappa)^2u''''_{(z)} = \frac{H^2}{K_{s1}+K_{s2}}\left[-f''''_{(z)} + \left(\eta_\varphi^2+\eta_\theta^2\right)f''_{(z)} - \left(\eta_\varphi^2\eta_\theta^2\right)f_{(z)}\right] \tag{256}$$

Assuming a general lateral load (Miranda E. , 1999):

$$f_{(x)} = \frac{W_{max}}{1-e^{-a}}\left(1 - e^{-a+a\frac{x}{H}}\right) \rightarrow f_{(z)} = \frac{W_{max}}{1-e^{-a}}(1 - e^{-a+az}) \tag{257}$$

Replacing the lateral load and introducing the fifth parameter:

$$u_{(z)}'''''' - (\alpha\kappa)^2 u_{(z)}'''' = \frac{\eta_\varphi^2 \eta_\theta^2}{24(\alpha\kappa)^2}\lambda + \frac{a^4 - \left(\eta_\varphi^2 + \eta_\theta^2\right)a^2 + \left(\eta_\varphi^2 \eta_\theta^2\right)}{a^4[a^2 - (\alpha\kappa)^2]}\lambda e^{-a+az} \tag{258}$$

In simplified form:

$$u_{(z)}'''''' - (\alpha\kappa)^2 u_{(z)}'''' = -\eta_\varphi^2 \eta_\theta^2 \lambda + \left[a^4 - \left(\eta_\varphi^2 + \eta_\theta^2\right)a^2 + \left(\eta_\varphi^2 \eta_\theta^2\right)\right]\lambda e^{-a+az} \tag{259}$$

The expression for $u_{(z)}$, $\psi_{(z)}$ and $\theta_{(z)}$ is proposed:

$$\left\{\begin{aligned} u_{(z)} &= C_0 + C_1 z + C_2 z^2 + C_3 z^3 + C_4 \cosh(\alpha\kappa z) + C_5 \sinh(\alpha\kappa z) + C_6 z^4 + C_7 e^{-a+az} \\ H\psi_{(z)} &= C_8 + C_9 z + C_{10} z^2 + C_{11} \cosh(\alpha\kappa z) + C_{12} \sinh(\alpha\kappa z) + C_{13} e^{-a+az} \\ H\theta_{(z)} &= C_{14} + C_{15} z + C_{16} z^2 + C_{17} \cosh(\alpha\kappa z) + C_{18} \sinh(\alpha\kappa z) + C_{19} e^{-a+az} \end{aligned}\right\} \tag{260}$$

Where:

$$\left\{\begin{aligned} C_6 &= \frac{\eta_\varphi^2 \eta_\theta^2}{24(\alpha\kappa)^2} \\ C_7 &= \frac{a^4 - \left(\eta_\varphi^2 + \eta_\theta^2\right)a^2 + \left(\eta_\varphi^2 \eta_\theta^2\right)}{a^4[a^2 - (\alpha\kappa)^2]} \end{aligned}\right\} \tag{261}$$

Expressing the coefficients of $\psi_{(z)}$ and $\theta_{(z)}$ as a function of the coefficients of $u_{(z)}$:

$$\left\{\begin{aligned} &\left\{\begin{aligned} H\psi_{(z)} = C_1 + (2z)C_2 + \left(3z^2 + \frac{6}{\eta_\varphi^2}\right)C_3 - \left[\frac{(\alpha\kappa)\eta_\varphi^2}{(\alpha\kappa)^2 - \eta_\varphi^2}\sinh(\alpha\kappa z)\right]C_4 \\ + \left[\frac{(\alpha\kappa)\eta_\varphi^2}{(\alpha\kappa)^2 - \eta_\varphi^2}\cosh(\alpha\kappa z)\right]C_5 + \left(4z^3 + \frac{24}{\eta_\varphi^2}\right)C_6 - \left[\frac{a\eta_\varphi^2 e^{-a+az}}{a^2 - \eta_\varphi^2}\right]C_7 \end{aligned}\right\} \\ &\left\{\begin{aligned} H\theta_{(z)} = C_1 + (2z)C_2 + \left(3z^2 + \frac{6}{\eta_\theta^2}\right)C_3 - \left[\frac{(\alpha\kappa)\eta_\theta^2}{(\alpha\kappa)^2 - \eta_\theta^2}\sinh(\alpha\kappa z)\right]C_4 \\ + \left[\frac{(\alpha\kappa)\eta_\theta^2}{(\alpha\kappa)^2 - \eta_\theta^2}\cosh(\alpha\kappa z)\right]C_5 + \left(4z^3 + \frac{24}{\eta_\theta^2}\right)C_6 - \left[\frac{a\eta_\theta^2 e^{-a+az}}{a^2 - \eta_\theta^2}\right]C_7 \end{aligned}\right\} \end{aligned}\right\} \tag{262}$$

The constants are obtained by evaluating the relevant boundary conditions (the origin of x is at the base of the model):

$$\begin{Bmatrix} u_{(1)} = 0 \\ \psi_{(1)} = 0 \\ \theta_{(1)} = 0 \\ \psi'_{(0)} = 0 \\ \theta'_{(0)} = 0 \\ (K_{s1} + K_{s2})u'_{(0)} - K_{s1}\psi_{(0)} - K_{s2}\theta_{(0)} = 0 \end{Bmatrix} \tag{263}$$

Constants:

$$\begin{Bmatrix} C_2 \\ C_4 \end{Bmatrix} = \begin{bmatrix} 2 & -\dfrac{(\alpha\kappa)^2\eta_\varphi^2}{(\alpha\kappa)^2 - \eta_\varphi^2} \\ 2 & -\dfrac{(\alpha\kappa)^2\alpha_\theta^2}{(\alpha\kappa)^2 - \alpha_\theta^2} \end{bmatrix}^{-1} \begin{Bmatrix} -\dfrac{\eta_\theta^2}{(\alpha\kappa)^2} + \dfrac{a^2\eta_\varphi^2 e^{-a}}{a^2 - \eta_\varphi^2}\left[a^4 - \left(\eta_\varphi^2 + \eta_\theta^2\right)a^2 + \left(\eta_\varphi^2\eta_\theta^2\right)\right] \\ -\dfrac{\eta_\varphi^2}{(\alpha\kappa)^2} + \dfrac{a^2\eta_\theta^2 e^{-a}}{a^2 - \eta_\theta^2}\left[a^4 - \left(\eta_\varphi^2 + \eta_\theta^2\right)a^2 + \left(\eta_\varphi^2\eta_\theta^2\right)\right] \end{Bmatrix}$$

$$\begin{Bmatrix} C_1 \\ C_3 \\ C_5 \end{Bmatrix} = \begin{bmatrix} 1 & 3 + \dfrac{6}{\eta_\varphi^2} & -\dfrac{(\alpha\kappa)\eta_\varphi^2\cosh(\alpha\kappa)}{(\alpha\kappa)^2 - \eta_\varphi^2} \\ 1 & 3 + \dfrac{6}{\eta_\theta^2} & -\dfrac{(\alpha\kappa)\eta_\theta^2\cosh(\alpha\kappa)}{(\alpha\kappa)^2 - \eta_\theta^2} \\ 0 & -6\left(\dfrac{k_{S1}}{\eta_\varphi^2} + \dfrac{k_{S2}}{\eta_\theta^2}\right) & (\alpha\kappa)^3\left[\dfrac{k_{S1}}{(\alpha\kappa)^2 - \eta_\varphi^2} + \dfrac{k_{S2}}{(\alpha\kappa)^2 - \eta_\theta^2}\right] \end{bmatrix}^{-1} x$$

$$\begin{Bmatrix} -2C_2 + \dfrac{(\alpha\kappa)\eta_\varphi^2\sinh(\alpha\kappa)}{(\alpha\kappa)^2 - \eta_\varphi^2}C_4 - \left(4 + \dfrac{24}{\eta_\varphi^2}\right)C_6 + \dfrac{a\eta_\varphi^2}{a^2 - \eta_\varphi^2}C_7 \\ -2C_2 + \dfrac{(\alpha\kappa)\eta_\theta^2\sinh(\alpha\kappa)}{(\alpha\kappa)^2 - \eta_\theta^2}C_4 - \left(4 + \dfrac{24}{\eta_\theta^2}\right)C_6 + \dfrac{a\eta_\theta^2}{a^2 - \eta_\theta^2}C_7 \\ -\left(\dfrac{k_{S1}}{a^2 - \eta_\varphi^2} + \dfrac{k_{S2}}{a^2 - \eta_\theta^2}\right)a^3 e^{-a}C_7 \end{Bmatrix} \tag{264}$$

For the case of a uniformly distributed lateral load $(a \to \infty)$, the expression of $u_{(z)}$, $\psi_{(z)}$) and $\theta_{(z)}$ result:

$$\begin{Bmatrix} u_{(z)} = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + C_4\cosh(\alpha\kappa z) + C_5\sinh(\alpha\kappa z) + C_6 z^4 \\ \begin{Bmatrix} H\psi_{(z)} = C_1 + (2z)C_2 + \left(3z^2 + \dfrac{6}{\eta_\varphi^2}\right)C_3 - \left[\dfrac{(\alpha\kappa)\eta_\varphi^2}{(\alpha\kappa)^2 - \eta_\varphi^2}\sinh(\alpha\kappa z)\right]C_4 \\ + \left[\dfrac{(\alpha\kappa)\eta_\varphi^2}{(\alpha\kappa)^2 - \eta_\varphi^2}\cosh(\alpha\kappa z)\right]C_5 + \left(4z^3 + \dfrac{24}{\eta_\varphi^2}\right)C_6 \end{Bmatrix} \\ \begin{Bmatrix} H\theta_{(z)} = C_1 + (2z)C_2 + \left(3z^2 + \dfrac{6}{\eta_\theta^2}\right)C_3 - \left[\dfrac{(\alpha\kappa)\eta_\theta^2}{(\alpha\kappa)^2 - \eta_\theta^2}\sinh(\alpha\kappa z)\right]C_4 \\ + \left[\dfrac{(\alpha\kappa)\eta_\theta^2}{(\alpha\kappa)^2 - \eta_\theta^2}\cosh(\alpha\kappa z)\right]C_5 + \left(4z^3 + \dfrac{24}{\eta_\theta^2}\right)C_6 \end{Bmatrix} \end{Bmatrix} \tag{265}$$

Where:

$$\begin{Bmatrix} C_2 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\eta_\theta^4}{(\alpha\kappa)^2 - \alpha_\theta^2} + \dfrac{\eta_\varphi^4}{(\alpha\kappa)^2 - \eta_\varphi^2} \\ \dfrac{2\eta_\theta^2}{(\alpha\kappa)^2} + \dfrac{2\eta_\varphi^2}{(\alpha\kappa)^2} \end{Bmatrix}$$

$$\begin{Bmatrix} C_1 \\ C_3 \\ C_5 \end{Bmatrix} = \begin{bmatrix} 1 & 3 + \dfrac{6}{\eta_\varphi^2} & -\dfrac{(\alpha\kappa)\eta_\varphi^2 \cosh(\alpha\kappa)}{(\alpha\kappa)^2 - \eta_\varphi^2} \\ 1 & 3 + \dfrac{6}{\eta_\theta^2} & -\dfrac{(\alpha\kappa)\eta_\theta^2 \cosh(\alpha\kappa)}{(\alpha\kappa)^2 - \eta_\theta^2} \\ 0 & -6\left(\dfrac{k_{S1}}{\eta_\varphi^2} + \dfrac{k_{S2}}{\eta_\theta^2}\right) & \left[\dfrac{k_{S1}(\alpha\kappa)^3}{(\alpha\kappa)^2 - \eta_\varphi^2} + \dfrac{k_{S2}(\alpha\kappa)^3}{(\alpha\kappa)^2 - \eta_\theta^2}\right] \end{bmatrix}^{-1} \begin{Bmatrix} -2C_2 + \dfrac{(\alpha\kappa)\eta_\varphi^2 \sinh(\alpha\kappa)}{(\alpha\kappa)^2 - \eta_\varphi^2} C_4 - \left(4 + \dfrac{24}{\eta_\varphi^2}\right) C_6 \\ -2C_2 + \dfrac{(\alpha\kappa)\eta_\theta^2 \sinh(\alpha\kappa)}{(\alpha\kappa)^2 - \eta_\theta^2} C_4 - \left(4 + \dfrac{24}{\eta_\theta^2}\right) C_6 \\ 0 \end{Bmatrix}$$

(266)

- **Special Cases**

a) When $\frac{K_{S2}}{K_{S1}} \to \infty$ $\left(\frac{K_{S1}}{K_{S2}} \to 0\right)$. Evaluating the limit of of the differential equation:

$$u_{(z)}'''''' - \left[\left(\frac{K_{s1}}{K_{b1}} + \frac{K_{s1}}{K_{b2}}\right) H^2\right] u_{(z)}'''' = \frac{H^4}{K_{b2}}\left[f_{(z)}'' - H^2 \frac{K_{s1}}{K_{b1}} f_{(z)}\right] \quad (267)$$

The equation shows that if infinite shear stiffness is considered in the structure, the GSB1 beam has the same behavior as the two-field SWB beam. For the case of a uniformly distributed lateral load:

$$u_{(z)} = \frac{W_{max}H^4}{K_{b1} + K_{b2}}\left(\frac{1}{24}z^4 - \frac{1}{6}z + \frac{1}{8}\right) + \frac{1}{2k^4}\frac{W_{max}H^2}{K_{s1}}(1 - z^2)$$
$$-\frac{1}{\kappa^2}\frac{W_{max}H^4}{K_{b2}}\left\{\frac{1 - \cosh(\alpha\kappa z - \alpha\kappa) - (\alpha\kappa)[\sinh(\alpha\kappa z) - \sinh(\alpha\kappa)]}{(\alpha\kappa)^4 \cosh(\alpha\kappa)}\right\} \quad (268)$$

4.1.7.2 Case 2

- **Calculation of the Transfer Matrix**

According to the coupled differential equations assuming that the external loads act on the floors and not along the floor height, it is possible to write it as follows:

$$\begin{Bmatrix} K_{b1}\psi''_{(x)} + K_{s1}\left[u'_{(x)} - \psi_{(x)}\right] = 0 \\ K_{b2}\theta''_{(x)} + K_{s2}\left[u'_{(x)} - \theta_{(x)}\right] = 0 \\ K_{s1}\left(u''_{(x)} - \psi'_{(x)}\right) + K_{s2}\left(u''_{(x)} - \theta'_{(x)}\right) = 0 \end{Bmatrix} \tag{269}$$

The expression for $u_{(z)}$, $\psi_{(z)}$ and $\theta_{(z)}$ is proposed:

$$\begin{Bmatrix} u_{(z)} = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + C_4 \cosh(\alpha^* \kappa z) + C_5 \sinh(\alpha^* \kappa z) \\ \psi_{(z)} = C_6 + C_7 z + C_8 z^2 + C_9 \cosh(\alpha^* \kappa z) + C_{10} \sinh(\alpha^* \kappa z) \\ \theta_{(z)} = C_{11} + C_{12} z + C_{13} z^2 + C_{14} z^3 + C_{15} \cosh(\alpha^* \kappa z) + C_{16} \sinh(\alpha^* \kappa z) \end{Bmatrix} \tag{270}$$

Where:

$$\begin{Bmatrix} \alpha^* = \sqrt{\dfrac{K_{s1}K_{s2}}{K_{b2}(K_{s1}+K_{s2})}}, \kappa = \sqrt{1 + \dfrac{K_{b2}}{K_{b1}}}, \eta_\psi = \sqrt{\dfrac{K_{s1}}{K_{b1}}}\ , \eta_\theta = \sqrt{\dfrac{K_{s2}}{K_{b2}}} \\ r_\psi = \dfrac{(\alpha^*\kappa)\eta_\psi^2}{\eta_\psi^2 - (\alpha^*\kappa)^2}\ , r_\theta = \dfrac{(\alpha^*\kappa)\eta_\theta^2}{\eta_\theta^2 - (\alpha^*\kappa)^2}, r = (\alpha^*\kappa)^3\left[\dfrac{K_{s1}}{\eta_\psi^2 - (\alpha^*\kappa)^2} + \dfrac{K_{s2}}{\eta_\theta^2 - (\alpha^*\kappa)^2}\right] \end{Bmatrix} \tag{271}$$

Expressing the coefficients of $\psi_{(z)}$ and $\theta_{(z)}$ as a function of the coefficients of $u_{(z)}$:

$$\begin{Bmatrix} u_{(z)} = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + C_4 \cosh(\alpha^* \kappa z) + C_5 \sinh(\alpha^* \kappa z) \\ \psi_{(z)} = C_1 + (2z)C_2 + \left(3z^2 + \dfrac{6}{\eta_\psi^2}\right)C_3 + r_\psi \sinh(\alpha^*\kappa z)\, C_4 + r_\psi \cosh(\alpha^*\kappa z)\, C_5 \\ \theta_{(z)} = C_1 + (2z)C_2 + \left(3z^2 + \dfrac{6}{\eta_\theta^2}\right)C_3 + r_\theta \sinh(\alpha^*\kappa z)\, C_4 + r_\theta \cosh(\alpha^*\kappa z)\, C_5 \end{Bmatrix} \tag{272}$$

Internal forces such as bending moment and shear force associated with lateral displacement result:

$$\begin{Bmatrix} M_{\mathrm{l}(z)} = K_{b1}\psi'_{(x)} = (2K_{b1})C_2 + (6K_{b1}z)C_3 + r_\psi(\alpha^*\kappa)K_{b1}\cosh(\alpha^*\kappa z)\, C_4 + r_\psi(\alpha^*\kappa)K_{b1}\sinh(\alpha^*\kappa z)\, C_5 \\ M_{\mathrm{r}(z)} = K_{b2}\theta'_{(x)} = (2K_{b2})C_2 + (6K_{b2}z)C_3 + r_\theta(\alpha^*\kappa)K_{b2}\cosh(\alpha^*\kappa z)\, C_4 + r_\theta(\alpha^*\kappa)K_{b2}\sinh(\alpha^*\kappa z)\, C_5 \\ V_{(z)} = (K_{s1}+K_{s2})u'_{(x)} - K_{s1}\psi_{(x)} - K_{s2}\theta_{(x)} = -6(K_{b1}+K_{b2})C_3 - r\sinh(\alpha^*\kappa z)\, C_4 - r\cosh(\alpha^*\kappa z)\, C_5 \end{Bmatrix} \tag{273}$$

Writing the equations in matrix form:

$$\begin{Bmatrix} u_i(z_i) \\ \psi_i(z_i) \\ \theta_i(z_i) \\ M_{\mathrm{li}}(z_i) \\ M_{\mathrm{ri}}(z_i) \\ V_i(z_i) \end{Bmatrix} = K_i(z_i) \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{Bmatrix} \tag{274}$$

Where:

$$K_i(z_i) = \begin{bmatrix} 1 & z & z^2 & z^3 & \cosh(\alpha^* \kappa z) & \sinh(\alpha^* \kappa z) \\ 0 & 1 & 2z & 3z^2 + \dfrac{6}{\alpha_\psi^2} & r_\psi \sinh(\alpha^* \kappa z) & r_\psi \cosh(\alpha^* \kappa z) \\ 0 & 1 & 2z & 3z^2 + \dfrac{6}{\alpha_\theta^2} & r_\theta \sinh(\alpha^* \kappa z) & r_\theta \cosh(\alpha^* \kappa z) \\ 0 & 0 & 2K_{b1} & 6K_{b1} z & r_\psi (\alpha^* \kappa) K_{b1} \cosh(\alpha^* \kappa z) & r_\psi (\alpha^* \kappa) K_{b1} \sinh(\alpha^* \kappa z) \\ 0 & 0 & 2K_{b2} & 6K_{b2} z & r_\theta (\alpha^* \kappa) K_{b2} \cosh(\alpha^* \kappa z) & r_\theta (\alpha^* \kappa) K_{b2} \sinh(\alpha^* \kappa z) \\ 0 & 0 & 0 & -6(K_{b1} + K_{b2}) & -r \sinh(\alpha^* \kappa z) & -r \cosh(\alpha^* \kappa z) \end{bmatrix} \tag{275}$$

- **Static Analysis Under Static Point Loads Applied at Floor Level**

The relationship between forces and displacements between two consecutive floors is obtained by taking into account the transfer matrix and the vector of external point forces.

$$\begin{Bmatrix} u_n(0) \\ \psi_n(0) \\ \theta_n(0) \\ M_{\mathrm{ln}}(0) \\ M_{\mathrm{rn}}(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_k(0) \begin{Bmatrix} u_1(h_1) \\ \psi_1(h_1) \\ \theta_1(h_1) \\ M_{\mathrm{l1}}(h_1) \\ M_{\mathrm{r1}}(h_1) \\ V_1(h_1) \end{Bmatrix} - \sum_{s=0}^{n} \left[\prod_{k=s+1}^{n} T_k(0) \right] F_s - F_n \tag{276}$$

Expressing it in simplified form:

$$\begin{Bmatrix} u_n(0) \\ \psi_n(0) \\ \theta_n(0) \\ M_{\mathrm{ln}}(0) \\ M_{\mathrm{rn}}(0) \\ V_n(0) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ \psi_1(h_1) \\ \theta_1(h_1) \\ M_{\mathrm{l1}}(h_1) \\ M_{\mathrm{r1}}(h_1) \\ V_1(h_1) \end{Bmatrix} + f \tag{277}$$

Where:

$$\left\{\begin{matrix} \mathrm{t} = \prod_{k=1}^{n} T_k(0) \\ , f = -\sum_{s=0}^{n}\left[\prod_{k=s+1}^{n} T_k(0)\right] F_s - F_n \end{matrix}\right\} \tag{278}$$

This equation expresses the relationship between the part forces and displacements of the top and the base of the beam. An important point to note is that the size of the transfer matrix is 6x6 and remains constant for all floors.

According to the boundary conditions defined in case 1:

$$\left\{\begin{matrix} u_{(1)} = 0 \\ \psi_{(1)} = 0 \\ \theta_{(1)} = 0 \\ \psi'_{(0)} = 0 \\ \theta'_{(0)} = 0 \\ (K_{s1} + K_{s2})u'_{(0)} - K_{s1}\psi_{(0)} - K_{s2}\theta_{(0)} = 0 \end{matrix}\right\} \rightarrow \left\{\begin{matrix} u_1(h_1) = 0 \\ \psi_1(h_1) = 0 \\ \theta_1(h_1) = 0 \\ M_{ln\,(0)} = 0 \\ M_{rn\,(0)} = 0 \\ V_{n\,(0)} = 0 \end{matrix}\right\} \tag{279}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ \psi_n(0) \\ \theta_n(0) \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & t_{1,5} & t_{1,6} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} & t_{2,5} & t_{2,6} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} & t_{3,5} & t_{3,6} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,1} & t_{5,2} & t_{5,3} & t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,1} & t_{6,2} & t_{6,3} & t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ M_{11}(h_1) \\ M_{21}(h_1) \\ V_1(h_1) \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{Bmatrix} \tag{280}$$

By clearing the bending moment and the shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} M_{\mathrm{l}1}(h_1) \\ M_{\mathrm{r}1}(h_1) \\ V_1(h_1) \end{Bmatrix} + \begin{Bmatrix} f_4 \\ f_5 \\ f_6 \end{Bmatrix} \rightarrow \begin{Bmatrix} M_{\mathrm{l}1}(h_1) \\ M_{\mathrm{r}1}(h_1) \\ V_1(h_1) \end{Bmatrix} = -\begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix}^{-1} \begin{Bmatrix} f_4 \\ f_5 \\ f_6 \end{Bmatrix} \tag{281}$$

Substituting the internal forces gives the displacement and rotations at the top of the beam:

$$\begin{Bmatrix} u_n(0) \\ \psi_n(0) \\ \theta_n(0) \end{Bmatrix} = -\begin{bmatrix} t_{1,4} & t_{1,5} & t_{1,6} \\ t_{2,4} & t_{2,5} & t_{2,6} \\ t_{3,4} & t_{3,5} & t_{3,6} \end{bmatrix} \begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix}^{-1} \begin{Bmatrix} f_4 \\ f_5 \\ f_6 \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} \tag{282}$$

4.1.8 Generalized Sandwich Beam of Three-Field (GSB2)

The Generalized Sandwich Beam (GSB2) is presented, which considers that the structure consists of a series coupling of a sandwich beam (SWB) and a shear beam (SB). This beam (GSB2), results from the generalization of the sandwich beam (SWB) by including an additional rotational kinematic field, where the shear deformation in stiffer elements such as shear walls will be taken into account. Taking into account this new kinematic field and the introduction of shear stiffness, this beam (GSB2) can be used as a general replacement beam for all structural elements such as shear walls, coupled shear walls and portal frames.

Chesnais (2010) studied a 16-story shear wall building and concluded that the local shear stiffness of the vertical elements can have the same order of magnitude as the other characteristic stiffnesses of a sandwich beam. To overcome this problem he proposed this novel generalized sandwich replacement beam for the case of dynamic analysis and ignoring the rotational inertia. He performed a comparison of the dynamic analysis of the building using the sandwich beam and the generalized sandwich beam, finding an error reduction of 11% for the sandwich beam and 2% for the generalized sandwich beam with respect to the first fundamental period of the building, thus demonstrating the efficiency of this new replacement beam.

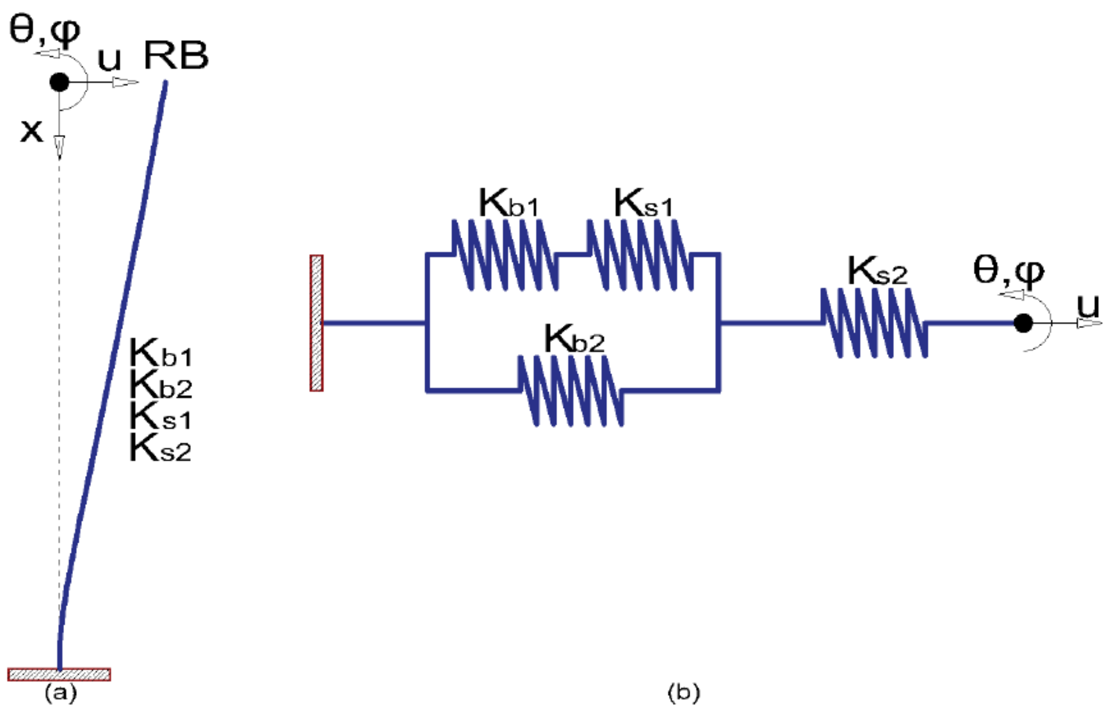
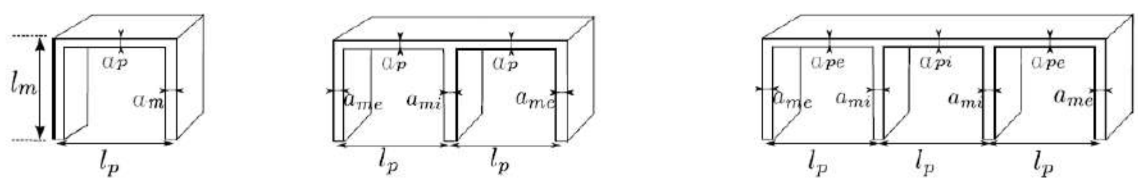


Figure 62. Generalized Sandwich Beam (GSB2) of three fields. a) Case 1, b) Case 2, c) Equivalent RB and d) Idealization of GSB2 stiffness.

The model has three kinematic fields, a single transverse motion u and a different rotation field (θ and ψ). In the model; K_{b1}, K_{s1}, K_{b2} and K_{s2} are the global bending stiffness, global shear stiffness, local bending stiffness and local shear stiffness respectively. It should be noted that the value of K_{s1} may be negligible with respect to K_{s2}; when the vertical elements are thin enough as columns.

Chesnais (2010) applied the HPDM method to one-, two- and three-span structures and calculated analytical expressions for the global shear stiffness (Table 3).

Single frame	Double frame	Triple frame



Single frame	Double frame	Triple frame
$\frac{2k_p k_m}{k_p + 2k_m}$	$\frac{k_p(k_p(2k_{me}+k_{mi})+12k_{me}k_{mi})}{k_p^2+2k_p(2k_{me}+k_{mi})+6k_{me}k_{mi}}$	$\frac{(2k_{mi}+2k_{me})k_{pe}(k_{pe}+2k_{pi})+3(2k_{mi}\,2k_{me}(k_{pi}+2k_{pe}))}{k_{pe}^2+3(2k_{me}(k_{pi}+2k_{mi}))+2k_{pe}(k_{pi}+2k_{me}+2k_{mi})}$
	with $k_j = \frac{12E_j I_j}{l_m l_j}$ and	$j = m, me, mi, p, pe$ or pi
(1.16)	(1.17)	(1.18)

Tabla.3 *Analytical expressions of the global shear stiffness Ks for single, double and triple portal frame structures (Chesnais, 2010).*

4.1.8.1 Case 1

The potential energy of the three-field GSB2 model is expressed as:

$$V = \frac{1}{2}\int_0^H \left\{K_{b1}{\theta'_{(x)}}^2 + K_{s1}\left[\theta_{(x)} - \psi_{(x)}\right]^2 + K_{b2}{\psi'_{(x)}}^2\right\}dx + \frac{1}{2}\int_0^H K_{s2}\left[\psi_{(x)} - u'_{(x)}\right]^2 dx \tag{283}$$

- Coupled shear wall:

$$\begin{cases} K_{b1} = \sum_{i=1}^{w} EA_{wi}c_i^2 \, , K_{b2} = r\sum_{i=1}^{w} EI_{wi} \\ K_{s1} = \sum_{i=1}^{b}\left[\frac{h}{L}\left(\frac{L^2}{12EI_b}+\frac{1}{GA'_b}\right)\right]^{-1} , K_{s2} = \sum_{i=1}^{w}\left[\frac{1}{2}\left(\frac{h^2}{12EI_w}+\frac{1}{GA'_w}\right)\right]^{-1} \end{cases} \tag{284}$$

- Frame:

$$\begin{cases} K_{b1} = \sum_{i=1}^{c} EA_{c,i}c_i^2 \, , K_{b2} = \sum_{i=1}^{c} rEI_{c,i} \\ K_{s1} = \sum_{i=1}^{b}\left[\frac{h}{L}\left(\frac{L^2}{12EI_b}+\frac{1}{GA'_b}\right)\right]^{-1} , K_{s2} = \sum_{i=1}^{c}\left[\frac{1}{2}\left(\frac{h^2}{12EI_c}+\frac{1}{GA'_c}\right)\right]^{-1} \end{cases} \tag{285}$$

- Dual (frame + shear wall):

$$\begin{cases} K_{b1} = \sum_{i=1}^{c} EA_{c,i}c_i^2 \, , K_{b2} = \sum_{i=1}^{c} rEI_{c,i} + \sum_{i=1}^{w} rEI_{w,i} \, , K_{s1} = \sum_{i=1}^{b}\left[\frac{h}{L}\left(\frac{L^2}{12EI_b}+\frac{1}{GA'_b}\right)\right]^{-1} \\ K_{s2} = \sum_{i=1}^{c}\left[\frac{1}{2}\left(\frac{h^2}{12EI_c}+\frac{1}{GA'_c}\right)\right]^{-1} + \sum_{i=1}^{w}\left[\frac{1}{2}\left(\frac{h^2}{12EI_w}+\frac{1}{GA'_w}\right)\right]^{-1} \end{cases} \tag{286}$$

The work done by the external force is:

$$W = \int_0^H f_{(x)}u_{(x)}\,dx \tag{287}$$

Consequently, the total potential energy of the three-field GSB2 beam subjected to a general lateral load distribution is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H \left\{K_{b1}{\theta'_{(x)}}^2 + K_{s1}\left[\theta_{(x)} - \psi_{(x)}\right]^2 + K_{b2}{\psi'_{(x)}}^2 + K_{s2}\left[\psi_{(x)} - u'_{(x)}\right]^2\right\}dx - \int_0^H f_{(x)}u_{(x)}\,dx \tag{288}$$

Closed-form solutions of the model on which a transverse load acts are achieved by solving the differential system arising from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\delta\mathcal{U} = \int_0^H \{K_{b1}\theta'_{(x)}\delta\theta'_{(x)} + K_{s1}[\theta_{(x)} - \psi_{(x)}]\delta\theta_{(x)} - K_{s1}[\theta_{(x)} - \psi_{(x)}]\delta\psi_{(x)} + K_{b2}\psi'_{(x)}\delta\psi'_{(x)}$$
$$+ K_{s2}[\psi_{(x)} - u'_{(x)}]\delta\psi_{(x)} - K_{s2}[\psi_{(x)} - u'_{(x)}]\delta u'_{(x)}\}dx - \int_0^H f(x)\delta u_{(x)}\,dx$$
$$- \int_0^H u_{(x)}\,\delta f(x)dx \tag{289}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\delta\mathcal{U} = -\{K_{s2}[\psi_{(x)} - u'_{(x)}]\delta u_{(x)}\}_0^H + [K_{b2}\psi'_{(x)}\delta\psi_{(x)}]_0^H + [K_{b1}\theta'_{(x)}\delta\theta_{(x)}]_0^H$$
$$- \int_0^H [K_{s2}[u''_{(x)} - \psi'_{(x)}] + f(x)]\,\delta u_{(x)}$$
$$- \int_0^H \{K_{b2}\psi''_{(x)} - (K_{s1} + K_{s2})\psi_{(x)} + K_{s1}\theta_{(x)} + K_{s2}u'_{(x)}\}\delta\psi_{(x)}$$
$$- \int_0^H \{K_{b1}\theta''_{(x)} - K_{s1}[\theta_{(x)} - \psi_{(x)}]\}\delta\theta_{(x)} - \int_0^H u_{(x)}\,\delta f(x)dx \tag{290}$$

Equating the terms to zero results in the following equations:

$$\begin{Bmatrix} K_{s2}[u''_{(x)} - \psi'_{(x)}] + f(x) = 0 \\ K_{b1}\theta''_{(x)} - K_{s1}[\theta_{(x)} - \psi_{(x)}] = 0 \\ K_{b2}\psi''_{(x)} - (K_{s1} + K_{s2})\psi_{(x)} + K_{s1}\theta_{(x)} + K_{s2}u'_{(x)} = 0 \end{Bmatrix} \tag{291}$$

And boundary conditions:

$$\begin{Bmatrix} \theta'_{(0)} = 0 \\ \psi'_{(0)} = 0 \\ \psi_{(0)} - u'_{(0)} = 0 \end{Bmatrix} \tag{292}$$

Using the method of differential operators for the solution of the system of equations:

$$\begin{Bmatrix} u_{(x)} \\ \theta_{(x)} \\ \psi_{(x)} \end{Bmatrix} = -\begin{bmatrix} K_{s2}D^2 & 0 & -K_{s2}D \\ 0 & K_{b1}D^2 - K_{s1} & K_{s1} \\ K_{s2}D & K_{s1} & K_{b2}D^2 - (K_{s1} + K_{s2}) \end{bmatrix}^{-1} \begin{Bmatrix} f_{(x)} \\ 0 \\ 0 \end{Bmatrix} \tag{293}$$

i.e.,

$$\begin{Bmatrix} u_{(x)}^{''''''} - \dfrac{K_{s1}(K_{b1}+K_{b2})}{K_{b1}K_{b2}} u_{(x)}^{''''} \\ \theta_{(x)}^{''''''} - \dfrac{K_{s1}(K_{b1}+K_{b2})}{K_{b1}K_{b2}} \theta_{(x)}^{''''} \\ \psi_{(x)}^{''''''} - \dfrac{K_{s1}(K_{b1}+K_{b2})}{K_{b1}K_{b2}} \psi_{(x)}^{''''} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{1}{K_{s2}} f_{(x)}^{''''} + \dfrac{K_{s1}K_{b2}+K_{s2}K_{b1}+K_{s1}K_{b1}}{K_{b1}K_{b2}K_{s2}} f_{(x)}^{''} - \dfrac{K_{s1}}{K_{b1}K_{b2}} f_{(x)} \\ \dfrac{K_{s1}}{K_{b1}K_{b2}} f_{(x)}^{'} \\ -\dfrac{1}{K_{b2}} f_{(x)}^{'''} + \dfrac{K_{s1}}{K_{b1}K_{b2}} f_{(x)}^{'} \end{Bmatrix} \tag{294}$$

A sixth order differential equation is obtained. Normalizing the differential equation by the variable $z = x/H$:

$$u_{(z)}^{''''''} - \frac{K_{s1}(K_{b1}+K_{b2})}{K_{b1}K_{b2}} H^2 u_{(z)}^{''''} = -\frac{H^2}{K_{s2}} f_{(z)}^{''''} + \frac{K_{s1}K_{b2}+K_{s2}K_{b1}+K_{s1}K_{b1}}{K_{b1}K_{b2}K_{s2}} H^4 f_{(z)}^{''} - \frac{K_{s1}H^6}{K_{b1}K_{b2}} f_{(z)} \tag{295}$$

Or its equivalent:

$$\frac{K_{b1}K_{b2}}{K_{s1}} u_{(z)}^{''''''} - (K_{b1}+K_{b2})H^2 u_{(z)}^{''''} = -\frac{K_{b1}K_{b2}}{K_{s1}K_{s2}} H^2 f_{(z)}^{''''} + \left[\frac{K_{b2}}{K_{s2}} + K_{b1}\left(\frac{1}{K_{s1}} + \frac{1}{K_{s2}}\right)\right] H^4 f_{(z)}^{''} - H^6 f_{(z)} \tag{296}$$

Defining four parameters:

$$\left\{\alpha = H\sqrt{\frac{K_{s1}}{K_{b2}}}\ , \kappa = \sqrt{1+\frac{K_{b2}}{K_{b1}}}, \eta_\psi = H\sqrt{\frac{K_{s2}}{K_{b2}}}, \lambda = \frac{W_{max}H^2}{K_{s2}(1-e^{-a})}\right\} \tag{297}$$

Replacing the first three parameters:

$$u_{(z)}^{''''''} - (\alpha\kappa)^2 u_{(z)}^{''''} = \frac{H^2}{K_{s2}}\left\{-f_{(z)}^{''''} + \left[(\alpha\kappa)^2 + \eta_\psi^2\right] f_{(z)}^{''} - \eta_\psi^2 \alpha^2 (\kappa^2 - 1) f_{(z)}\right\} \tag{298}$$

Assuming a general lateral load (Miranda E. , 1999):

$$f_{(x)} = \frac{W_{max}}{1-e^{-a}}\left(1 - e^{-a+a\frac{x}{H}}\right) \rightarrow f_{(z)} = \frac{W_{max}}{1-e^{-a}}(1 - e^{-a+az}) \tag{299}$$

Replacing the lateral load and introducing the fourth parameter:

$$u_{(z)}^{''''''} - (\alpha\kappa)^2 u_{(z)}^{''''} = -\eta_\varphi^2 \alpha^2 (\kappa^2 - 1)\lambda + \left\{a^4 - \left[(\alpha\kappa)^2 + \eta_\psi^2\right] a^2 + \eta_\varphi^2 \alpha^2 (\kappa^2 - 1)\right\}\lambda e^{-a+az} \tag{300}$$

The expression for $u_{(z)}$, $\psi_{(z)}$ and $\theta_{(z)}$ is proposed:

$$\begin{Bmatrix} u_{(z)} = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + C_4 \cosh(\alpha\kappa z) + C_5 \sinh(\alpha\kappa z) + C_6 z^4 + C_7 e^{-a+az} \\ H\theta_{(z)} = C_8 + C_9 z + C_{10} z^2 + C_{11} z^3 + C_{12} \cosh(\alpha\kappa z) + C_{13} \sinh(\alpha\kappa z) + C_{14} e^{-a+az} \\ H\psi_{(z)} = C_{15} + C_{16} z + C_{17} z^2 + C_{18} z^3 + C_{19} \cosh(\alpha\kappa z) + C_{20} \sinh(\alpha\kappa z) + C_{21} e^{-a+az} \end{Bmatrix} \quad (301)$$

Where:

$$\left\{ C_6 = \frac{\eta_\varphi^2(\kappa^2 - 1)}{24\kappa^2}, C_7 = \frac{a^4 - \left[(\alpha\kappa)^2 + \eta_\psi^2\right]a^2 + \eta_\varphi^2 \alpha^2(\kappa^2 - 1)}{a^4[a^2 - (\alpha\kappa)^2]} \right\} \quad (302)$$

Expressing the coefficients of $\psi_{(z)}$ and $\theta_{(z)}$ as a function of the coefficients of $u_{(z)}$:

$$\begin{Bmatrix} u_{(z)} = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + C_4 \cosh(\alpha\kappa z) + C_5 \sinh(\alpha\kappa z) + C_6 z^4 + C_7 e^{-a+az} \\ \begin{Bmatrix} H\theta_{(z)} = C_1 + (2z)C_2 + (3z^2 + 6p_1)C_3 + [(\alpha\kappa)p_3 \sinh(\alpha\kappa z)]C_4 \\ +[(\alpha\kappa)p_3 \cosh(\alpha\kappa z)]C_5 + (4z^3 + 24p_1 z)C_6 + (p_5 a e^{-a+az})C_7 \end{Bmatrix} \\ \begin{Bmatrix} H\psi_{(z)} = C_1 + (2z)C_2 + (3z^2 + 6p_2)C_3 + [(\alpha\kappa)p_4 \sinh(\alpha\kappa z)]C_4 \\ +[(\alpha\kappa)p_4 \cosh(\alpha\kappa z)]C_5 + (4z^3 + 24p_2 z)C_6 + (p_6 a e^{-a+az})C_7 \end{Bmatrix} \end{Bmatrix} \quad (303)$$

Where:

$$\begin{Bmatrix} p_1 = \frac{K_{s1}K_{b2} + K_{s2}K_{b1} + K_{s1}K_{b1}}{K_{s1}K_{s2}H^2}, p_2 = \frac{K_{s1}K_{b2} + K_{s1}K_{b1}}{K_{s1}K_{s2}H^2} \\ p_3 = \frac{K_{s1}K_{s2}H^4}{K_{b1}K_{b2}(\alpha\kappa)^4 - [K_{s1}K_{b2} + K_{s2}K_{b1} + K_{s1}K_{b1}]H^2(\alpha\kappa)^2 + K_{s1}K_{s2}H^4} \\ p_4 = \frac{[K_{s1}H^2 - (\alpha\kappa)^2 K_{b1}]K_{s2}H^2}{K_{b1}K_{b2}(\alpha\kappa)^4 - [K_{s1}K_{b2} + K_{s2}K_{b1} + K_{s1}K_{b1}]H^2(\alpha\kappa)^2 + K_{s1}K_{s2}H^4} \\ p_5 = \frac{K_{s1}K_{s2}H^4}{K_{b1}K_{b2}a^4 - [K_{s1}K_{b2} + K_{s2}K_{b1} + K_{s1}K_{b1}]H^2 a^2 + K_{s1}K_{s2}H^4} \\ p_6 = \frac{[K_{s1} - a^2 K_{b1}]K_{s2}H^2}{K_{b1}K_{b2}a^4 - [K_{s1}K_{b2} + K_{s2}K_{b1} + K_{s1}K_{b1}]H^2 a^2 + K_{s1}K_{s2}H^4} \end{Bmatrix} \quad (304)$$

The constants are obtained by evaluating the relevant boundary conditions (the origin of x is at the top of the model):

$$\begin{Bmatrix} u_{(1)} = 0 \\ \theta_{(1)} = 0 \\ \psi_{(1)} = 0 \\ \theta'_{(0)} = 0 \\ \psi'_{(0)} = 0 \\ \psi_{(0)} - u'_{(0)} = 0 \end{Bmatrix} \quad (305)$$

Constants:

$$\begin{Bmatrix} C_2 \\ C_4 \end{Bmatrix} = \begin{bmatrix} 2 & (\alpha\kappa)^2 p_3 \\ 2 & (\alpha\kappa)^2 p_4 \end{bmatrix}^{-1} \left\{-24\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} C_6 - \begin{Bmatrix} p_5 \\ p_6 \end{Bmatrix} a^2 e^{-a} C_7\right\}$$

$$\begin{Bmatrix} C_1 \\ C_3 \\ C_5 \end{Bmatrix} = \begin{bmatrix} 1 & 3+6p_1 & (\alpha\kappa)p_3\cosh(\alpha\kappa) \\ 1 & 3+6p_2 & (\alpha\kappa)p_4\cosh(\alpha\kappa) \\ 0 & 6p_2 & (\alpha\kappa)(p_4-1) \end{bmatrix}^{-1} x \begin{Bmatrix} -2C_2 + (\alpha\kappa)p_3\sinh(\alpha\kappa)\,C_4 - (4+24p_1)C_6 - p_5 a C_7 \\ -2C_2 + (\alpha\kappa)p_4\sinh(\alpha\kappa)\,C_4 - (4+24p_2)C_6 - p_6 a C_7 \\ (1-p_6)ae^{-a}C_7 \end{Bmatrix}$$

$$C_0 = -(C_1 + C_2 + C_3 + C_6 + C_7) - C_4\cosh(\alpha\kappa) - C_5\sinh(\alpha\kappa) \qquad (306)$$

For the case of a uniformly distributed lateral load ($a \to \infty$), the expression of $u_{(z)}$, $\psi_{(z)}$ and $\theta_{(z)}$ result:

$$\begin{Bmatrix} u_{(z)} = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + C_4\cosh(\alpha\kappa z) + C_5\sinh(\alpha\kappa z) + C_6 z^4 \\ \begin{Bmatrix} H\theta_{(z)} = C_1 + (2z)C_2 + (3z^2 + 6p_1)C_3 + [(\alpha\kappa)p_3\sinh(\alpha\kappa z)]C_4 \\ +[(\alpha\kappa)p_3\cosh(\alpha\kappa z)]C_5 + (4z^3 + 24p_1 z)C_6 \end{Bmatrix} \\ \begin{Bmatrix} H\psi_{(z)} = C_1 + (2z)C_2 + (3z^2 + 6p_2)C_3 + [(\alpha\kappa)p_4\sinh(\alpha\kappa z)]C_4 \\ +[(\alpha\kappa)p_4\cosh(\alpha\kappa z)]C_5 + (4z^3 + 24p_2 z)C_6 \end{Bmatrix} \end{Bmatrix} \qquad (307)$$

Where:

$$\begin{Bmatrix} \begin{Bmatrix} C_2 \\ C_4 \end{Bmatrix} = -24\begin{bmatrix} 2 & (\alpha\kappa)^2 p_3 \\ 2 & (\alpha\kappa)^2 p_4 \end{bmatrix}^{-1} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} C_6 \\ \begin{Bmatrix} C_1 \\ C_3 \\ C_5 \end{Bmatrix} = \begin{bmatrix} 1 & 3+6p_1 & (\alpha\kappa)p_3\cosh(\alpha\kappa) \\ 1 & 3+6p_2 & (\alpha\kappa)p_4\cosh(\alpha\kappa) \\ 0 & 6p_2 & (\alpha\kappa)(p_4-1) \end{bmatrix}^{-1} x \begin{Bmatrix} -2C_2 + (\alpha\kappa)p_3\sinh(\alpha\kappa)\,C_4 - (4+24p_1)C_6 \\ -2C_2 + (\alpha\kappa)p_4\sinh(\alpha\kappa)\,C_4 - (4+24p_2)C_6 \\ 0 \end{Bmatrix} \\ C_0 = -(C_1 + C_2 + C_3 + C_6) - C_4\cosh(\alpha\kappa) - C_5\sinh(\alpha\kappa) \end{Bmatrix} \qquad (308)$$

4.1.8.2 Case 2

- **Calculation of the Transfer Matrix**

According to the coupled differential equations assuming that the external loads act on the floors and not along the floor height, it is possible to write it as follows:

$$\begin{Bmatrix} K_{s2}\left[u''_{(x)} - \psi'_{(x)}\right] = 0 \\ K_{b1}\theta''_{(x)} - K_{s1}\left[\theta_{(x)} - \psi_{(x)}\right] = 0 \\ K_{b2}\psi''_{(x)} - (K_{s1} + K_{s2})\psi_{(x)} + K_{s1}\theta_{(x)} + K_{s2}u'_{(x)} = 0 \end{Bmatrix} \qquad (309)$$

The expression for $u_{(z)}$, $\psi_{(z)}$ and $\theta_{(z)}$ is proposed:

$$\begin{Bmatrix} u_{(z)} = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + C_4 \cosh(\alpha^* \kappa z) + C_5 \sinh(\alpha^* \kappa z) \\ \theta_{(z)} = C_8 + C_9 z + C_{10} z^2 + C_{12} \cosh(\alpha^* \kappa z) + C_{13} \sinh(\alpha^* \kappa z) \\ \psi_{(z)} = C_{15} + C_{16} z + C_{17} z^2 + C_{19} \cosh(\alpha^* \kappa z) + C_{20} \sinh(\alpha^* \kappa z) \end{Bmatrix} \quad (310)$$

Where:

$$\left\{ \alpha^* = \sqrt{\frac{K_{s1}}{K_{b2}}}, \kappa = \sqrt{1 + \frac{K_{b2}}{K_{b1}}} \right\} \quad (311)$$

Expressing the coefficients of $\psi_{(z)}$ and $\theta_{(z)}$ as a function of the coefficients of $u_{(z)}$:

$$\begin{Bmatrix} u_{(z)} = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + C_4 \cosh(\alpha^* \kappa z) + C_5 \sinh(\alpha^* \kappa z) \\ \begin{Bmatrix} \theta_{(z)} = C_1 + (2z)C_2 + (3z^2 + 6p_1)C_3 + [(\alpha^* \kappa)p_3 \sinh(\alpha^* \kappa z)]C_4 + [(\alpha\kappa)p_3 \cosh(\alpha^* \kappa z)]C_5 \\ \psi_{(z)} = C_1 + (2z)C_2 + (3z^2 + 6p_2)C_3 + [(\alpha^* \kappa)p_4 \sinh(\alpha^* \kappa z)]C_4 + [(\alpha\kappa)p_4 \cosh(\alpha^* \kappa z)]C_5 \end{Bmatrix} \end{Bmatrix} \quad (312)$$

Where:

$$\begin{Bmatrix} p_1 = \dfrac{K_{s1}K_{b2} + K_{s2}K_{b1} + K_{s1}K_{b1}}{K_{s1}K_{s2}}, p_2 = \dfrac{K_{s1}K_{b2} + K_{s1}K_{b1}}{K_{s1}K_{s2}} \\ p_3 = \dfrac{K_{s1}K_{s2}}{K_{b1}K_{b2}(\alpha^*\kappa)^4 - [K_{s1}K_{b2} + K_{s2}K_{b1} + K_{s1}K_{b1}](\alpha^*\kappa)^2 + K_{s1}K_{s2}} \\ p_4 = \dfrac{[K_{s1} - (\alpha\kappa)^2 K_{b1}]K_{s2}}{K_{b1}K_{b2}(\alpha^*\kappa)^4 - [K_{s1}K_{b2} + K_{s2}K_{b1} + K_{s1}K_{b1}](\alpha^*\kappa)^2 + K_{s1}K_{s2}} \end{Bmatrix}$$

Internal forces such as bending moment and shear force associated with lateral displacement result:

$$\begin{Bmatrix} M_{\mathrm{l}(z)} = K_{b1}\theta'_{(x)} = (2K_{b1})C_2 + (6K_{b1}z)C_3 + (\alpha^*\kappa)^2 p_3 K_{b1} \cosh(\alpha^*\kappa z)\, C_4 + (\alpha^*\kappa)^2 p_3 K_{b1} \sinh(\alpha^*\kappa z)\, C_5 \\ M_{\mathrm{r}(z)} = K_{b2}\psi'_{(x)} = (2K_{b2})C_2 + (6K_{b2}z)C_3 + (\alpha^*\kappa)^2 p_4 K_{b2} \cosh(\alpha^*\kappa z)\, C_4 + (\alpha^*\kappa)^2 p_4 K_{b2} \sinh(\alpha^*\kappa z)\, C_5 \\ V_{(z)} = K_{s2}\left(u'_{(x)} - \psi_{(x)}\right) = 6p_3 K_{s2} C_3 + (\alpha^*\kappa)(p_4 - 1)K_{s2} \sinh(\alpha^*\kappa z)\, C_4 + (\alpha^*\kappa)(p_4 - 1)K_{s2} \cosh(\alpha^*\kappa z)\, C_5 \end{Bmatrix}$$

$$(313)$$

Writing the equations in matrix form:

$$
\begin{Bmatrix} u_i(z_i) \\ \theta_i(z_i) \\ \psi_i(z_i) \\ M_{\mathrm{li}}(z_i) \\ M_{\mathrm{ri}}(z_i) \\ V_i(z_i) \end{Bmatrix} = K_i(z_i) \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{Bmatrix} \tag{314}
$$

Where:

$$
K_i(z_i) = \begin{bmatrix} 1 & z & z^2 & z^3 & \cosh(\alpha^*\kappa z) & \sinh(\alpha^*\kappa z) \\ 0 & 1 & 2z & 3z^2 + 6p_1 & (\alpha^*\kappa)p_3 \sinh(\alpha^*\kappa z) & (\alpha\kappa)p_3 \cosh(\alpha^*\kappa z) \\ 0 & 1 & 2z & 3z^2 + 6p_2 & (\alpha^*\kappa)p_4 \sinh(\alpha^*\kappa z) & (\alpha\kappa)p_4 \cosh(\alpha^*\kappa z) \\ 0 & 0 & 2K_{b1} & 6K_{b1}z & (\alpha^*\kappa)^2 p_3 K_{b1} \cosh(\alpha^*\kappa z) & (\alpha^*\kappa)^2 p_3 K_{b1} \sinh(\alpha^*\kappa z) \\ 0 & 0 & 2K_{b2} & 6K_{b2}z & (\alpha^*\kappa)^2 p_4 K_{b2} \cosh(\alpha^*\kappa z) & (\alpha^*\kappa)^2 p_4 K_{b2} \sinh(\alpha^*\kappa z) \\ 0 & 0 & 0 & 6p_3K_{s2} & (\alpha^*\kappa)(p_4 - 1)K_{s2} \sinh(\alpha^*\kappa z) & (\alpha^*\kappa)(p_4 - 1)K_{s2} \cosh(\alpha^*\kappa z) \end{bmatrix} \tag{315}
$$

- **Static Analysis Under Static Point Loads Applied at Floor Level**

The relationship between forces and displacements between two consecutive floors is obtained by taking into account the transfer matrix and the vector of external point forces.

$$
\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ \psi_n(0) \\ M_{\mathrm{ln}}(0) \\ M_{\mathrm{rn}}(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_k(0) \begin{Bmatrix} u_1(h_1) \\ \theta_1(h_1) \\ \psi_1(h_1) \\ M_{\mathrm{l1}}(h_1) \\ M_{\mathrm{r1}}(h_1) \\ V_1(h_1) \end{Bmatrix} - \sum_{s=0}^{n} \left[\prod_{k=s+1}^{n} T_k(0) \right] F_s - F_n \tag{316}
$$

Expressing it in simplified form:

$$
\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ \psi_n(0) \\ M_{\mathrm{ln}}(0) \\ M_{\mathrm{rn}}(0) \\ V_n(0) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ \theta_1(h_1) \\ \psi_1(h_1) \\ M_{\mathrm{l1}}(h_1) \\ M_{\mathrm{r1}}(h_1) \\ V_1(h_1) \end{Bmatrix} + f \tag{317}
$$

Where:

$$\left\{\begin{array}{c} \mathrm{t}=\prod_{k=1}^{n} T_k(0) \\ f=-\sum_{s=0}^{n}\left[\prod_{k=s+1}^{n} T_k(0)\right] F_s-F_n \end{array}\right\} \tag{318}$$

This equation expresses the relationship between the part forces and displacements of the top and the base of the beam. An important point to note is that the size of the transfer matrix is 6x6 and remains constant for all floors.

According to the boundary conditions defined in case 1:

$$\left\{\begin{array}{c} u_{(1)}=0 \\ \theta_{(1)}=0 \\ \psi_{(1)}=0 \\ \psi'_{(0)}=0 \\ \theta'_{(0)}=0 \\ (K_{s1}+K_{s2})u'_{(0)}-K_{s1}\psi_{(0)}-K_{s2}\theta_{(0)}=0 \end{array}\right\} \rightarrow \left\{\begin{array}{c} u_1(h_1)=0 \\ \theta_1(h_1)=0 \\ \psi_1(h_1)=0 \\ M_{ln\,(0)}=0 \\ M_{rn\,(0)}=0 \\ V_{n\,(0)}=0 \end{array}\right\} \tag{319}$$

Replacing:

$$\left\{\begin{array}{c} u_n(0) \\ \theta_n(0) \\ \psi_n(0) \\ 0 \\ 0 \\ 0 \end{array}\right\}=\begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & t_{1,5} & t_{1,6} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} & t_{2,5} & t_{2,6} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} & t_{3,5} & t_{3,6} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,1} & t_{5,2} & t_{5,3} & t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,1} & t_{6,2} & t_{6,3} & t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix}\left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ M_{11}(h_1) \\ M_{21}(h_1) \\ V_1(h_1) \end{array}\right\}+\left\{\begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{array}\right\} \tag{320}$$

By clearing the bending moment and the shear force at the base of the model:

$$\left\{\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right\}=\begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix}\left\{\begin{array}{c} M_{\mathrm{l}1}(h_1) \\ M_{\mathrm{r}1}(h_1) \\ V_1(h_1) \end{array}\right\}+\left\{\begin{array}{c} f_4 \\ f_5 \\ f_6 \end{array}\right\} \rightarrow \left\{\begin{array}{c} M_{\mathrm{l}1}(h_1) \\ M_{\mathrm{r}1}(h_1) \\ V_1(h_1) \end{array}\right\}=-\begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix}^{-1}\left\{\begin{array}{c} f_4 \\ f_5 \\ f_6 \end{array}\right\} \tag{321}$$

Substituting the internal forces gives the displacement and rotations at the top of the beam:

$$\left\{\begin{array}{c} u_n(0) \\ \theta_n(0) \\ \psi_n(0) \end{array}\right\}=-\begin{bmatrix} t_{1,4} & t_{1,5} & t_{1,6} \\ t_{2,4} & t_{2,5} & t_{2,6} \\ t_{3,4} & t_{3,5} & t_{3,6} \end{bmatrix}\begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix}^{-1}\left\{\begin{array}{c} f_4 \\ f_5 \\ f_6 \end{array}\right\}+\left\{\begin{array}{c} f_1 \\ f_2 \\ f_3 \end{array}\right\} \tag{322}$$

4.1.9 Modified Generalized Sandwich Beam of Two-Field (MGSB)

Due to the complexity in Generalized Sandwich Beam (GSB) calculations, a new replacement beam model "Modified Generalized Sandwich Beam (MGSB)" is developed suitable for modeling coupled shear walls. Moghadasi (2015), performed a parametric analysis to compare the rotations in coupled shear walls against a uniformly distributed lateral load using SAP 2000 Software and concluded that the rotation fields in coupled shear walls can be assumed to be almost identical as long as the wall width ratio is between 0.25 and 4:

$$0.25 \leq \frac{B_1}{B_2} \leq 4 \qquad (323)$$

Taking this criterion into account, the three-field generalized sandwich beam (GSB) is reduced to a two-field modified generalized sandwich beam (MGSB).

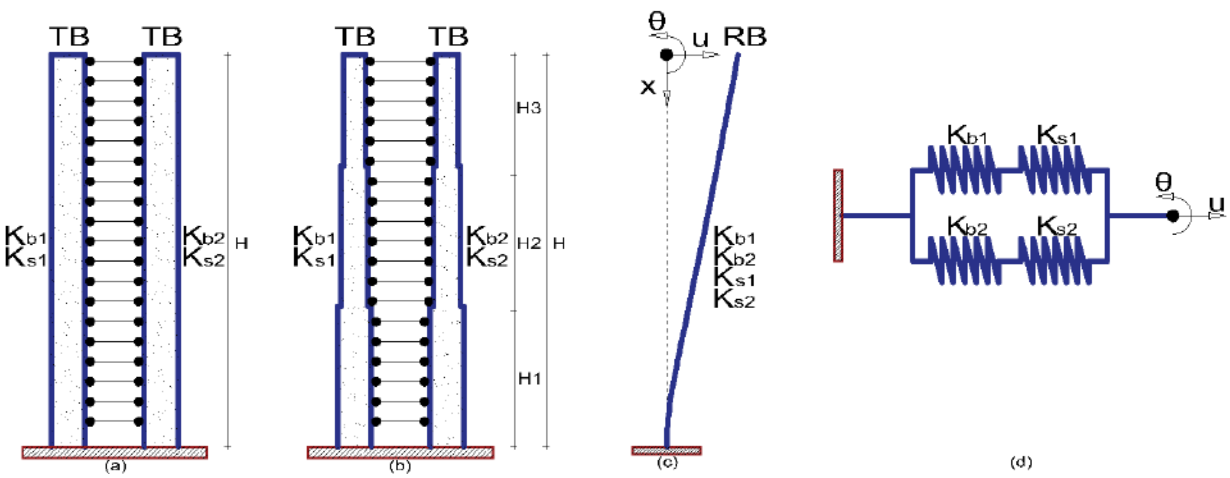


Figure 63. Modified Generalized Sandwich Beam of two-field (MGSB). a) Case 1, b) Case 2, c) equivalent RB and d) Idealization of MGSB stiffness.

4.1.9.1 Case 1

The potential energy of the two-field MGSB model is expressed as follows:

$$V = \frac{1}{2}\int_0^H \left\{K_{b1}{\theta'_{(x)}}^2 + K_{s1}\left[u'_{(x)} - \theta_{(x)}\right]^2 + K_{b2}{\theta'}^2 + K_{s2}\left[u'_{(x)} - \theta_{(x)}\right]^2\right\} dx \qquad (324)$$

Taking into account common terms:

$$V = \frac{1}{2}\int_0^H \left\{(K_{b1} + K_{b2}){\theta'_{(x)}}^2 + (K_{s1} + K_{s2})\left[u'_{(x)} - \theta_{(x)}\right]^2\right\} dx \tag{325}$$

Where:

$$\left\{\begin{matrix} K_{b1} = \sum_{i=1}^{c} rEI_{c,i}\,, K_{b2} = \sum_{i=1}^{c} EA_{c,i}c_i^2\,, K_{s1} = \sum_{i=1}^{c} GA_{c,i}\,, K_{s2} = \left(K_b^{-1} + K_c^{-1}\right)^{-1} \\ K_b = \sum_{i=1}^{b} \frac{6EI_{b,i}[(l^* + S_1)^2 + (l^* + S_2)^2]}{l^{*3}h\left(1 + 12\frac{kEI_{b,i}}{l^{*2}GA_{b,i}}\right)}, K_c = \sum_{i=1}^{c} \frac{12EI_{c,i}}{h^2}, r = \frac{K_c}{K_c + K_b} \end{matrix}\right\} \tag{326}$$

This new expression of the potential energy is identical to the potential energy of a Timoshenko beam (TB) where the bending and shear stiffnesses are coupled in series. By cancelling the effect of the axially stiff members that transmitted the horizontal forces between the two beams, the two-field MGSB beam becomes stiffer than the original GSB. To overcome this problem it is necessary to modify the flexural stiffnesses by means of an appropriate coefficient η. This coefficient η will be determined by displacement compatibility at z=0.

The horizontal displacement of a sandwich beam (SWB) against a lateral load proposed by Miranda (1999) is:

$$u_{(z)} = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + C_4 \cosh(\alpha\kappa z) + C_5 \sinh(\alpha\kappa z) + C_6 z^4 + C_7 e^{-a+az} \tag{327}$$

Where the coefficients are known; evaluating it at $z = 0$ and ordering appropriately, we obtain:

$$u_{(0)} = \lambda K_t \rightarrow K_t = \frac{u_{(0)}}{\lambda} = f_{(\alpha,\kappa)} \tag{328}$$

The parameter K_t is only a function of α and κ. The graph shows the trend of the K_t parameter for different values of the connection beam superelevation: 0.5, 0.6, 0.7, 0.8, 0.9 and 1.0 m. It can be observed that the increase of the camber of the connecting beams has a positive effect by increasing the lateral stiffness and consequently reducing the total displacement of the structural element. This is the importance of considering the contribution of these coupling beams in the elevator shafts, where their consideration in the analysis significantly increases the lateral and torsional stiffness.

The figure shows that the higher the degree of coupling, the tendency of the displacement is towards global bending behavior, disregarding the contribution of stiffness to local bending; and even for a lower degree of coupling, this tendency is rapidly approaching the global bending situation.

The local bending behavior, disregarding global bending, is obtained by evaluating $h_v \rightarrow 0$ and consequently $\alpha \rightarrow 0$. In this case the function of the connecting beams is primarily to transmit the horizontal loads and force the shear walls to work together. The maximum displacement of the building for this case will be denoted as $u_{(h_v \rightarrow 0\ )}$. Evaluating this condition on the maximum displacement of the building subject to lateral loads, we obtain:

$$u_{(h_v \rightarrow 0)} = \frac{wH^4}{8K_{b1}} \tag{329}$$

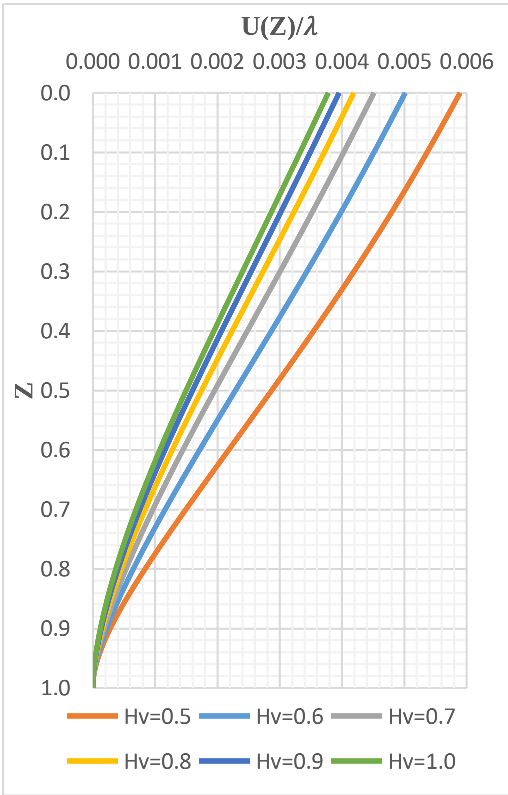


Figure 64. Displacement of the building as a function of the depth of the connecting beams.

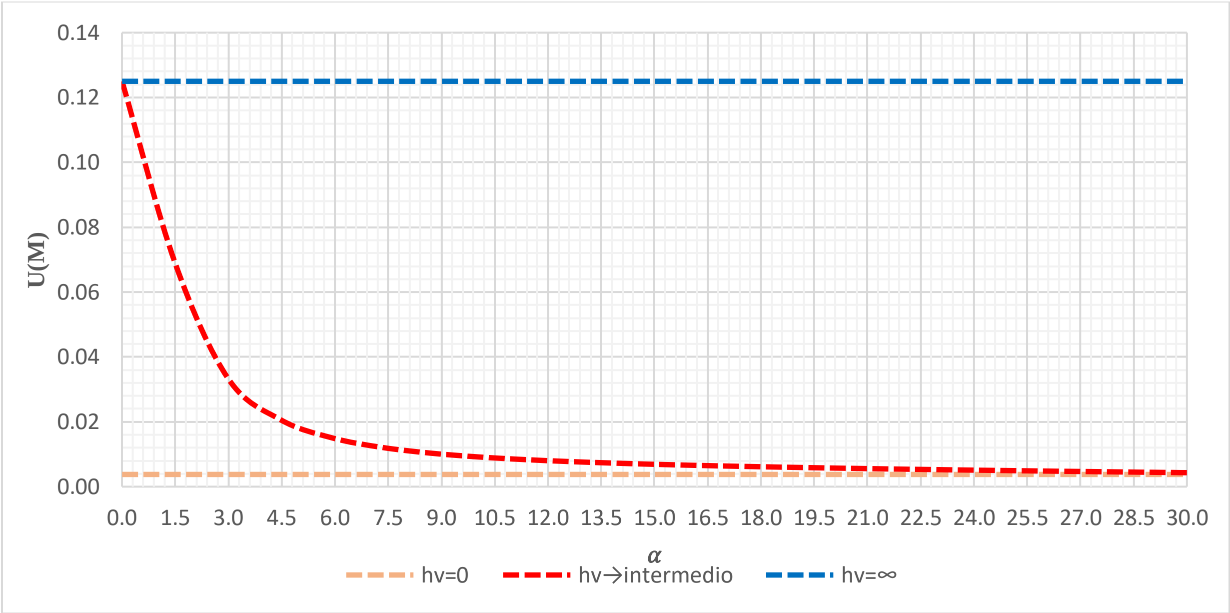


Figure 65. Maximum building displacement as a function of the degree of coupling.

The total bending behavior (global + local), is obtained by evaluating $h_v \to \infty$ and as a consequence $\alpha \to \infty$. This is the case for tall and slender shear walls (With a very high height/width ratio). The maximum displacement of the building for this case will be denoted as $u_{(h_v \to \infty\,)}$. Evaluating this condition on the maximum displacement of the building subject to lateral loads, we obtain:

$$u_{(h_v \to \infty)} = \frac{wH^4}{8(K_{b1} + K_{b2})} \qquad (330)$$

The maximum displacement of the building will be denoted as:

$$u_{(0)} = (\eta)u_{(\,0,h_v \to 0\,\,)} + (1-\eta)u_{(\,0,h_v \to \infty)} \qquad (331)$$

Substituting the above expressions and clearing we can obtain the coefficient η:

$$\eta = \frac{u_{(0)} - u_{(\,0,h_v \to \infty)}}{u_{(\,0,h_v \to 0\,)} - u_{(\,0,h_v \to \infty)}} \qquad (332)$$

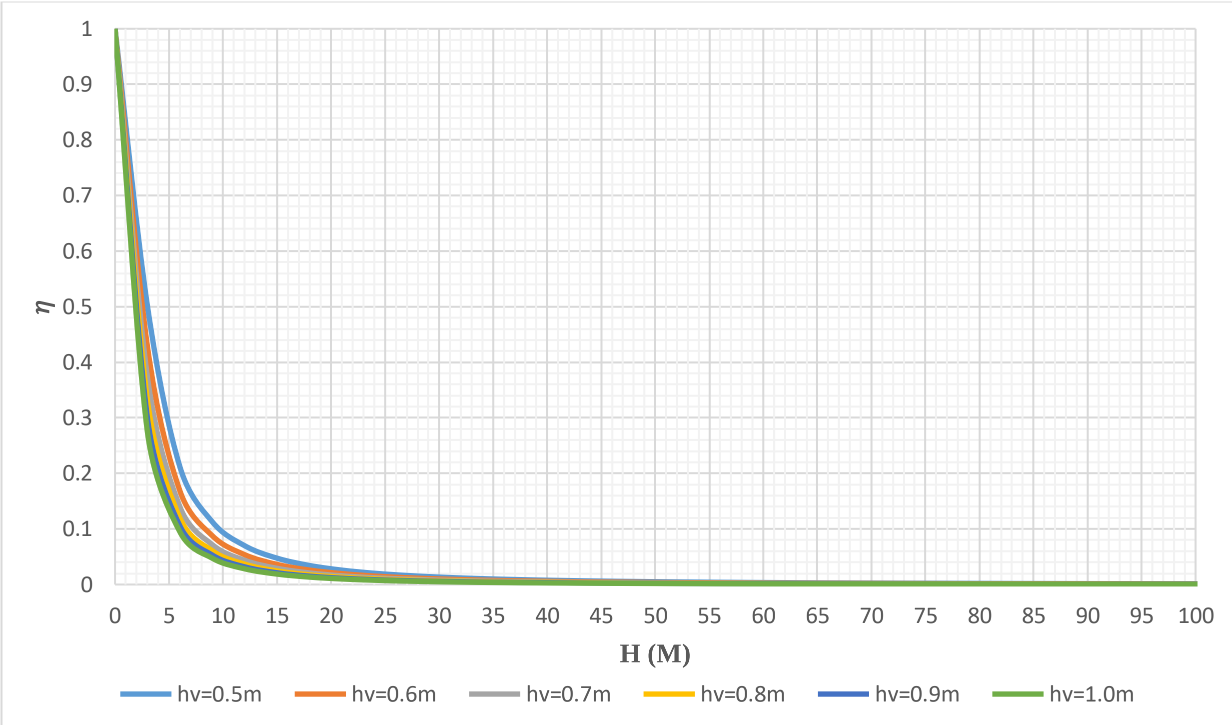


Figure 66. Trend of parameter η with building height.

The figure shows that the higher the height, the tendency of the parameter η is close to zero. It means that when the number of floors increases, the global bending becomes more and more important and therefore the connecting beams increase the stiffness of the coupled walls with respect to the situation of uncoupled walls; this is understandable, as pointed out by Zalka (2020) as the number of floors increases the global bending dominates the behavior of the structure. In addition, it is observed that the tendency of the parameter η to zero value increases with the increase of the span of the connecting beams.

The potential energy of the two-field MGSB (TB + TB) becomes:

$$V = \frac{1}{2}\int_0^H \left\{[\eta K_{b1} + (1-\eta)K_{b2}]\theta_{(x)}'^{\,2} + (K_{s1} + K_{s2})\left[u_{(x)}' - \theta_{(x)}\right]^2\right\} dx \tag{333}$$

i.e.,

$$V = \frac{1}{2}\int_0^H \left\{K_b^* \theta_{(x)}'^{\,2} + K_s^*\left[u_{(x)}' - \theta_{(x)}\right]^2\right\} dx \tag{334}$$

Where:

$$\left\{\begin{array}{c} K_b^* = \eta K_{b1} + (1-\eta)K_{b2} = \eta \sum_{i=0}^{c} rEI_{c,i} + (1-\eta)\sum_{i=0}^{c} EA_{c,i}c_i^2 \\ K_s^* = K_{s1} + K_{s2} = \sum_{i=0}^{c} GA_{c,i} + \left(K_b^{-1} + K_c^{-1}\right)^{-1} \\ K_b = \sum_{i=0}^{b} \dfrac{6EI_{b,i}[(l^* + S_1)^2 + (l^* + S_2)^2]}{l^{*3}h\left(1 + 12\dfrac{kEI_{b,i}}{l^{*2}GA_{b,i}}\right)}, K_c = \sum_{i=0}^{n} \dfrac{12EI_c}{h^2} \end{array}\right\} \tag{335}$$

As initially observed, if the stiffnesses K_b^* and K_s^* are replaced by their lateral equivalents K_b and K_s respectively, it follows that the equations of the two-field MGSB beam are identical to those of the TB beam; that is, the same conclusions obtained for the TB beam are applicable for the two-field MGSB beam.

The expression of $u_{(Z)}$:

$$\begin{aligned} u_{(z)} = \lambda\left\{\left[\frac{1}{8} + \frac{1}{a^3}\left(\frac{1}{a} - 1\right) + \frac{e^{-a}}{a^2}\left(\frac{1}{2} - \frac{1}{a}\right)\right] + \frac{1}{\alpha^2}\left(\frac{1}{2} - \frac{1}{a^2} + \frac{e^{-a}}{a}\right)\right\} \\ + \lambda\left[\left(\frac{1}{a^3} - \frac{1}{6}\right) - \frac{e^{-a}}{a}\left(\frac{1}{a} + \frac{1}{2} + \frac{1}{\alpha^2}\right)\right]z - \frac{\lambda}{2\alpha^2}\left(1 - \frac{\alpha^2}{a^2}e^{-a}\right)z^2 + \frac{\lambda e^{-a}}{6a}z^3 \\ + \frac{\lambda}{24}z^4 + \left[\frac{\lambda(a^2 - \alpha^2)}{a^4\alpha^2}\right]e^{-a+az} \end{aligned} \tag{336}$$

For the case of a uniformly distributed lateral load $(a \to \infty)$:

$$u_{(z)} = \frac{W_{max}H^4}{K_b^*}\left(\frac{1}{24}z^4 - \frac{1}{6}z + \frac{1}{8}\right) + \frac{W_{max}H^2}{2K_s^*}(1 - z^2) \tag{337}$$

4.1.9.2 Case 2

- **Calculation of the Transfer Matrix**

Recalling the transfer matrix of the TB beam, we have:

$$K_i(z_i) = \begin{bmatrix} 1 & z_i & z_i^2 & z_i^3 \\ 0 & 1 & 2z_i & 3z_i^2 + \dfrac{6}{\alpha^{*2}} \\ 0 & 0 & 2K_b^* & 6K_b^* z_i \\ 0 & 0 & 0 & -\dfrac{6}{\alpha^{*2}} K_s^* \end{bmatrix}_i \tag{338}$$

Moreover;

$$\left\{ \begin{array}{c} T_i(\mathrm{z}) = K_i(z_i) K_i^{-1}(h_i) \\ \begin{Bmatrix} u_i(0) \\ \theta_i(0) \\ M_i(0) \\ V_i(0) \end{Bmatrix} = T_i(0) \begin{Bmatrix} u_i(h_i) \\ \theta_i(h_i) \\ M_i(h_i) \\ V_i(h_i) \end{Bmatrix} \end{array} \right\}$$

- **Static Analysis Under Static Point Loads Applied at Floor Level**

The relationship between the forces and displacements of the top and the base of the beam, according to the TB beam results:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ \theta_1(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} + f \tag{339}$$

Where:

$$\left\{ \begin{array}{c} \mathrm{t} = \prod_{k=1}^{n} T_k(0) \\ f = -\sum_{s=0}^{n} \left[\prod_{k=s+1}^{n} T_k(0) \right] F_s - F_n \end{array} \right\} \tag{340}$$

Moreover:

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \end{Bmatrix} = -\begin{bmatrix} t_{1,3} & t_{1,4} \\ t_{2,3} & t_{2,4} \end{bmatrix} \begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix}^{-1} \begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{341}$$

4.1.10 Parallel Coupling of Shear Beam and Timoshenko Beam (MCTB)

Due to the mathematical complexity that results when analyzing the "Generalized Sandwich Beam (GSB)", a mathematical model is developed that neglects the global bending stiffness. According to Mangione and Migliorati (2015), this "Modified Generalized Sandwich Beam (MGSB)" model is suitable for modeling flat portal frames that in practice are used for buildings less than 25 stories.

The "modified two-beam coupling (MCTB) of two fields" results from the parallel coupling of a Timoshenko beam (TB) and a shear beam (SB). A single transverse motion u and an equal rotational field θ in both beams are taken into account.

In the model; K_{s1} is the global shear stiffness in the left SB and K_{b1} and K_{s1} are the local bending and local shear stiffnesses in the right TB.

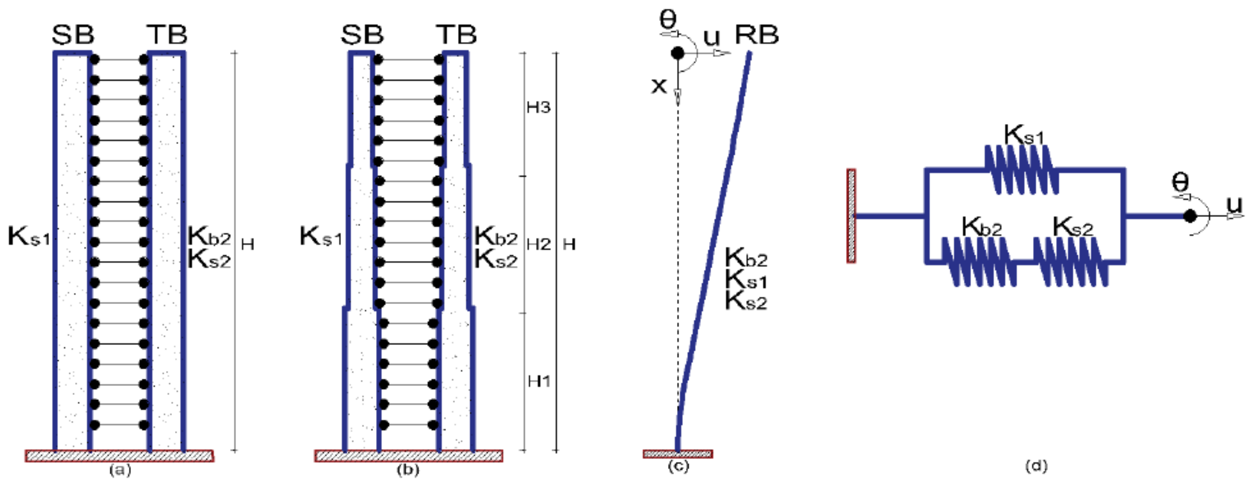


Figure 67. Modified two-beam coupling (MCTB) of two fields. a) Case 1, b) Case 2, c) equivalent RB and d) MCTB stiffness idealization.

4.1.10.1 Case 1

The potential energy of the two-field MCTB model is expressed as follows:

$$V = \frac{1}{2}\int_0^H \left[K_{s1}{u'_{(x)}}^2\right] dx + \frac{1}{2}\int_0^H \left\{K_{b2}{\theta'_{(x)}}^2 + K_{s2}\left[\theta_{(x)} - u'_{(x)}\right]^2\right\} dx \tag{342}$$

Where:

$$\left\{K_{s1} = (K_b^{-1} + K_c^{-1})^{-1}\,, K_{b2} = r\sum_{i=0}^{c} EI_{c,i}\ , K_{s2} = \sum_{i=0}^{c} GA_{c,i}\,, K_b = \sum_{i=1}^{b} \frac{12EI_{b,i}}{hL}\ , K_c = \sum_{i=0}^{c} \frac{12EI_{c,i}}{h^2}\right\} \quad (343)$$

The work done by the external force is:

$$W = \int_0^H f_{(x)} u_{(x)}\, dx \quad (344)$$

Consequently, the total potential energy of the two-field MCTB beam subjected to a general lateral load distribution is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H \left\{K_{s1} {u'_{(x)}}^2 + K_{b2} {\theta'_{(x)}}^2 + K_{s2}\left[\theta_{(x)} - u'_{(x)}\right]^2\right\} dx - \int_0^H f_{(x)} u_{(x)}\, dx \quad (345)$$

Closed-form solutions of the model on which a transverse load acts are achieved by solving the differential system arising from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\begin{aligned}\delta\mathcal{U} = \int_0^H &\{K_{s1} u'_{(x)} \delta u'_{(x)} + K_{b2}\theta'_{(x)}\delta\theta'_{(x)} + K_{s2}\left[\theta_{(x)} - u'_{(x)}\right]\delta\theta_{(x)} \\ &- K_{s2}\left[\theta_{(x)} - u'_{(x)}\right]\delta u'_{(x)}\} dx - \int_0^H f(x)\delta u_{(x)}\, dx - \int_0^H u_{(x)}\, \delta f(x) dx\end{aligned} \quad (346)$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\begin{aligned}\delta\mathcal{U} = &\left[K_{b2}\theta'_{(x)}\delta\theta_{(x)}\right]_0^H + \left\{(K_{s1} + K_{s2})u'_{(x)} - K_{s2}\theta_{(x)}\right\}\delta u_{(x)}{}_0^H \\ &- \int_0^H \{K_{b2}\theta''_{(x)} - K_{s2}\left[\theta_{(x)} - u'_{(x)}\right]\}\delta\theta_{(x)} \\ &- \int_0^H \left\{(K_{s1} + K_{s2})u''_{(x)} - K_{s2}\theta'_{(x)} + f(x)\right\}\delta u_{(x)} - \int_0^H u_{(x)}\, \delta f(x) dx\end{aligned} \quad (347)$$

Equating the terms to zero results in the following equations:

$$\left\{\begin{matrix} K_{b2}\theta''_{(x)} - K_{s2}\left[\theta_{(x)} - u'_{(x)}\right] = 0 \\ (K_{s1} + K_{s2})u''_{(x)} - K_{s2}\theta'_{(x)} + f(x) = 0 \end{matrix}\right\} \quad (348)$$

Boundary conditions:

$$\begin{Bmatrix} \theta'_{(0)} = 0 \\ (K_{s1}+K_{s2})u'_{(0)} - K_{s2}\theta_{(0)} = 0 \end{Bmatrix} \tag{349}$$

Using the method of coefficients for the solution of the system of equations:

$$\begin{Bmatrix} u_{(x)} \\ \theta_{(x)} \end{Bmatrix} = -\begin{bmatrix} K_{s2}D & K_{b2}D^2 - K_{s2} \\ (K_{s1}+K_{s2})D^2 & -K_{s2}D \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ f_{(x)} \end{Bmatrix} \tag{350}$$

i.e.,

$$\begin{Bmatrix} u''''_{(x)} - \dfrac{K_{s1}K_{s2}}{K_{b2}(K_{s1}+K_{s2})} u''_{(x)} \\ \theta''''_{(x)} - \dfrac{K_{s1}K_{s2}}{K_{b2}(K_{s1}+K_{s2})} \theta''_{(x)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{1}{K_{s1}+K_{s2}}\left[-f''_{(x)} + \dfrac{K_{s2}}{K_{b2}} f_{(x)}\right] \\ \dfrac{K_{s2}}{K_{b2}(K_{s1}+K_{s2})} f'_{(x)} \end{Bmatrix} \tag{351}$$

A fourth order differential equation is obtained. Normalizing the differential equation by the variable $z = x/H$:

$$u''''_{(z)} - \left[\frac{K_{s1}K_{s2}}{K_{b2}(K_{s1}+K_{s2})} H^2\right] u''_{(z)} = \frac{H^2}{K_{s1}+K_{s2}}\left[-f''_{(z)} + H^2 \frac{K_{s2}}{K_{b2}} f_{(z)}\right] \tag{352}$$

Defining three parameters:

$$\left\{ \alpha = H\sqrt{\frac{K_{s2}^2}{K_{b2}(K_{s1}+K_{s2})}}\ , \kappa = \sqrt{\frac{K_{s1}}{K_{s2}}}, \lambda = \frac{W_{max}H^2}{(K_{s1}+K_{s2})(1-e^{-a})} \right\} \tag{353}$$

Replacing the first two parameters:

$$u''''_{(z)} - (\alpha\kappa)^2 u''_{(z)} = \frac{H^2}{K_{s1}+K_{s2}}\left[-f''_{(z)} + \alpha^2(k^2+1) f_{(z)}\right] \tag{354}$$

Assuming a general lateral load (Miranda E. , 1999):

$$f_{(x)} = \frac{W_{max}}{1-e^{-a}}\left(1 - e^{-a+a\frac{x}{H}}\right) \to f_{(z)} = \frac{W_{max}}{1-e^{-a}}(1 - e^{-a+az}) \tag{355}$$

Replacing the lateral load and the third parameter:

$$u''''_{(z)} - (\alpha\kappa)^2 u''_{(z)} = \lambda\alpha^2(k^2+1) - \lambda[\alpha^2(k^2+1) - a^2]e^{-a+az} \tag{356}$$

The expression for $u_{(z)}$ is proposed:

$$u_{(z)} = C_0 + C_1 z + C_2\cosh(\alpha\kappa z) + C_3\sinh(\alpha\kappa z) - \frac{\lambda(k^2+1)}{2\kappa^2}z^2 - \frac{\lambda[\alpha^2(k^2+1)-a^2]}{a^2[a^2-(\alpha\kappa)^2]}e^{-a+az} \tag{357}$$

The constants are obtained by evaluating the relevant boundary conditions (the origin of x is at the base of the model):

$$\left\{\begin{array}{c} u_{(1)} = 0 \\ u'_{(1)} = -\lambda\left(1 - \dfrac{1}{a} + \dfrac{e^{-a}}{a}\right) \\ u''_{(0)} = -\lambda(1-e^{-a}) \\ u'''_{(1)} = \lambda\left[\alpha^2\left(1-\dfrac{1}{a}+\dfrac{e^{-a}}{a}\right)+a\right] \end{array}\right\} \tag{358}$$

Constants:

$$\left\{\begin{array}{c} C_0 = u_{(1)} - C_1 - [C_2\cosh(\alpha\kappa) + C_3\sinh(\alpha\kappa)] + \lambda\left\{\dfrac{1}{2} + \dfrac{1}{2k^2} + \dfrac{\alpha^2(\kappa^2+1)-a^2}{a^2[a^2-(\alpha\kappa)^2]}\right\} \\ C_1 = u'_{(1)} - (\alpha\kappa)[C_2\sinh(\alpha\kappa) + C_3\cosh(\alpha\kappa)] + \lambda\left\{1 + \dfrac{1}{k^2} + \dfrac{\alpha^2(\kappa^2+1)-a^2}{a[a^2-(\alpha\kappa)^2]}\right\} \\ C_2 = \dfrac{\lambda}{(\alpha\kappa)^2}\left\{\dfrac{1}{k^2} + \dfrac{e^{-a}\alpha^2}{a^2-(\alpha\kappa)^2}\right\} \\ C_3 = -C_2\tanh(\alpha\kappa) + \dfrac{\lambda\alpha^2}{(\alpha\kappa)^3\cosh(\alpha\kappa)}\left[\left(1-\dfrac{1}{a}+\dfrac{e^{-a}}{a}\right) + \dfrac{a}{a^2-(\alpha\kappa)^2}\right] \end{array}\right\} \tag{359}$$

For the case of a uniformly distributed lateral load ($a \to \infty$), the expression for $u_{(z)}$ results:

$$u_{(z)} = \frac{W_{max}H^2}{K_{s1}}(1-z^2) - \frac{1}{\kappa^2}\frac{W_{max}H^2}{K_{s1}+K_{s2}}\left\{\frac{1-\cosh(\alpha\kappa z - \alpha\kappa) - (\alpha\kappa)[\sinh(\alpha\kappa z) - \sinh(\alpha\kappa)]}{(\alpha\kappa)^2\cosh(\alpha\kappa)}\right\} \tag{360}$$

This deflection expression shows that the lateral displacement is composed of the shear deflection and a deflection due to the interaction between bending and shear.

4.1.10.2 Case 2

- **Calculation of the Transfer Matrix**

According to the coupled differential equations and assuming that the external loads act on the floors and not along the floor height, it is possible to write it as follows:

$$\begin{Bmatrix} K_{b2}\theta''_{(x)} - K_{s2}\left[\theta_{(x)} - u'_{(x)}\right] = 0 \\ (K_{s1} + K_{s2})u''_{(x)} - K_{s2}\theta'_{(x)} = 0 \end{Bmatrix} \tag{361}$$

The expression for $u_{(z)}$ and $\theta_{(z)}$ is proposed:

$$\begin{Bmatrix} u_{(z)} = C_0 + C_1 z + C_2\cosh(\alpha^*\kappa z) + C_3\sinh(\alpha^*\kappa z) \\ \theta_{(z)} = C_4 + C_5\cosh(\alpha^*\kappa z) + C_6\sinh(\alpha^*\kappa z) \end{Bmatrix} \tag{362}$$

Where:

$$\begin{Bmatrix} \alpha^* = \sqrt{\dfrac{K_{s2}^2}{K_{b2}(K_{s1} + K_{s2})}}\ , \kappa = \sqrt{\dfrac{K_{s1}}{K_{s2}}} \end{Bmatrix} \tag{363}$$

Expressing the coefficients of the function $\theta_{(z)}$ as a function of the coefficients of $u_{(z)}$:

$$\begin{Bmatrix} u_{(z)} = C_0 + C_1 z + C_2\cosh(\alpha^*\kappa z) + C_3\sinh(\alpha^*\kappa z) \\ \theta_{(z)} = C_1 + C_2\dfrac{\alpha^*\kappa}{1-\kappa^2}\sinh(\alpha^*\kappa z) + C_3\dfrac{\alpha^*\kappa}{1-\kappa^2}\cosh(\alpha^*\kappa z) \end{Bmatrix} \tag{364}$$

Internal forces such as bending moment and shear force associated with lateral displacement result:

$$\begin{Bmatrix} M_{1(z)} = K_{b2}\theta'_{(x)} = C_2\dfrac{(\alpha^*\kappa)^2}{1-\kappa^2}K_{b2}\cosh(\alpha^*\kappa z) + C_3\dfrac{(\alpha^*\kappa)^2}{1-\kappa^2}K_{b2}\sinh(\alpha^*\kappa z) \\ V_{(z)} = (K_{s1} + K_{s2})u'_{(x)} - K_{s2}\theta_{(x)} = K_{s1}C_1 \end{Bmatrix} \tag{365}$$

Writing the equations in matrix form:

$$\begin{Bmatrix} u_i(z_i) \\ \theta_i(z_i) \\ M_{\mathrm{i}}(z_i) \\ V_i(z_i) \end{Bmatrix} = K_i(z_i)\begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{Bmatrix} \tag{366}$$

Where:

$$K_i(z_i) = \begin{bmatrix} 1 & z & \cosh(\alpha^*\kappa z) & \sinh(\alpha^*\kappa z) \\ 0 & 1 & \dfrac{\alpha^*\kappa}{1-\kappa^2}\sinh(\alpha^*\kappa z) & \dfrac{\alpha^*\kappa}{1-\kappa^2}\cosh(\alpha^*\kappa z) \\ 0 & 0 & \dfrac{(\alpha^*\kappa)^2}{1-\kappa^2}K_{b2}\cosh(\alpha^*\kappa z) & \dfrac{(\alpha^*\kappa)^2}{1-\kappa^2}K_{b2}\sinh(\alpha^*\kappa z) \\ 0 & K_{s1} & 0 & 0 \end{bmatrix} \tag{367}$$

- **Static Analysis Under Static Point Loads Applied at Floor Level**

The relationship between forces and displacements between two consecutive floors is obtained by taking into account the transfer matrix and the vector of external point forces.

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_k(0) \begin{Bmatrix} u_1(h_i) \\ \theta_1(h_i) \\ M_1(h_i) \\ V_1(h_i) \end{Bmatrix} - \sum_{s=0}^{n} \left[\prod_{k=s+1}^{n} T_k(0) \right] F_s - F_n \tag{368}$$

Expressing it in simplified form:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_i) \\ \theta_1(h_i) \\ M_1(h_i) \\ V_1(h_i) \end{Bmatrix} + f \tag{369}$$

Where:

$$\begin{Bmatrix} \mathrm{t} = \prod_{k=1}^{n} T_k(0) \\ f = -\sum_{s=0}^{n} \left[\prod_{k=s+1}^{n} T_k(0) \right] F_s - F_n \end{Bmatrix} \tag{370}$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 4x4 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\left\{\begin{matrix} u_{(1)} = 0 \\ \theta_{(1)} = 0 \\ K_{b2}\theta'_{(0)} = 0 \\ (K_{s1} + K_{s2})u'_{(0)} - K_{s2}\theta_{(0)} = 0 \end{matrix}\right\} \rightarrow \left\{\begin{matrix} u_1(h_1) = 0 \\ \theta_1(h_1) = 0 \\ M_{rn\,(0)} = 0 \\ V_{n\,(0)} = 0 \end{matrix}\right\} \tag{371}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} \tag{372}$$

Solving for bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{3,3} & t_{3,5} \\ t_{4,3} & t_{4,5} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} + \begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix}$$

$$\begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} = -\begin{bmatrix} t_{3,3} & t_{3,5} \\ t_{4,3} & t_{4,5} \end{bmatrix}^{-1} \begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix}$$

Substituting the internal forces we get the displacement and rotation at the top:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \end{Bmatrix} = -\begin{bmatrix} t_{1,1} & t_{1,2} \\ t_{2,1} & t_{2,2} \end{bmatrix} \begin{bmatrix} t_{3,3} & t_{3,5} \\ t_{4,3} & t_{4,5} \end{bmatrix}^{-1} \begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{373}$$

4.1.11 Generalized Parallel Coupling of Two Beams and Three Field (GCTB)

The 3-field GCTB beam is developed, which considers that the structure consists of a parallel coupling of an extensible Timoshenko beam (Timoshenko beam due to the effect of the shear walls and extensibility due to the axial extensibility of the shear walls) and a rotation restraint beam (due to the continuous core resulting from the presence of the connecting beam). The beams are assumed to be connected in parallel by means of axially rigid members that only transmit the horizontal forces and do not deform.

Moghadasi (2015) proposed this novel replacement beam for lateral deflection analysis of coupled shear walls subjected to uniformly distributed loading and then used a one-dimensional FEM formulation for dynamic analysis.

The stiffness of a coupled shear wall is strongly influenced by the properties of the connecting beams: if there are connecting beams, the total stiffness of the system exceeds the sum of the individual shear wall stiffnesses.

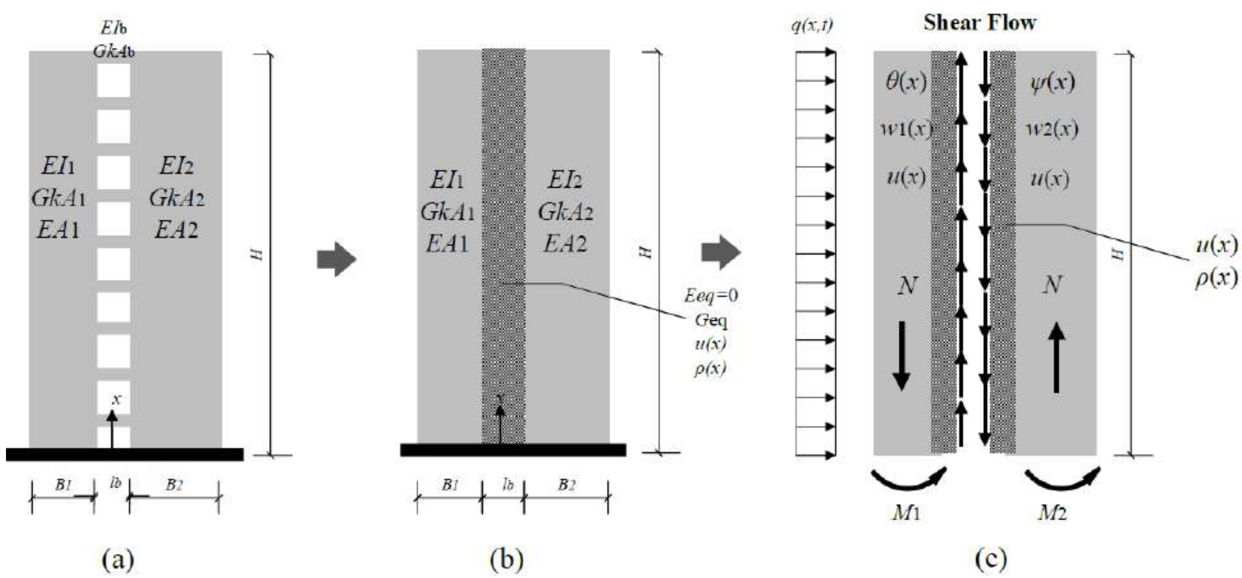


Figure 68. a) Coupled shear wall, (b) equivalent continuous model and (c) force balance and consistent kinematic fields (Moghadasi, 2015).

To obtain the equivalent stiffness of the rotational restraining beam, it is necessary to model the connecting beam as a Timoshenko beam and equate the potential energy of a connecting beam and its continuous equivalent. The connection beam was modeled as a Timoshenko beam to take into account the bending and shear stiffness (not negligible in deep beams).

Potential energy of the connecting beam:

$$V_b = \frac{1}{2}\left\{\int_0^{l_b}\left[\frac{M_{(x)}^2}{EI_b} + \frac{V_{(x)}^2}{kGA_b}\right]dx + 2\frac{M_0^2}{K_\theta}\right\} = \frac{V^2 l_b}{2}\left[\frac{{l_b}^2}{12EI_b} + \frac{1}{kGA_b} + \frac{l_b}{2K_\theta}\right] \tag{374}$$

Where $M_0 = V\frac{l_b}{2}$ is the bending moment at the ends.

Potential energy of the continuous equivalent, according to Capuani (1994):

$$V_{eq} = \frac{V^2 l_b}{2}\frac{1}{G_{eq} t_b h} \tag{375}$$

Equating both potential energies, we get:

$$G_{eq} = \frac{1}{t_b h}\left(\frac{{l_b}^2}{12EI_b} + \frac{1}{kGA_b} + \frac{l_b}{2K_\theta}\right)^{-1} \tag{376}$$

It is very complicated to establish a closed equation taking into account the rotational effect at the ends of the connecting beam. Capuani (1994) suggests:

$$K_\theta = \left\{\pi + \frac{1}{\pi}[log(3-4\nu)]^2\right\}\frac{Eh^2 t_b}{16} \tag{377}$$

Moghadasi (2015) proposes to take into account the rotational effect at the ends of the connecting beam by adding an additional length:

$$l'_b = l_b\left(1 + \mu\frac{h_b}{l_b}\right) \approx l_b\left(1 + 0.50\frac{h_b}{l_b}\right) \tag{378}$$

In this research project, the proposal of Moghadasi (2015) will be taken into account. The equivalent stiffness of the restraining beam to rotation is equal to:

$$G_{eq} = \frac{1}{t_b h}\left(\frac{{l'_b}^2}{12EI_b} + \frac{1}{kGA_b}\right)^{-1} \tag{379}$$

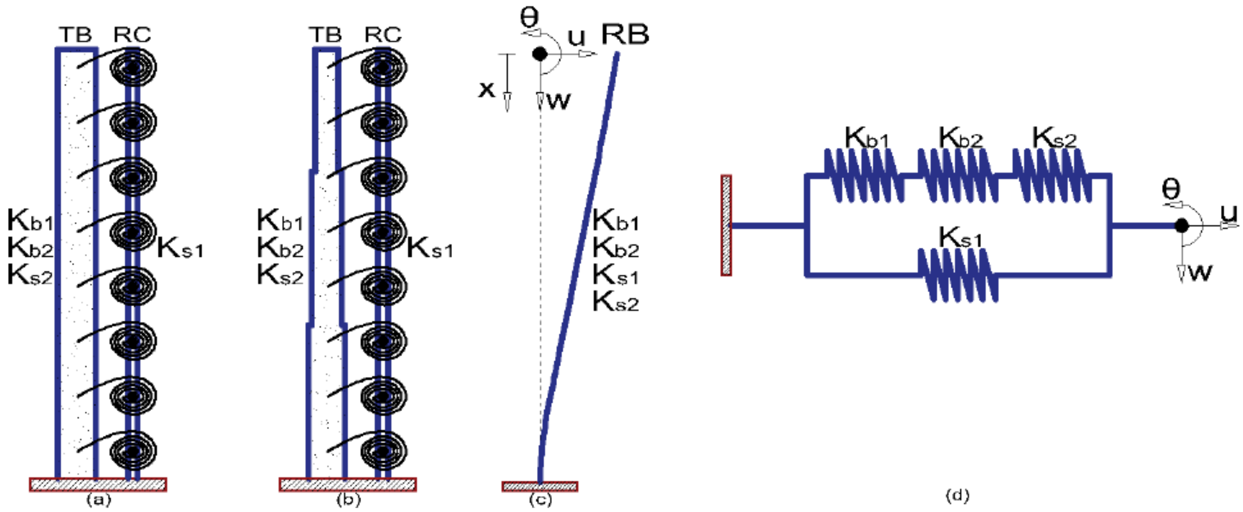


Figure 69. Three-field CTB beam. a) Case 1, b) Case 2 and c) Equivalent RB and d) Idealization of the three-field CTB stiffness

The 3-field GCTB beam model takes into account three kinematic fields: a transverse motion u and a rotational motion θ and an axial extensibility w.

The basic hypotheses of the model are:

- Shear walls are in plane tension.
- Shear walls have a rigid cross section.
- The connecting beams are axially inextensible.
- The rotational fields at each connecting wall are assumed to be the same.

As shown by Moghadasi (2015) this last condition is verified when the following relation is fulfilled:

$$0.25 \leq \frac{B_1}{B_2} \leq 4 \qquad (380)$$

Where B_1 and B_2 are the widths of the left and right shear walls, respectively. The rotation compatibility between the shear walls and the equivalent core is evaluated by means of the parameter ρ:

$$\rho = -\frac{B_1\theta_{(x)} + B_2\varphi_{(x)}}{2l_b} \rightarrow \rho = -\frac{(B_1 + B_1)}{2l_b}\theta_{(x)} \qquad (381)$$

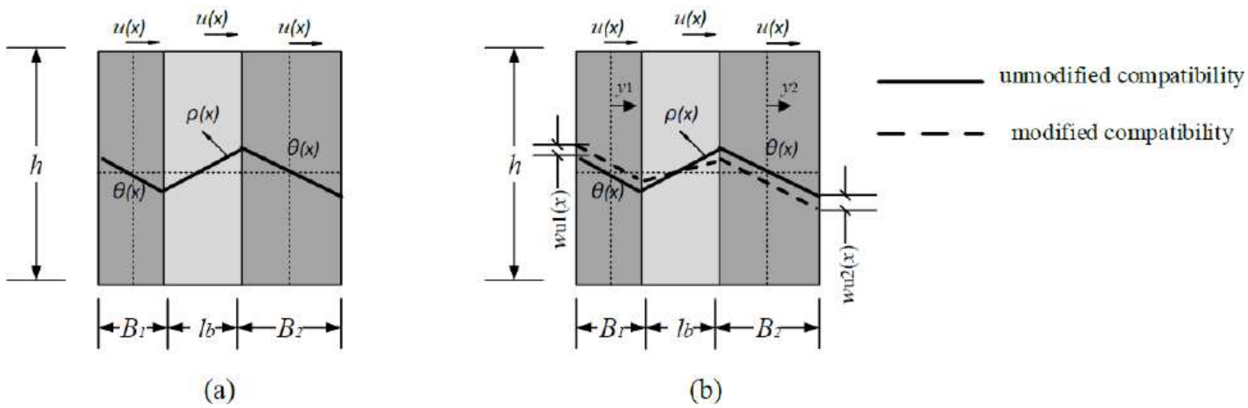


Figure 70. (a) Unmodified and (b) modified rotation compatibility with axial extensibility in a typical portion of the continuum model Moghadasi (2015).

The shear flow acting vertically inside the core generates an axial extensibility that is distributed along the height. The local kinematic fields are established:

$$\begin{Bmatrix} w_1(x) = w_{u1}(x) + y_1\theta_{(x)} \\ w_2(x) = w_{u2}(x) + y_2\theta_{(x)} \\ \gamma_c(x) = u'_{(x)} + \rho\theta_{(x)} - \dfrac{w_{u1}(x) + w_{u2}(x)}{l_b} w_{(x)} \end{Bmatrix} \tag{382}$$

Since the shear flow generates a secondary rotation in the shear wall, an additional condition can be established between both kinematic fields:

$$w_1 = -\frac{A_2}{A_1} w_2 \rightarrow \begin{Bmatrix} w_1 = -\dfrac{A_2}{A_1} w \\ w_2 = w \end{Bmatrix} \tag{383}$$

Taking this condition into account, the local kinematic fields are:

$$\begin{Bmatrix} w_1(x) = -\dfrac{A_2}{A_1} w_{(x)} + y_1\theta_{(x)} \\ w_2(x) = -w_{(x)} + y_2\theta_{(x)} \\ \gamma_c(x) = u'_{(x)} + \dfrac{B_1 + B_1}{2l_b}\theta_{(x)} - \dfrac{\frac{A_2}{A_1} + 1}{l_b} w_{(x)} \end{Bmatrix} \tag{384}$$

4.1.11.1 Case 1

The potential energy of the three-field GCTB model is expressed as:

$$V = \frac{1}{2}\int_0^H \left\{K_{b1}{w'_{(x)}}^2 + K_{b2}{\theta'_{(x)}}^2 + K_{s2}\left[u'_{(x)} - \theta_{(x)}\right]^2\right\} dx + \frac{1}{2}\int_0^H K_{s1}\gamma_c(x)^2\, dx \tag{385}$$

Where:

$$\left\{K_{b1} = E\left(A_2 + \frac{A_2^2}{A_1}\right), K_{b2} = E(I_1 + I_2), K_{s1} = G_{eq}t_b l_b, K_{s2} = G\kappa(A_1 + A_2)\right\} \tag{386}$$

Denoting:

$$\left\{\gamma_c = u'_{(x)} + m\theta_{(x)} - nw_{(x)}, m = \frac{B_1 + B_2}{2l_b}, n = \frac{1 + \frac{A_2}{A_1}}{l_b}\right\} \tag{387}$$

Rewriting:

$$V = \frac{1}{2}\int_0^H \left\{K_{b1}{w'_{(x)}}^2 + K_{b2}{\theta'_{(x)}}^2 + K_{s2}\left[u'_{(x)} - \theta_{(x)}\right]^2\right\} dx + \frac{1}{2}\int_0^H K_{s1}\left[u'_{(x)} + m\theta_{(x)} - nw_{(x)}\right]^2 dx \tag{388}$$

The work done by the external force is:

$$W = \int_0^H f_{(x)}u_{(x)}\, dx \tag{389}$$

Consequently, the total potential energy of the 3-field beam GCTB subjected to a general lateral load distribution is expressed as:

$$\begin{aligned}\mathcal{U} = \frac{1}{2}\int_0^H &\left\{K_{b1}{w'_{(x)}}^2 + K_{b2}{\theta'_{(x)}}^2 + K_{s2}\left[u'_{(x)} - \theta_{(x)}\right]^2\right\} dx \\ &+ \frac{1}{2}\int_0^H K_{s1}\left[u'_{(x)} + m\theta_{(x)} - nw_{(x)}\right]^2 dx - \int_0^H f_{(x)}u_{(x)}\, dx\end{aligned} \tag{390}$$

Closed-form solutions of the model acted on by a transverse load are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\delta \mathcal{U} = \int_0^H \{K_{b1} w'_{(x)} \delta w'_{(x)} + K_{b2} \theta'_{(x)} \delta \theta'_{(x)} + K_{s2} [u'_{(x)} - \theta_{(x)}][\delta u'_{(x)} - \delta \theta_{(x)}]\} dx$$
$$+ \int_0^H K_{s1} [u'_{(x)} + m\theta_{(x)} - n w_{(x)}][\delta u'_{(x)} + m\delta\theta_{(x)} - n\delta w_{(x)}] dx$$
$$- \int_0^H f(x) \delta u_{(x)}\, dx - \int_0^H u_{(x)}\, \delta f(x) dx \tag{391}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\delta \mathcal{U} = K_{b1} [w'_{(x)} \delta w_{(x)}]_0^H + K_{b2} [\theta'_{(x)} \delta \theta_{(x)}]_0^H + \{[(K_{s1} + K_{s2}) u'_{(x)} - (K_{s2} - mK_{s1}) \theta_{(x)} - nK_{s1} w_{(x)}] \delta \theta_{(x)}\}_0^H$$
$$+ \int_0^H \{-K_{b1} w''_{(x)} - nK_{s1} u'_{(x)} - mnK_{s1} \theta_{(x)} + n^2 K_{s1} w_{(x)}\} \delta w_{(x)}$$
$$+ \int_0^H \{-K_{b2} \theta''_{(x)} - (K_{s2} - mK_{s1}) u'_{(x)} + (K_{s2} + m^2 K_{s1}) \theta_{(x)} - mnK_{s1} w_{(x)}\} \delta \theta_{(x)}$$
$$+ \int_0^H \{-(K_{s1} + K_{s2}) u''_{(x)} + (K_{s2} - mK_{s1}) \theta'_{(x)} + nK_{s1} w'_{(x)} - f(x)\} \delta u_{(x)} - \int_0^H u_{(x)}\, \delta f(x) dx \tag{392}$$

Setting the terms equal to zero, the following equations result:

$$\begin{Bmatrix} -K_{b1} w''_{(x)} - nK_{s1} u'_{(x)} - mnK_{s1} \theta_{(x)} + n^2 K_{s1} w_{(x)} = 0 \\ -K_{b2} \theta''_{(x)} - (K_{s2} - mK_{s1}) u'_{(x)} + (K_{s2} + m^2 K_{s1}) \theta_{(x)} - mnK_{s1} w_{(x)} = 0 \\ -(K_{s1} + K_{s2}) u''_{(x)} + (K_{s2} - mK_{s1}) \theta'_{(x)} + nK_{s1} w'_{(x)} - f(x) = 0 \end{Bmatrix} \tag{393}$$

And boundary conditions:

$$\begin{Bmatrix} \theta'_{(0)} = 0 \\ w'_{(0)} = 0 \\ (K_{s1} + K_{s2}) u'_{(0)} - (K_{s2} - mK_{s1}) \theta_{(0)} - nK_{s1} w_{(0)} = 0 \end{Bmatrix} \tag{394}$$

Using the method of coefficients for the solution of the system of equations:

$$\begin{Bmatrix} u_{(x)} \\ w_{(x)} \\ \theta_{(x)} \end{Bmatrix} = \begin{bmatrix} -nK_{s1} D & -K_{b1} D^2 + n^2 K_{s1} & -mnK_{s1} \\ -(K_{s2} - mK_{s1}) D & -mnK_{s1} & -K_{b2} D^2 + (K_{s2} + m^2 K_{s1}) \\ -(K_{s1} + K_{s2}) D^2 & nK_{s1} D & (K_{s2} - mK_{s1}) D \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 0 \\ f_{(x)} \end{Bmatrix} \tag{395}$$

i.e.,

$$\begin{Bmatrix} u_{(x)}'''''' - \dfrac{K_{s1}K_{s2}[n^2K_{b2} + (m+1)^2K_{b1}]}{K_{b1}K_{b2}(K_{s1}+K_{s2})} u_{(x)}'''' \\ w_{(x)}'''''' - \dfrac{K_{s1}K_{s2}[n^2K_{b2} + (m+1)^2K_{b1}]}{K_{b1}K_{b2}(K_{s1}+K_{s2})} w_{(x)}'''' \\ \theta_{(x)}'''''' - \dfrac{K_{s1}K_{s2}[n^2K_{b2} + (m+1)^2K_{b1}]}{K_{b1}K_{b2}(K_{s1}+K_{s2})} \theta_{(x)}'''' \end{Bmatrix}$$
$$= \begin{Bmatrix} -\dfrac{1}{(K_{s1}+K_{s2})} f_{(x)}'''' + \dfrac{K_{b1}(K_{s2}+m^2K_{s1}) + n^2K_{s1}K_{b2}}{K_{b1}K_{b2}(K_{s1}+K_{s2})} f_{(x)}'' - \dfrac{n^2K_{s1}K_{s2}}{K_{b1}K_{b2}(K_{s1}+K_{s2})} f_{(x)} \\ \dfrac{nK_{s1}}{K_{b1}(K_{s1}+K_{s2})} f_{(x)}''' - \dfrac{K_{s1}K_{s2}n(m+1)}{K_{b1}K_{b2}(K_{s1}+K_{s2})} f_{(x)}' \\ \dfrac{(K_{s2}-mK_{s1})}{K_{b2}(K_{s1}+K_{s2})} f_{(x)}''' - \dfrac{K_{s1}K_{s2}n^2}{K_{b1}K_{b2}(K_{s1}+K_{s2})} f_{(x)}' \end{Bmatrix} \tag{396}$$

A sixth order differential equation is obtained. Normalizing the differential equation by the variable $z = x/H$:

$$u_{(z)}'''''' - \frac{K_{s1}K_{s2}[n^2K_{b2} + (m+1)^2K_{b1}]}{K_{b1}K_{b2}(K_{s1}+K_{s2})} H^2 u_{(z)}'''' = -\frac{H^2}{(K_{s1}+K_{s2})} f_{(z)}'''' + \frac{K_{b1}(K_{s2}+m^2K_{s1}) + n^2K_{s1}K_{b2}}{K_{b1}K_{b2}(K_{s1}+K_{s2})} H^4 f_{(z)}'' - \frac{n^2K_{s1}K_{s2}}{K_{b1}K_{b2}(K_{s1}+K_{s2})} H^6 f_{(z)} \tag{397}$$

Defining six parameters:

$$\begin{Bmatrix} \alpha = H\sqrt{\dfrac{K_{s1}K_{s2}}{K_{b2}(K_{s1}+K_{s2})}}, \kappa = \sqrt{(m+1)^2 + n^2\dfrac{K_{b2}}{K_{b1}}} \\ \eta_w = H\sqrt{\dfrac{K_{s1}}{K_{b1}}}, \eta_\theta = H\sqrt{\dfrac{K_{s2}}{K_{b2}}}, \eta_\phi = H\sqrt{\dfrac{K_{s1}}{K_{b2}}}, \lambda = \dfrac{W_{max}H^2}{(K_{s1}+K_{s2})(1-e^{-a})} \end{Bmatrix} \tag{398}$$

Replacing the first six parameters:

$$u_{(z)}'''''' - (\alpha\kappa)^2 u_{(z)}'''' = \frac{H^2}{(K_{s1}+K_{s2})}\left[-f_{(z)}'''' + \left(\eta_\theta^2 + m^2\eta_\phi^2 + n^2\eta_w^2\right) f_{(z)}'' - \left(n^2\eta_w^2\eta_\theta^2\right) f_{(z)}\right] \tag{399}$$

Assuming a general lateral load (Miranda E. , 1999):

$$f_{(x)} = \frac{W_{max}}{1-e^{-a}}\left(1 - e^{-a+a\frac{x}{H}}\right) \rightarrow f_{(z)} = \frac{W_{max}}{1-e^{-a}}(1 - e^{-a+az}) \tag{400}$$

Replacing the lateral load and introducing the sixth parameter:

$$u_{(z)}'''''' - (\alpha\kappa)^2 u_{(z)}'''' = -n^2\eta_w^2\eta_\theta^2\lambda + \left[a^4 - \left(\eta_\theta^2 + m^2\eta_\phi^2 + n^2\eta_w^2\right)a^2 + n^2\eta_w^2\eta_\theta^2\right]\lambda \tag{401}$$

The expression for $u_{(z)}, w_{(z)}$ and $\theta_{(z)}$ is proposed:

$$\begin{Bmatrix} u_{(z)} = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + C_4\cosh(\alpha\kappa z) + C_5\sinh(\alpha\kappa z) + C_6 z^4 + C_7 e^{-a+az} \\ w_{(z)} = C_8 + C_9 z + C_{10} z^2 + C_{11}\cosh(\alpha\kappa z) + C_{12}\sinh(\alpha\kappa z) + C_{13} z^3 + C_{14} e^{-a+az} \\ \theta_{(z)} = C_{15} + C_{16} z + C_{17} z^2 + C_{18}\cosh(\alpha\kappa z) + C_{19}\sinh(\alpha\kappa z) + C_{20} z^3 + C_{21} e^{-a+az} \end{Bmatrix} \tag{402}$$

Where:

$$\begin{Bmatrix} C_6 = \dfrac{n^2\eta_w^2\eta_\theta^2}{24(\alpha\kappa)^2} \\ C_7 = \dfrac{a^4 - \left(\eta_\theta^2 + m^2\eta_\phi^2 + n^2\eta_w^2\right)a^2 + n^2\eta_w^2\eta_\theta^2}{a^4[a^2 - (\alpha\kappa)^2]} \end{Bmatrix} \tag{403}$$

Expressing $w_{(z)}$ and $\theta_{(z)}$ as a function of the coefficients of $u_{(z)}$:

$$\begin{Bmatrix} w_{(z)} = (p_3)C_1 + (2p_3 z)C_2 + (p_8 + 3p_3 z^2)C_3 + [p_6(\alpha\kappa)\sinh(\alpha\kappa z)]C_4 \\ +[p_6(\alpha\kappa)\cosh(\alpha\kappa z)]C_5 + (p_4 z + 4p_3 z^3)C_6 + ap_1 e^{-a+az}C_7 \end{Bmatrix}$$

$$\begin{Bmatrix} \theta_{(z)} = C_1 + (2z)C_2 + (p_9 + 3z^2)C_3 + [p_7(\alpha\kappa)\sinh(\alpha\kappa z)]C_4 \\ +[p_7(\alpha\kappa)\cosh(\alpha\kappa z)]C_5 + (p_5 z + 4z^3)C_6 + ap_2 e^{-a+az}C_7 \end{Bmatrix} \tag{404}$$

Where:

$$\begin{Bmatrix} p_1 = \dfrac{n\eta_w^2\left[(m+1)\eta_\theta^2 - a^2\right]}{a^4 - \left(n^2\eta_w^2 + \eta_\theta^2 + m^2\eta_\phi^2\right)a^2 + n^2\eta_w^2\eta_\theta^2}, p_2 = \dfrac{n\eta_w^2\eta_\theta^2 - \left(\eta_\theta^2 - m\eta_\phi^2\right)a^2}{a^4 - \left(n^2\eta_w^2 + \eta_\theta^2 + m^2\eta_\phi^2\right)a^2 + n^2\eta_w^2\eta_\theta^2} \\ p_3 = \dfrac{m+1}{n}, p_4 = \dfrac{24\left[p_3\left(\eta_\theta^2 + m^2\eta_\phi^2\right) + mn\eta_w^2\right]}{n^2\eta_w^2\eta_\theta^2}, p_5 = \dfrac{24\left[p_3 m\eta_\phi^2 + n\eta_w^2\right]}{n\eta_w^2\eta_\theta^2} \\ p_6 = \dfrac{n\eta_w^2\left[(m+1)\eta_\theta^2 - (\alpha\kappa)^2\right]}{a^4 - \left(n^2\eta_w^2 + \eta_\theta^2 + m^2\eta_\phi^2\right)a^2 + n^2\eta_w^2\eta_\theta^2}, p_7 = \dfrac{n\eta_w^2\eta_\theta^2 - \left(\eta_\theta^2 - m\eta_\phi^2\right)(\alpha\kappa)^2}{a^4 - \left(n^2\eta_w^2 + \eta_\theta^2 + m^2\eta_\phi^2\right)a^2 + n^2\eta_w^2\eta_\theta^2} \\ p_8 = \dfrac{6\left[p_3\left(\eta_\theta^2 + m^2\eta_\phi^2\right) + mn\eta_w^2\right]}{n\eta_w^2\eta_\theta^2}, p_9 = \dfrac{6\left[p_3 m\eta_\phi^2 + n\eta_w^2\right]}{n\eta_w^2\eta_\theta^2}, \end{Bmatrix} \tag{405}$$

The constants are obtained by evaluating the relevant boundary conditions (the origin of x is at the base of the model). The constants:

- We evaluate two boundary conditions at the top:

$$\begin{Bmatrix} \theta'_{(0)} = 0 \\ w'_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} C_2 \\ C_4 \end{Bmatrix} = -\begin{bmatrix} 2p_3 & p_6(\alpha\kappa)^2 \\ 2 & p_7(\alpha\kappa)^2 \end{bmatrix}^{-1} \cdot \left[\begin{Bmatrix} p_4 \\ p_5 \end{Bmatrix} C_6 + \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} a^2 e^{-a} C_7\right] \tag{406}$$

- We evaluate three boundary conditions at the base:

$$\begin{Bmatrix} \theta_{(1)} = 0 \\ w_{(1)} = 0 \\ (K_{s1} + K_{s2})u'_{(0)} - (K_{s2} - mK_{s1})\theta_{(0)} - nK_{s1}w_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} \theta_{(1)} = 0 \\ w_{(1)} = 0 \\ V_{(0)} = 0 \end{Bmatrix} \tag{407}$$

i.e.,

$$\begin{Bmatrix} C_1 \\ C_3 \\ C_5 \end{Bmatrix} = -\begin{bmatrix} 1 & p_9 + 3 & p_7(\alpha\kappa)\cosh(\alpha\kappa) \\ p_3 & p_8 + 3p_3 & p_6(\alpha\kappa)\cosh(\alpha\kappa) \\ 0 & K_{s1}(mp_9 - np_8) - K_{s2}p_9 & (\alpha\kappa)[K_{s1}(1 + mp_7 - np_6) - K_{s2}(1 - p_7)] \end{bmatrix}^{-1} x$$

$$\begin{Bmatrix} 2C_2 + p_7(\alpha\kappa)\sinh(\alpha\kappa)\,C_4 + (p_5 + 4)C_6 + p_2 aC_7 \\ 2p_3C_2 + p_6(\alpha\kappa)\sinh(\alpha\kappa)\,C_4 + (p_4 + 4p_3)C_6 + p_1 aC_7 \\ [K_{s1}(1 + mp_2 - np_1) + K_{s2}(1 - p_2)]ae^{-a}C_7 \end{Bmatrix} \tag{408}$$

- We evaluate a zero displacement in the base:

$$\{u_{(1)} = 0\} \rightarrow \{C_0 = -(C_1 + C_2 + C_3) - [C_4\cosh(\alpha\kappa) + C_5\sinh(\alpha\kappa)] - (C_6 + C_7)\} \tag{409}$$

4.1.11.2 Case 2

- **Calculation of the Transfer Matrix**

According to the coupled differential equations and assuming that the external loads act on the stories and not along the height of the story, it is possible to write it as follows:

$$\begin{Bmatrix} -K_{b1}w''_{(z)} - nK_{s1}u'_{(z)} - mnK_{s1}\theta_{(z)} + n^2K_{s1}w_{(z)} = 0 \\ -K_{b2}\theta''_{(z)} - (K_{s2} - mK_{s1})u'_{(z)} + (K_{s2} + m^2K_{s1})\theta_{(z)} - mnK_{s1}w_{(z)} = 0 \\ -(K_{s1} + K_{s2})u''_{(z)} + (K_{s2} - mK_{s1})\theta'_{(z)} + nK_{s1}w'_{(z)} = 0 \end{Bmatrix} \tag{410}$$

The expression for $u_{(z)}, w_{(z)}$ and $\theta_{(z)}$ is proposed:

$$\left\{\begin{array}{c} u_{(z)} = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + C_4 \cosh(\alpha^* \kappa z) + C_5 \sinh(\alpha^* \kappa z) \\ w_{(z)} = C_8 + C_9 z + C_{10} z^2 + C_{11} \cosh(\alpha^* \kappa z) + C_{12} \sinh(\alpha^* \kappa z) \\ \theta_{(z)} = C_{15} + C_{16} z + C_{17} z^2 + C_{18} \cosh(\alpha^* \kappa z) + C_{19} \sinh(\alpha^* \kappa z) \end{array}\right\} \tag{411}$$

Defining five parameters:

$$\left\{\begin{array}{c} \alpha = \sqrt{\dfrac{K_{s1} K_{s2}}{K_{b2}(K_{s1} + K_{s2})}}, \kappa = \sqrt{(m+1)^2 + n^2 \dfrac{K_{b2}}{K_{b1}}} \\ \eta_w = \sqrt{\dfrac{K_{s1}}{K_{b1}}}, \eta_\theta = \sqrt{\dfrac{K_{s2}}{K_{b2}}}, \eta_\phi = \sqrt{\dfrac{K_{s1}}{K_{b2}}} \end{array}\right\} \tag{412}$$

Expressing the coefficients of the function $w_{(z)}$ and $\theta_{(z)}$ as a function of the coefficients of $u_{(z)}$:

$$\left\{\begin{array}{c} u_{(z)} = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + C_4 \cosh(\alpha^* \kappa z) + C_5 \sinh(\alpha^* \kappa z) \\ w_{(z)} = (p_3) C_1 + (2 p_3 z) C_2 + (p_8 + 3 p_3 z^2) C_3 + [p_6 (\alpha^* \kappa) \sinh(\alpha^* \kappa z)] C_4 + [p_6 (\alpha^* \kappa) \cosh(\alpha^* \kappa z)] C_5 \\ \theta_{(z)} = C_1 + (2z) C_2 + (p_9 + 3 z^2) C_3 + [p_7 (\alpha^* \kappa) \sinh(\alpha^* \kappa z)] C_4 + [p_7 (\alpha^* \kappa) \cosh(\alpha^* \kappa z)] C_5 \end{array}\right\} \tag{413}$$

Where:

$$\left\{\begin{array}{c} p_6 = \dfrac{n \eta_w^2 [(m+1) \eta_\theta^2 - (\alpha \kappa)^2]}{a^4 - (n^2 \eta_w^2 + \eta_\theta^2 + m^2 \eta_\phi^2) a^2 + n^2 \eta_w^2 \eta_\theta^2}, p_7 = \dfrac{n \eta_w^2 \eta_\theta^2 - (\eta_\theta^2 - m \eta_\phi^2)(\alpha \kappa)^2}{a^4 - (n^2 \eta_w^2 + \eta_\theta^2 + m^2 \eta_\phi^2) a^2 + n^2 \eta_w^2 \eta_\theta^2} \\ p_3 = \dfrac{m+1}{n}, p_8 = \dfrac{6[p_3 (\eta_\theta^2 + m^2 \eta_\phi^2) + m n \eta_w^2]}{n \eta_w^2 \eta_\theta^2}, p_9 = \dfrac{6[p_3 m \eta_\phi^2 + n \eta_w^2]}{n \eta_w^2 \eta_\theta^2}, \end{array}\right\}$$

The bending moment and the shear force associated with the lateral displacement result:

$$\left\{\begin{array}{c} M_1 = K_{b1} w'_{(x)} = (2 p_3 K_{b1}) C_2 + (6 p_3 K_{b1} \mathrm{z}) C_3 + [p_6 K_{b1} (\alpha^* \kappa)^2 \cosh(\alpha^* \kappa z)] C_4 \\ + [p_6 K_{b1} (\alpha^* \kappa)^2 \sinh(\alpha^* \kappa z)] C_5 \end{array}\right\}$$

$$\left\{\begin{array}{c} M_2 = K_{b2} \theta'_{(x)} = (2 K_{b2}) C_2 + (6 K_{b2} \mathrm{z}) C_3 + [p_7 K_{b2} (\alpha^* \kappa)^2 \cosh(\alpha^* \kappa z)] C_4 \\ + [p_7 K_{b2} (\alpha^* \kappa)^2 \sinh(\alpha^* \kappa z)] C_5 \end{array}\right\}$$

$$\left\{\begin{array}{c} V = (K_{s1} + K_{s2}) u'_{(x)} - (K_{s2} - m K_{s1}) \theta_{(x)} - n K_{s1} w_{(x)} = p_{10} C_1 + (2 p_{10} z) C_2 \\ + (p_{11} + 3 p_{10} z^2) C_3 + [p_{12} (\alpha^* \kappa) \sinh(\alpha^* \kappa z)] C_4 + [p_{12} (\alpha^* \kappa) \cosh(\alpha^* \kappa z)] C_5 \end{array}\right\} \tag{414}$$

Where:

$$\begin{Bmatrix} p_{10} = (1 + m - np_3)K_{s1}, p_{11} = (mp_9 - np_8)K_{s1} - p_9 K_{s2} \\ p_{12} = (1 + mp_7 - np_6)K_{s1} - (1 - p_7)K_{s2} \end{Bmatrix} \tag{415}$$

Writing the equations in matrix form:

$$\begin{Bmatrix} u_i(z_i) \\ w_i(z_i) \\ \theta_i(z_i) \\ M_{1i}(z_i) \\ M_{2i}(z_i) \\ V_i(z_i) \end{Bmatrix} = K_i(z_i) \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{Bmatrix} \tag{416}$$

Where:

$$K_i(z_i) = \begin{bmatrix} 1 & z & z^2 & z^3 & \cosh(\alpha^*\kappa z) & \sinh(\alpha^*\kappa z) \\ 0 & p_3 & 2p_3 z & p_8 + 3p_3 z^2 & p_6(\alpha^*\kappa)\sinh(\alpha^*\kappa z) & p_6(\alpha^*\kappa)\cosh(\alpha^*\kappa z) \\ 0 & 1 & 2 & p_9 + 3z^2 & p_7(\alpha^*\kappa)\sinh(\alpha^*\kappa z) & p_7(\alpha^*\kappa)\cosh(\alpha^*\kappa z) \\ 0 & 0 & 2p_3 K_{b1} & 6p_3 K_{b1}\mathrm{z} & p_6 K_{b1}(\alpha^*\kappa)^2\cosh(\alpha^*\kappa z) & p_6 K_{b1}(\alpha^*\kappa)^2\sinh(\alpha^*\kappa z) \\ 0 & 0 & 2K_{b2} & 6K_{b2}\mathrm{z} & p_7 K_{b2}(\alpha^*\kappa)^2\cosh(\alpha^*\kappa z) & p_7 K_{b2}(\alpha^*\kappa)^2\sinh(\alpha^*\kappa z) \\ 0 & p_{10} & 2p_{10}z & p_{11} + 3p_{10}z^2 & p_{12}(\alpha^*\kappa)\sinh(\alpha^*\kappa z) & p_{12}(\alpha^*\kappa)\cosh(\alpha^*\kappa z) \end{bmatrix} \tag{417}$$

- **Static Analysis Under Point Static Loads Applied at Floor Level**

The relationship between forces and displacements between two consecutive floors is obtained by taking into account the transfer matrix and the vector of external point forces.

$$\begin{Bmatrix} u_n(0) \\ w_n(0) \\ \theta_n(0) \\ M_{1n}(0) \\ M_{2n}(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_k(0) \begin{Bmatrix} u_1(h_i) \\ w_1(h_i) \\ \theta_1(h_i) \\ M_{11}(h_i) \\ M_{21}(h_i) \\ V_1(h_i) \end{Bmatrix} - \sum_{s=0}^{n}\left[\prod_{k=s+1}^{n} T_k(0)\right] F_s - F_n \tag{418}$$

Expressing it in simplified form:

$$\begin{Bmatrix} u_n(0) \\ w_n(0) \\ \theta_n(0) \\ M_{1n}(0) \\ M_{2n}(0) \\ V_n(0) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_i) \\ w_1(h_i) \\ \theta_1(h_i) \\ M_{11}(h_i) \\ M_{21}(h_i) \\ V_1(h_i) \end{Bmatrix} + f \tag{419}$$

Where:

$$\begin{Bmatrix} t = \prod_{k=1}^{n} T_k(0) \\ f = -\sum_{s=0}^{n} \left[\prod_{k=s+1}^{n} T_k(0) \right] F_s - F_n \end{Bmatrix} \tag{420}$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 6x6 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ w_{(1)} = 0 \\ \theta_{(1)} = 0 \\ w'_{(0)} = 0 \\ \theta'_{(0)} = 0 \\ (K_{s1} + K_{s2})u'_{(0)} - (K_{s2} - mK_{s1})\theta_{(0)} - nK_{s1}w_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1(h_1) = 0 \\ w_1(h_1) = 0 \\ \theta_1(h_1) = 0 \\ M_{1n\,(0)} = 0 \\ M_{2n\,(0)} = 0 \\ V_{n\,(0)} = 0 \end{Bmatrix} \tag{421}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ w_n(0) \\ \theta_n(0) \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & t_{1,5} & t_{1,6} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} & t_{2,5} & t_{2,6} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} & t_{3,5} & t_{3,6} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,1} & t_{5,2} & t_{5,3} & t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,1} & t_{6,2} & t_{6,3} & t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ M_{11}(h_1) \\ M_{21}(h_1) \\ V_1(h_1) \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{Bmatrix} \tag{422}$$

Solving for bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} M_{11}(h_1) \\ M_{21}(h_1) \\ V_1(h_1) \end{Bmatrix} + \begin{Bmatrix} f_4 \\ f_5 \\ f_6 \end{Bmatrix} \rightarrow \begin{Bmatrix} M_{11}(h_1) \\ M_{21}(h_1) \\ V_1(h_1) \end{Bmatrix} = -\begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix}^{-1} \begin{Bmatrix} f_4 \\ f_5 \\ f_6 \end{Bmatrix} \tag{423}$$

Substituting the internal forces we obtain the displacement, the axial strain and the rotation at the top of the beam:

$$\begin{Bmatrix} u_n(0) \\ w_n(0) \\ \theta_n(0) \end{Bmatrix} = -\begin{bmatrix} t_{1,4} & t_{1,5} & t_{1,6} \\ t_{2,4} & t_{2,5} & t_{2,6} \\ t_{3,4} & t_{3,5} & t_{3,6} \end{bmatrix} \begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix}^{-1} \begin{Bmatrix} f_4 \\ f_5 \\ f_6 \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} \tag{424}$$

4.1.12 Modified Generalized Parallel Coupling of Two Beams and Two Field (GCTB)

The two-field GCTB beam is developed, which considers that the structure consists of a parallel coupling of an extensible Euler Bernoulli beam (Bernoulli beam because it only considers the bending effect of the walls and extensibility due to the extensibility axial shear walls) and a rotation restraint beam (due to the continuous core resulting from the presence of the connecting beam). The beams are assumed to be connected in parallel by means of axially rigid members that only transmit the horizontal forces and do not deform. This two-field CTB is suitable for intermediate to high shear walls, where it is generally possible to neglect the effect of the shear stiffness of the shear walls.

The two-field beam GCTB results from ignoring the local shear strain and considering $u'_{(x)} = \theta_{(x)}$ in the three-field beam GCTB. The two-field GCTB beam model takes into account two kinematic fields: a transverse motion u and an axial extensibility w. Also; K_{b1}, K_{b2} and K_{s1} are the global bending stiffness, global shear stiffness, local bending stiffness and equivalent local shear stiffness of the connecting beams, respectively.

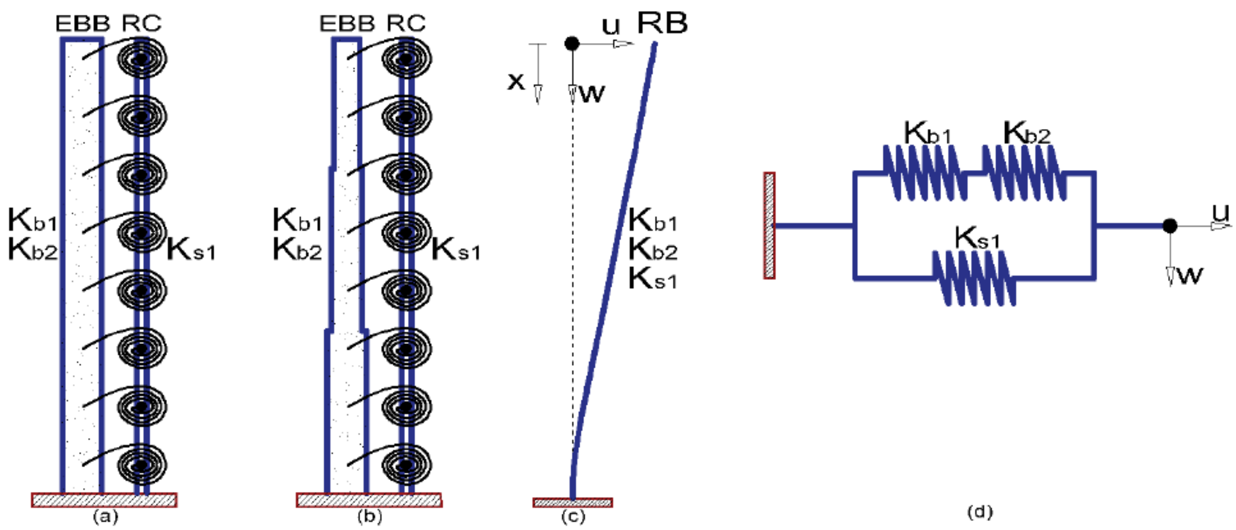


Figure 71. GCTB beam of two fields. a) Case 1, b) Case 2 and c) Equivalent RB and d) Idealization of the GCTB stiffness of two fields.

4.1.12.1 Case 1

The potential energy of the two-field GCTB model is expressed as:

$$V = \frac{1}{2}\int_0^H \left[K_{b1} {w'_{(x)}}^2 + K_{b2} {u''_{(x)}}^2\right] dx + \frac{1}{2}\int_0^H K_{s1}\gamma_c(x)^2\, dx \tag{425}$$

Where:

$$\left\{K_{b1} = E\left(A_2 + \frac{A_2^2}{A_1}\right), K_{b2} = E(I_1 + I_2), K_{s1} = G_{eq} t_w l_b\right\} \tag{426}$$

Denoting:

$$\left\{\gamma_c = (m+1)u'_{(x)} - n w_{(x)}, m = \frac{B_1 + B_2}{2 l_b}, n = \frac{1 + A_2/A_1}{l_b}\right\} \tag{427}$$

Rewriting:

$$V = \frac{1}{2}\int_0^H \left[K_{b1} {w'_{(x)}}^2 + K_{b2} {u''_{(x)}}^2\right] dx + \frac{1}{2}\int_0^H K_{s1}\left[(m+1)u'_{(x)} - n w_{(x)}\right]^2 dx \tag{428}$$

The work done by the external force is:

$$W = \int_0^H f_{(x)} u_{(x)}\, dx \tag{429}$$

Consequently, the total potential energy of the two-field beam GCTB subjected to a general lateral load distribution is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H \left[K_{b1} {w'_{(x)}}^2 + K_{b2} {u''_{(x)}}^2\right] dx + \frac{1}{2}\int_0^H K_{s1}\left[(m+1)u'_{(x)} - n w_{(x)}\right]^2 dx - \int_0^H f_{(x)} u_{(x)}\, dx \tag{430}$$

Closed-form solutions of the model acted on by a transverse load are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\delta\mathcal{U} = \int_0^H \{K_{b1} w'_{(x)} \delta w'_{(x)} + K_{b2} u''_{(x)} \delta u''_{(x)}\} dx$$
$$+ \int_0^H K_{s1}\left[(m+1)u'_{(x)} - n w_{(x)}\right]\left[(m+1)\delta u'_{(x)} - n\delta w_{(x)}\right] dx$$
$$- \int_0^H f(x)\delta u_{(x)}\, dx - \int_0^H u_{(x)}\, \delta f(x) dx \tag{431}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\delta\mathcal{U} = K_{b1}\left[w'_{(x)}\delta w_{(x)}\right]_0^H + K_{b2}\left[u''_{(x)}\delta u'_{(x)}\right]_0^H$$
$$+ \left\{\left[-K_{b2}u'''_{(x)} + (m+1)^2 K_{s1} u'_{(x)} - n(m+1)K_{s1} w_{(x)}\right]\delta u_{(x)}\right\}_0^H$$
$$+ \int_0^H \{-K_{b1} w''_{(x)} - n(m+1)K_{s1}u'_{(x)} + n^2 K_{s1} w_{(x)}\}\delta w_{(x)}$$
$$+ \int_0^H \{K_{b2} u''''_{(x)} - (m+1)^2 K_{s1} u''_{(x)} + n(m+1)K_{s1} w'_{(x)} - f(x)\}\delta u_{(x)}$$
$$- \int_0^H u_{(x)}\, \delta f(x) dx \tag{432}$$

Setting the terms equal to zero, the following equations result:

$$\begin{Bmatrix} -K_{b1} w''_{(x)} - n(m+1)K_{s1} u'_{(x)} + n^2 K_{s1} w_{(x)} = 0 \\ K_{b2} u''''_{(x)} - (m+1)^2 K_{s1} u''_{(x)} + n(m+1) K_{s1} w'_{(x)} - f(x) = 0 \end{Bmatrix} \tag{433}$$

And boundary conditions:

$$\begin{Bmatrix} w'_{(0)} = 0 \\ u''_{(0)} = 0 \\ -K_{b2} u'''_{(0)} + (m+1)K_{s1} u'_{(0)} - n K_{s1} w_{(0)} = 0 \end{Bmatrix} \tag{434}$$

Using the method of coefficients for the solution of the system of equations:

$$\begin{Bmatrix} u_{(x)} \\ w_{(x)} \end{Bmatrix} = \begin{bmatrix} -n(m+1)K_{s1}D & -K_{b1}D^2 + n^2 K_{s1} \\ K_{b2}D^4 - (m+1)^2 K_{s1} D^2 & n(m+1)K_{s1} D \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ f_{(x)} \end{Bmatrix} \tag{435}$$

i.e.,

$$\begin{Bmatrix} u'''''''_{(x)} - \dfrac{K_{s1}[n^2 K_{b2} + (m+1)^2 K_{b1}]}{K_{b1} K_{b2}} u''''_{(x)} \\ w'''''''_{(x)} - \dfrac{K_{s1}[n^2 K_{b2} + (m+1)^2 K_{b1}]}{K_{b1} K_{b2}} w''''_{(x)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{1}{K_{b2}} \left[f''_{(x)} - n^2 \dfrac{K_{s1}}{K_{b1}} f_{(x)} \right] \\ -n^2 \dfrac{K_{s1}}{K_{b1} K_{b2}} f'_{(x)} \end{Bmatrix} \tag{436}$$

A sixth order differential equation is obtained. Normalizing the differential equation by the variable $z = x/H$:

$$u'''''''_{(x)} - \frac{K_{s1}[n^2 K_{b2} + (m+1)^2 K_{b1}]}{K_{b1} K_{b2}} H^2 u''''_{(x)} = \frac{H^4}{K_{b2}} \left[f''_{(x)} - n^2 \frac{K_{s1}}{K_{b1}} H^2 f_{(x)} \right] \tag{437}$$

Defining three parameters:

$$\left\{ \alpha = H \sqrt{\frac{K_{s1}}{K_{b2}}}, \kappa = \sqrt{(m+1)^2 + n^2 \frac{K_{b2}}{K_{b1}}}, \lambda = \frac{W_{max} H^4}{K_{b2}(1 - e^{-a})} \right\} \tag{438}$$

Replacing the first two parameters:

$$u'''''''_{(z)} - (\alpha\kappa)^2 u''''_{(z)} = \frac{H^4}{K_{b2}} \{ f''_{(z)} - \alpha^2 [\kappa^2 - (m+1)^2] f_{(z)} \} \tag{439}$$

Assuming a general lateral load (Miranda E. , 1999):

$$f_{(x)} = \frac{W_{max}}{1 - e^{-a}} \left(1 - e^{-a + a\frac{x}{H}} \right) \rightarrow f_{(z)} = \frac{W_{max}}{1 - e^{-a}} (1 - e^{-a + az}) \tag{440}$$

Substituting the lateral load and the third parameter:

$$u'''''''_{(z)} - (\alpha\kappa)^2 u''''_{(z)} = -\lambda \alpha^2 [\kappa^2 - (m+1)^2] + \lambda \{ \alpha^2 [\kappa^2 - (m+1)^2] - a^2 \} e^{-a + az} \tag{441}$$

The expression for $u_{(z)}$ is proposed:

$$\begin{aligned} u_{(z)} = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + C_4 \cosh(\alpha\kappa z) + C_5 \sinh(\alpha\kappa z) + \frac{\lambda[\kappa^2 - (m+1)^2]}{24\kappa^2} z^4 \\ + \frac{\lambda\{\alpha^2[\kappa^2 - (m+1)^2] - a^2\}}{a^4[a^2 - (\alpha\kappa)^2]} e^{-a + az} \end{aligned} \tag{442}$$

The constants are obtained by evaluating the relevant boundary conditions (the origin of x is at the base of the model):

$$\left\{\begin{array}{c} u_{(1)} = 0 \\ u'_{(1)} = 0 \\ u''_{(0)} = 0 \\ u'''_{(1)} = \lambda\left(1 - \dfrac{1}{a} + \dfrac{e^{-a}}{a}\right) \\ u''''_{(0)} = \lambda(1 - e^{-a}) \\ u'''''_{(1)} = \lambda\left[\alpha^2(m+1)^2\left(1 - \dfrac{1}{a} + \dfrac{e^{-a}}{a}\right) - a\right] \end{array}\right\} \tag{443}$$

The constants:

$$\left\{\begin{array}{c} C_0 = -[C_1 + C_2 + C_3 + C_4\cosh(\alpha\kappa) + C_5\sinh(\alpha\kappa)] - \lambda\left\{\dfrac{1}{24}\left[1 - \dfrac{(m+1)^2}{k^2}\right] + \dfrac{\alpha^2[\kappa^2 - (m+1)^2] - a^2}{a^4[a^2 - (\alpha\kappa)^2]}\right\} \\ C_1 = -[2C_2 + 3C_3 + (\alpha\kappa)C_4\sinh(\alpha\kappa) + (\alpha\kappa)C_5\cosh(\alpha\kappa)] - \lambda\left\{\dfrac{1}{6}\left[1 - \dfrac{(m+1)^2}{k^2}\right] + \dfrac{\alpha^2[\kappa^2 - (m+1)^2] - a^2}{a^3[a^2 - (\alpha\kappa)^2]}\right\} \\ C_2 = -C_4\dfrac{(\alpha\kappa)^2}{2} - \dfrac{\lambda\{\alpha^2[\kappa^2 - (m+1)^2] - a^2\}}{2a^2[a^2 - (\alpha\kappa)^2]}e^{-a} \\ C_3 = -\dfrac{(\alpha\kappa)^3}{6}[C_4\sinh(\alpha\kappa) + C_5\cosh(\alpha\kappa)] + \dfrac{\lambda}{6}\left\{\dfrac{e^{-a}}{a} + \dfrac{(m+1)^2}{k^2} + \dfrac{(m+1)^2\alpha^2}{a[a^2 - (\alpha\kappa)^2]}\right\} \\ C_4 = \dfrac{\lambda}{(\alpha\kappa)^4}\left\{\dfrac{(m+1)^2}{k^2} - e^{-a}\left[\dfrac{\alpha^2[\kappa^2 - (m+1)^2] - a^2}{a^2 - (\alpha\kappa)^2} + 1\right]\right\} \\ C_5 = -C_4\tanh(\alpha\kappa) + \dfrac{\lambda\alpha^2(m+1)^2}{(\alpha\kappa)^5\cosh(\alpha\kappa)}\left[\left(1 - \dfrac{1}{a} + \dfrac{e^{-a}}{a}\right) + \dfrac{a}{a^2 - (\alpha\kappa)^2}\right] \end{array}\right\} \tag{444}$$

For the case of a uniformly distributed lateral load $(a \to \infty)$, the constants are:

$$\left\{\begin{array}{c} C_0 = \lambda\left\{-\dfrac{(m+1)^2}{\alpha^3\kappa^5\cosh(\alpha\kappa)}\left[\dfrac{1}{(\alpha\kappa)} + \sinh(\alpha\kappa)\right] + \dfrac{(m+1)^2}{2\alpha^2\kappa^4} + \dfrac{1}{8} - \dfrac{(m+1)^2}{8\kappa^2}\right\} \\ C_1 = -\lambda\left[\dfrac{\kappa^2 - (m+1)^2}{6\kappa^2}\right] \\ C_2 = -\lambda\dfrac{(m+1)^2}{2\alpha^2\kappa^4} \\ C_3 = 0 \\ C_4 = \lambda\dfrac{(m+1)^2}{\alpha^4\kappa^6} \\ C_5 = \lambda\dfrac{(m+1)^2}{\alpha^3\kappa^5\cosh(\alpha\kappa)}\left[1 - \dfrac{\sinh(\alpha\kappa)}{(\alpha\kappa)}\right] \end{array}\right\} \tag{445}$$

The expression for $u_{(z)}$:

$$u_{(z)} = \lambda\left[\frac{\kappa^2-(m+1)^2}{\kappa^2}\right]\left(\frac{1}{24}z^4-\frac{1}{6}z+\frac{1}{8}\right)+\frac{(m+1)^2}{2\kappa^4\alpha^2}\lambda(1-z^2)$$
$$-\frac{(m+1)^2}{\kappa^2}\lambda\left\{\frac{1-\cosh(\alpha\kappa z-\alpha\kappa)-(\alpha\kappa)[\sinh(\alpha\kappa z)-\sinh(\alpha\kappa)]}{(\alpha\kappa)^4\cosh(\alpha\kappa)}\right\} \tag{446}$$

i.e.,

$$u_{(z)} = \frac{W_{max}H^4}{\left(\frac{m+1}{n}\right)^2 K_{b1}+K_{b2}}\left(\frac{1}{24}z^4-\frac{1}{6}z+\frac{1}{8}\right)+\frac{(m+1)^2}{2k^4}\frac{W_{max}H^2}{K_{s1}}(1-z^2)$$
$$-\frac{(m+1)^2}{\kappa^2}\frac{W_{max}H^4}{K_{b2}}\left\{\frac{1-\cosh(\alpha\kappa z-\alpha\kappa)-(\alpha\kappa)[\sinh(\alpha\kappa z)-\sinh(\alpha\kappa)]}{(\alpha\kappa)^4\cosh(\alpha\kappa)}\right\} \tag{447}$$

Where:

$$\left\{\begin{array}{c} u_{(flexión)} = \dfrac{W_{max}H^4}{\left(\frac{m+1}{n}\right)^2 K_{b1}+K_{b2}}\left(\dfrac{1}{24}z^4-\dfrac{1}{6}z+\dfrac{1}{8}\right) \\ u_{(corte)} = \dfrac{(m+1)^2}{2k^4}\dfrac{W_{max}H^2}{K_{s1}}(1-z^2) \\ u_{(interacción)} = -\dfrac{(m+1)^2}{\kappa^2}\dfrac{W_{max}H^4}{K_{b2}}\left\{\dfrac{1-\cosh(\alpha\kappa z-\alpha\kappa)-(\alpha\kappa)[\sinh(\alpha\kappa z)-\sinh(\alpha\kappa)]}{(\alpha\kappa)^4\cosh(\alpha\kappa)}\right\} \end{array}\right\} \tag{448}$$

Evaluating the maximum deflection when $z = 0$:

$$u_{(z)} = \frac{W_{max}H^4}{8\left[\left(\frac{m+1}{n}\right)^2 K_{b1}+K_{b2}\right]}+\frac{(m+1)^2}{k^4}\frac{W_{max}H^2}{2K_{s1}}-\frac{(m+1)^2}{\kappa^2}\frac{W_{max}H^4}{K_{b2}}\left[\frac{1-Cosh(\alpha\kappa)+(\alpha\kappa)\sinh(\alpha\kappa)}{(\alpha\kappa)^4\cosh(\alpha\kappa)}\right] \tag{449}$$

4.1.12.2 Case 2

- **Calculation of the Transfer Matrix**

According to the coupled differential equations and assuming that the external loads act on the stories and not along the height of the story, it is possible to write it as follows:

$$\begin{Bmatrix} -K_{b1}w''_{(x)} - n(m+1)K_{s1}u'_{(x)} + n^2K_{s1}w_{(x)} = 0 \\ K_{b2}u''''_{(x)} - (m+1)^2K_{s1}u''_{(x)} + n(m+1)K_{s1}w'_{(x)} = 0 \end{Bmatrix} \tag{450}$$

The expression for $u_{(z)}$ and $w_{(z)}$ is proposed:

$$\begin{Bmatrix} u_{(z)} = C_0 + C_1z + C_2z^2 + C_3z^3 + C_4\cosh(\alpha^*\kappa z) + C_5\sinh(\alpha^*\kappa z) \\ w_{(z)} = C_6 + C_7z + C_8z^2 + C_9\cosh(\alpha^*\kappa z) + C_{10}\sinh(\alpha^*\kappa z) \end{Bmatrix} \tag{451}$$

Where:

$$\left\{ \alpha^* = \sqrt{\frac{K_{s1}}{K_{b2}}}, \kappa = \sqrt{(m+1)^2 + n^2\frac{K_{b2}}{K_{b1}}} \right\} \tag{452}$$

Expressing the coefficients of the function $w_{(z)}$ as a function of the coefficients of $u_{(z)}$:

$$\begin{aligned} w_{(z)} = \left(\frac{m+1}{n}\right)C_1 + \left[2\left(\frac{m+1}{n}\right)z\right]C_2 + \left\{3\left(\frac{m+1}{n}\right)z^2 + \left(\frac{m+1}{n}\right)\frac{6}{\alpha^{*2}[\kappa^2-(m+1)^2]}\right\}C_3 \\ - \left\{\frac{\alpha^*\kappa}{n(m+1)}[\kappa^2-(m+1)^2]\sinh(\alpha^*\kappa z)\right\}C_4 \\ - \left\{\frac{\alpha^*\kappa}{n(m+1)}[\kappa^2-(m+1)^2]\cosh(\alpha^*\kappa z)\right\}C_5 \end{aligned} \tag{453}$$

Internal forces such as bending moment and shear force associated with lateral displacement result:

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} M_{\mathrm{l}(z)} = K_{b1}\theta'_{(z)} = (2K_{b1})C_2 + \left[6\left(\frac{m+1}{n}\right)K_{b1}z\right]C_3 - \left\{\frac{(\alpha^*\kappa)^2}{n(m+1)}[\kappa^2-(m+1)^2]\cosh(\alpha^*\kappa z)\,K_{b1}\right\}C_4 \\ - \left\{\frac{(\alpha^*\kappa)^2}{n(m+1)}[\kappa^2-(m+1)^2]\sinh(\alpha^*\kappa z)\,K_{b1}\right\}C_5 \end{array} \right\} \\ M_{\mathrm{r}(z)} = K_{b2}u''_{(x)} = (2K_{b2})C_2 + (6K_{b2}z)C_3 + [(\alpha^*\kappa)^2\cosh(\alpha^*\kappa z)\,K_{b2}]C_4 + [(\alpha^*\kappa)^2\sinh(\alpha^*\kappa z)K_{b2}]C_5 \\ V_{(z)} = K_{s1}\left[\theta_{(x)} - u'_{(x)}\right] + K_{b2}u'''_{(x)} = \left[6\left(\frac{m+1}{n}\right)^2K_{b1} + K_{b2}\right]C_3 \end{array} \right\} \tag{454}$$

Writing the equations in matrix form:

$$
\begin{Bmatrix} u_i(z_i) \\ u_i'(z_i) \\ w_i(z_i) \\ M_{\mathrm{l}}(z_i) \\ M_{\mathrm{r}}(z_i) \\ V_i(z_i) \end{Bmatrix} = K_i(z_i) \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{Bmatrix} \tag{455}
$$

Where:

$$
K_i(z_i) = \begin{bmatrix}
1 & z & z^2 & z^3 & \cosh(\alpha^*\kappa z) & \sinh(\alpha^*\kappa z) \\
0 & 1 & 2z & 3z^2 & (\alpha^*\kappa)\sinh(\alpha^*\kappa z) & (\alpha^*\kappa)\cosh(\alpha^*\kappa z) \\
0 & \left(\frac{m+1}{n}\right) & 2\left(\frac{m+1}{n}\right)z & 3\left(\frac{m+1}{n}\right)z^2 + \left(\frac{m+1}{n}\right)\frac{6}{\alpha^{*2}[\kappa^2-(m+1)^2]} & -\frac{\alpha^*\kappa}{n(m+1)}[\kappa^2-(m+1)^2]\sinh(\alpha^*\kappa z) & -\frac{\alpha^*\kappa}{n(m+1)}[\kappa^2-(m+1)^2]\cosh(\alpha^*\kappa z) \\
0 & 0 & 2K_{b1} & 6\left(\frac{m+1}{n}\right)K_{b1}z & -\frac{(\alpha^*\kappa)^2}{n(m+1)}[\kappa^2-(m+1)^2]\cosh(\alpha^*\kappa z)\,K_{b1} & -\frac{(\alpha^*\kappa)^2}{n(m+1)}[\kappa^2-(m+1)^2]\sinh(\alpha^*\kappa z)\,K_{b1} \\
0 & 0 & 2K_{b2} & 6K_{b2}z & (\alpha^*\kappa)^2\cosh(\alpha^*\kappa z)\,K_{b2} & (\alpha^*\kappa)^2\sinh(\alpha^*\kappa z)K_{b2} \\
0 & 0 & 0 & 6\left(\frac{m+1}{n}\right)^2 K_{b1} + K_{b2} & 0 & 0
\end{bmatrix} \tag{456}
$$

- **Static Analysis Under Point Static Loads Applied at Floor Level**

The relationship between forces and displacements between two consecutive floors is obtained by taking into account the transfer matrix and the vector of external point forces.

$$
\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ \theta_n(0) \\ M_{\mathrm{ln}}(0) \\ M_{\mathrm{rn}}(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_k(0) \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ \theta_1(h_1) \\ M_{\mathrm{l1}}(h_1) \\ M_{\mathrm{r1}}(h_1) \\ V_1(h_1) \end{Bmatrix} - \sum_{s=0}^{n}\left[\prod_{k=s+1}^{n} T_k(0)\right]F_s - F_n \tag{457}
$$

Expressing it in simplified form:

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ \theta_n(0) \\ M_{\mathrm{ln}}(0) \\ M_{\mathrm{rn}}(0) \\ V_n(0) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ \theta_1(h_1) \\ M_{\mathrm{l}1}(h_1) \\ M_{\mathrm{r}1}(h_1) \\ V_1(h_1) \end{Bmatrix} + f \tag{458}$$

Where:

$$\begin{Bmatrix} \mathrm{t} = \prod_{k=1}^{n} T_k(0) \\ f = -\sum_{s=0}^{n} \left[\prod_{k=s+1}^{n} T_k(0) \right] F_s - F_n \end{Bmatrix} \tag{459}$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 6x6 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ u'_{(1)} = 0 \\ \theta_{(1)} = 0 \\ \theta'_{(0)} = 0 \\ u''_{(0)} = 0 \\ -K_{b2}u'''_{(0)} + (m+1)^2 K_{s1} u'_{(0)} - n(m+1)K_{s1} w_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1(h_1) = 0 \\ u_1'(h_1) = 0 \\ \theta_1(h_1) = 0 \\ M_{ln}(0) = 0 \\ M_{rn}(0) = 0 \\ V_n(0) = 0 \end{Bmatrix} \tag{460}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ \theta_n(0) \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & t_{1,5} & t_{1,6} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} & t_{2,5} & t_{2,6} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} & t_{3,5} & t_{3,6} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,1} & t_{5,2} & t_{5,3} & t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,1} & t_{6,2} & t_{6,3} & t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ M_{\mathrm{l}1}(h_1) \\ M_{\mathrm{r}1}(h_1) \\ V_1(h_1) \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{Bmatrix} \tag{461}$$

Solving for bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} M_{l1}(h_1) \\ M_{r1}(h_1) \\ V_1(h_1) \end{Bmatrix} + \begin{Bmatrix} f_4 \\ f_5 \\ f_6 \end{Bmatrix} \rightarrow \begin{Bmatrix} M_{l1}(h_1) \\ M_{r1}(h_1) \\ V_1(h_1) \end{Bmatrix} = -\begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix}^{-1} \begin{Bmatrix} f_4 \\ f_5 \\ f_6 \end{Bmatrix} \qquad (462)$$

Substituting the internal forces, we obtain the displacement, its derivative and the rotation at the top of the beam:

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ \theta_n(0) \end{Bmatrix} = -\begin{bmatrix} t_{1,4} & t_{1,5} & t_{1,6} \\ t_{2,4} & t_{2,5} & t_{2,6} \\ t_{3,4} & t_{3,5} & t_{3,6} \end{bmatrix} \begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix}^{-1} \begin{Bmatrix} f_4 \\ f_5 \\ f_6 \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} \qquad (463)$$

4.1.13 Generalized Parallel Coupling of Two Beams of a Field (GCTB)

The GCTB beam of a field is developed, which considers that the structure consists of a parallel coupling of an Euler Bernoulli beam (Bernoulli beam because it only considers the bending effect of the walls) and a rotation restraint beam (due to the continuous core resulting from the presence of the connecting beam). The beams are assumed to be connected in parallel by means of axially rigid members that only transmit the horizontal forces and do not deform. This 1-field GCTB beam is suitable for shear walls where the effect of axial strains and shear strains are negligible.

The one-field beam model GCTB takes into account the transverse motion (u) as the kinematic field. K_{b1} and K_{s2}; are the bending stiffness and the equivalent stiffness of the connecting beams, respectively.

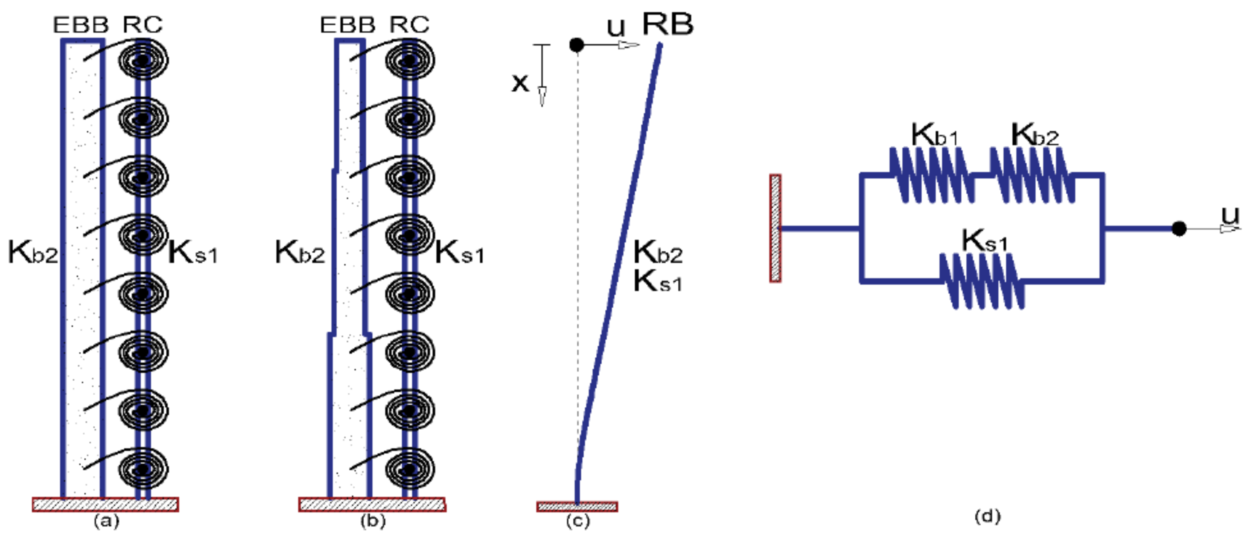


Figure 72. GCTB beam of a field. a) Case 1, b) Case 2 and c) Equivalent RB and d) Idealization of the GCTB stiffness of a field.

4.1.13.1 Case 1

The potential energy of the GCTB model of a field is expressed as:

$$V = \frac{1}{2}\int_0^H K_b {u''_{(x)}}^2\, dx + \frac{1}{2}\int_0^H K_s \gamma_c(x)^2\, dx \qquad (464)$$

Where:

$$\{K_b = E(I_1 + I_2), K_s = G_{eq} t_w l_b\} \tag{465}$$

Denoting:

$$\left\{\gamma_c = (m+1)u'_{(x)}, m = \frac{B_1 + B_2}{2l_b}\right\} \tag{466}$$

Rewriting:

$$V = \frac{1}{2} K_b u''_{(x)}{}^2 dx + \frac{1}{2}\int_0^H (m+1)^2 K_s u'_{(x)}{}^2\, dx \tag{467}$$

The work done by the external force is:

$$W = \int_0^H f_{(x)} u_{(x)}\, dx \tag{468}$$

Consequently, the total potential energy of the beam GCTB of a field subjected to a general lateral load distribution is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H \left[K_b u''_{(x)}{}^2 + (m+1)^2 K_s u'_{(x)}{}^2\right] dx - \int_0^H f(x) u_{(x)}\, dx \tag{469}$$

Closed-form solutions of the model acted on by a transverse load are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\delta\mathcal{U} = \int_0^H \{K_b u''_{(x)} \delta u''_{(x)} + (m+1)^2 K_s u'_{(x)} \delta u'_{(x)}\} dx - \int_0^H f(x) \delta u_{(x)}\, dx - \int_0^H u_{(x)}\, \delta f(x) dx \tag{470}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\begin{aligned}\delta\mathcal{U} = {} & \left[K_b u''_{(x)} \delta u'_{(x)}\right]_0^H - \left\{\left[K_b u'''_{(x)} - (m+1)^2 K_s u'_{(x)}\right] \delta u_{(x)}\right\}_0^H \\ & + \int_0^H \left[K_b u''''_{(x)} - (m+1)^2 K_s u''_{(x)} - f(x)\right] \delta u_{(x)} dx - \int_0^H u_{(x)}\, \delta f(x) dx\end{aligned} \tag{471}$$

Setting the terms equal to zero, the following equations result:

$$K_b u''''_{(x)} - (m+1)^2 K_s u''_{(x)} - f(x) = 0 \tag{472}$$

Boundary conditions:

$$\begin{Bmatrix} u''_{(0)} = 0 \\ K_b u'''_{(0)} - (m+1)^2 K_s u'_{(0)} = 0 \end{Bmatrix} \tag{473}$$

A sixth order differential equation is obtained. Normalizing the differential equation by the variable $z = x/H$:

$$K_b u''''_{(z)} - (m+1)^2 K_s H^2 u''_{(z)} - H^4 f(z) = 0 \tag{474}$$

Defining two parameters:

$$\left\{ \alpha = H\sqrt{\frac{(m+1)^2 K_s}{K_b}}\ , \lambda = \frac{W_{max} H^4}{K_b(1-e^{-a})} \right\} \tag{475}$$

Replacing the first parameter:

$$u''''_{(z)} - \alpha^2 u''_{(z)} = \frac{H^4}{K_b} f(z) \tag{476}$$

Assuming a general lateral load (Miranda E. , 1999):

$$f_{(x)} = \frac{W_{max}}{1-e^{-a}}\left(1 - e^{-a+a\frac{x}{H}}\right) \rightarrow f_{(z)} = \frac{W_{max}}{1-e^{-a}}(1 - e^{-a+az}) \tag{477}$$

Substituting the lateral load and the second parameter:

$$u''''_{(z)} - \alpha^2 u''_{(z)} = \lambda(1 - e^{-a+az}) \tag{478}$$

The expression for $u_{(z)}$ is proposed:

$$u_{(z)} = C_0 + C_1 z + C_2\cosh(\alpha z) + C_3\sinh(\alpha z) - \frac{\lambda}{2\alpha^2} z^2 - \frac{\lambda}{a^2(a^2-\alpha^2)} e^{-a+az} \tag{479}$$

The constants are obtained by evaluating the relevant boundary conditions (the origin of x is at the base of the model):

$$\begin{Bmatrix} u_{(1)} = 0 \\ u'_{(1)} = 0 \\ u''_{(0)} = 0 \\ u'''_{(x)} - \alpha^2 u'_{(x)} = 0 \end{Bmatrix} \qquad (480)$$

The constants:

$$\begin{Bmatrix} C_0 = \lambda\left[\frac{1}{2\alpha^2} + \frac{1}{a^2(a^2-\alpha^2)}\right] - (C_1 + C_2 Cosh\alpha + C_3 Senh\alpha) \\ C_1 = -\lambda\left(\frac{e^{-a}}{a\alpha^2}\right), C_2 = \frac{\lambda}{\alpha^2}\left(\frac{1}{\alpha^2} + \frac{e^{-a}}{a^2-\alpha^2}\right) \\ C_3 = \frac{1}{\alpha Cosh\alpha}\left\{\lambda\left[\frac{1}{\alpha^2} + \frac{1}{a(a^2-\alpha^2)}\right] - (C_1 + C_2 \alpha Senh\alpha)\right\} \end{Bmatrix} \qquad (481)$$

For the case of a uniformly distributed lateral load $(a \to \infty)$, the expression for $u_{(z)}$ is:

$$u_{(z)} = \lambda\left(\frac{1-z^2}{2\alpha^2}\right) + \lambda\left\{\frac{\alpha[\sinh(\alpha z) - \sinh\alpha] - 1 + \cosh[\alpha(z-1)]}{\alpha^4 \cosh\alpha}\right\} \qquad (482)$$

i.e.,

$$u_{(z)} = \frac{W_{max}H^2}{K_s}\left(\frac{1-z^2}{2}\right) + \frac{W_{max}H^4}{K_b}\left\{\frac{\alpha[\sinh(\alpha z) - \sinh\alpha] - 1 + \cosh[\alpha(z-1)]}{\alpha^4 \cosh\alpha}\right\} \qquad (483)$$

This expression clearly shows how the bending and shear contributors interact; producing an interaction between them.

$$\begin{Bmatrix} u_{(shear)} = \frac{W_{max}H^2}{K_s}\left(\frac{1-z^2}{2}\right) \\ u_{(interaction)} = \frac{W_{max}H^4}{K_b}\left\{\frac{\alpha[\sinh(\alpha z) - \sinh\alpha] - 1 + \cosh[\alpha(z-1)]}{\alpha^4 \cosh\alpha}\right\} \end{Bmatrix} \qquad (484)$$

Evaluating the maximum deflection when z = 0:

$$u_{(0)} = \frac{W_{max}H^2}{2K_s} - \frac{W_{max}H^4}{K_b}\left[\frac{1 + \alpha\sinh\alpha - \cosh\alpha}{\alpha^4 \cosh\alpha}\right] \qquad (485)$$

Since the two-field beam GCTB and the classical CTB beam are identical in solution, the conclusions obtained for the classical CTB beam are also applicable to the one-field beam GCTB.

4.1.13.2 Case 2

- **Calculation of the Transfer Matrix**

According to the differential equation and since the external loads are assumed to act on the stories and not along the height of the story, it is possible to write it as follows:

$$K_b u''''_{(x)} - (m+1)^2 K_s u''_{(x)} = 0 \tag{486}$$

The expression for $u_{(z)}$ and $u'_{(z)}$ is proposed:

$$\begin{Bmatrix} u_{(z)} = C_0 + C_1 z + C_2 \cosh(\alpha^* z) + C_3 \sinh(\alpha^* z) \\ u'_{(z)} = C_1 + C_2 \alpha^* \sin(\alpha^* z) + C_3 \alpha^* \cosh(\alpha^* z) \end{Bmatrix} \tag{487}$$

Where:

$$\alpha^* = \sqrt{\frac{(m+1)^2 K_s}{K_b}} \tag{488}$$

Internal forces such as bending moment and shear force associated with lateral displacement result in:

$$\begin{Bmatrix} M_{(z)} = K_b u''_{(z)} = \alpha^{*2} \cosh(\alpha^* z) K_b C_2 + \alpha^{*2} \sinh(\alpha^* z) K_b C_3 \\ V_{(z)} = K_b u'''_{(z)} - (m+1)^2 K_s u'_{(x)} = (-\alpha^{*2} K_b) C_1 \end{Bmatrix} \tag{489}$$

Writing the equations in matrix form:

$$\begin{Bmatrix} u_i(z_i) \\ u'_i(z_i) \\ M_i(z_i) \\ V_i(z_i) \end{Bmatrix} = K_i(z_i) \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{Bmatrix} \tag{490}$$

Where:

$$K_i(z_i) = \begin{bmatrix} 1 & z_i & \cosh(\alpha^* z) & \sinh(\alpha^* z) \\ 0 & 1 & \alpha^* \sin(\alpha^* z) & \alpha^* \cosh(\alpha^* z) \\ 0 & 0 & \alpha^{*2} \cosh(\alpha^* z) K_b & \alpha^{*2} \sinh(\alpha^* z) K_b \\ 0 & -\alpha^{*2} K_b & 0 & 0 \end{bmatrix}_i \tag{491}$$

- **Static Analysis Under Point Static Loads Applied at Floor Level**

The relationship between forces and displacements between two consecutive floors is obtained by taking into account the transfer matrix and the vector of external point forces.

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_k(0) \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} - \sum_{s=0}^{n} \left[\prod_{k=s+1}^{n} T_k(0) \right] F_s - F_n \qquad (492)$$

Expressing it in simplified form:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ \theta_1(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} + f \qquad (493)$$

Where:

$$\begin{Bmatrix} \mathrm{t} = \prod_{k=1}^{n} T_k(0) \\ f = -\sum_{s=0}^{n} \left[\prod_{k=s+1}^{n} T_k(0) \right] F_s - F_n \end{Bmatrix} \qquad (494)$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 4x4 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ u_{(1)}' = 0 \\ u_{(0)}'' = 0 \\ K_b u_{(0)}''' - K_s u_{(0)}' = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1(h_1) = 0 \\ u_1'(h_1) = 0 \\ M_n(0) = 0 \\ V_n(0) = 0 \end{Bmatrix} \qquad (495)$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} \tag{496}$$

Solving for bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} + \begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix} \rightarrow \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} = -\begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix}^{-1} \begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix} \tag{497}$$

Substituting the internal forces we obtain the displacement and its derivative at the top:

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \end{Bmatrix} = -\begin{bmatrix} t_{1,3} & t_{1,4} \\ t_{2,3} & t_{2,4} \end{bmatrix} \begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix}^{-1} \begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix} + \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{498}$$

4.2 DYNAMIC ANALYSIS OF INDIVIDUAL STRUCTURAL SYSTEMS

The objective of this section is to develop the dynamic analysis of the replacement beams presented in the previous section. Through free vibration analysis, the natural frequencies of the main vibration modes are calculated and graphs are built that will be very useful for practical application in engineering offices.

- **Case 1**: A continuous analysis is considered because the method used is based solely on the continuous method and a uniformly distributed vertical load is assumed over the height of the element.

  To take into account that the vertical load is applied at the level of the floors and that it is not distributed over the height of the building, Zalka (2020), using Dunkerley's theorem, proposes considering a correction factor in the dynamic analysis.

  $$r_f = \frac{n}{n + 2.06} \tag{499}$$

  Where n is the number of floors of the building. It is true that this effect is negligible in tall buildings, but for medium and low buildings, not considering this correction coefficient is not conservative because the centroid of the total vertical load shifts downwards, resulting in critical load values greater than the actual load.

  The main disadvantage is that it is only applicable to structures where the cross section is uniform in height. The main advantage is that continuous closed-form solutions are obtained that allow parametric analysis.

- **Case 2**: A discrete analysis is considered because the methods used are the continuous method and the transfer matrix method, and an arbitrary point vertical load applied at floor level is assumed.

  The main disadvantage is that closed continuous solutions that allow parametric analysis are not obtained. The main advantage is that it allows the analysis of non-uniform structures with mass and stiffness distributed variably along the height and/or for structures where the loads are applied at the level of the floors; that is, it is considered a case of general analysis because it even serves as a verification of case 1.

4.2.1 Bending Beam of a Field (EBB)

4.2.1.1 Case 1

The potential energy and kinetic energy of the EBB model of a field are:

$$V = \frac{1}{2}\int_0^H K_b (u'')^2\, dx, T = \frac{1}{2}\int_0^H \gamma_u (\dot{u})^2\, dx \tag{500}$$

Where:

$$\gamma_u = \sum_{i=1}^{n} \rho A_i \tag{501}$$

Consequently, the total potential energy of the EBB beam of a field is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H \left[\gamma_u {\dot{u}_{(x,t)}}^2 - K_b {u''_{(x,t)}}^2\right] dx \tag{502}$$

Closed-form solutions of the model are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\delta\mathcal{U} = \int_0^H \left[\gamma_u \dot{u}_{(x,t)} \delta\dot{u}_{(x,t)} - K_b u''_{(x,t)} \delta u''_{(x,t)}\right] dx \tag{503}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\delta\mathcal{U} = \left[\gamma_u \dot{u}_{(x,t)} + K_b u'''_{(x,t)}\right] \delta {u_{(x,t)}}_0^H - \left[K_b u''_{(x,t)}\right] \delta {u'_{(x,t)}}_0^H - \int_0^H \left[\gamma_u \ddot{u}_{(x,t)} + K_b u''''_{(x,t)}\right] \delta u \tag{504}$$

Setting the terms equal to zero, the following equation results:

$$\gamma_u \ddot{u}_{(x,t)} + K_b u''''_{(x,t)} = 0 \tag{505}$$

And boundary conditions:

$$\begin{Bmatrix} u'''_{(H)} = 0 \\ u''_{(H)} = 0 \end{Bmatrix} \tag{506}$$

The partial differential equation can be solved by separating variables, applying a solution of the following type:

$$u_{(x,t)} = \emptyset_{(x)} q_{(t)} \tag{507}$$

Where $\emptyset_{(x)}$ defines the variation of the displacement along the length of the beam, while $q_{(t)}$ does so with time. Replacing and collecting similar terms:

$$\frac{\ddot{q}_{(t)}}{q_{(t)}} + \frac{K_b}{\gamma_u} \cdot \frac{1}{\emptyset_{(x)}} \emptyset''''_{(x)} = 0 \tag{508}$$

Because the time and height coordinates are independent variables, each of the terms must equal a constant with opposite signs, to ensure that the net result is zero. Consequently, it can be divided into two ordinary differential equations:

$$\begin{Bmatrix} \ddot{q}_{(t)} + w^2 q_{(t)} = 0 \\ K_b \emptyset''''_{(x)} - \gamma_u w^2 \emptyset_{(x)} = 0 \end{Bmatrix} \tag{509}$$

Where the first equation is the same that governs the behavior of an SDOF system with vibration frequency w.

A sixth order differential equation is obtained. Normalizing the length by the variable $z = x/H$, we obtain:

$$\emptyset''''_{(z)} - \left(\frac{\gamma_u w^2 H^4}{K_b}\right) \emptyset_{(z)} = 0 \tag{510}$$

The equation is rewritten:

$$\emptyset''''_{(z)} - \delta^4 \emptyset_{(z)} = 0 \tag{511}$$

Where:

$$\delta^2 = \sqrt{\frac{\gamma_u H^4}{K_b} w^2} \tag{512}$$

A solution can be obtained in the following way for the mode forms:

$$\emptyset_{(z)} = C_1 \cos(\delta z) + C_2 \sin(\delta z) + C_3 \cosh(\delta z) + C_4 \sinh(\delta z) \tag{513}$$

- **Frequency and Periods of Vibration**

The following boundary conditions are considered:

$$\begin{Bmatrix} \emptyset_{(0)} = 0 \\ \emptyset'_{(0)} = 0 \\ \emptyset''_{(1)} = 0 \\ \emptyset'''_{(1)} = 0 \end{Bmatrix} \tag{514}$$

Writing in matrix form the linear algebraic system resulting from expanding the boundary conditions:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -\cos\delta & -\sin\delta & \cosh\delta & \sinh\delta \\ \sin\delta & -\cos\delta & \sinh\delta & \cosh\delta \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{515}$$

Which has a different solution than the trivial one if the determinant is equal to zero (the matrix of coefficients is singular).

After some simple manipulations, the modal characteristic equation is obtained, whose roots define a set of particular solutions that satisfy the differential equation of motion and the boundary conditions.

$$\cos\delta \cosh\delta + 1 = 0 \tag{516}$$

The eigenvalue δ is derived by numerically solving the characteristic equation. Knowing the value of δ, the frequencies and periods of vibration of the model are obtained.

$$w = \frac{\delta}{H^2}\sqrt{\frac{K_b}{\gamma_u}} \rightarrow T = \frac{2\pi}{w} = \frac{2\pi H^2}{\delta}\sqrt{\frac{\gamma_u}{K_b}} \tag{517}$$

- **Eigenvalues**

Solving the value of δ numerically, the vibration frequencies and periods are calculated.

$$\begin{Bmatrix} \delta_1 = 1.87510 \rightarrow w_1 = \dfrac{3.51602}{H^2}\sqrt{\dfrac{K_b}{\gamma_u}} \rightarrow T_1 = 1.78702H^2\sqrt{\dfrac{\gamma_u}{K_b}} \\ \delta_2 = 4.69409 \rightarrow w_2 = \dfrac{22.03449}{H^2}\sqrt{\dfrac{K_b}{\gamma_u}} \rightarrow T_2 = 0.28515H^2\sqrt{\dfrac{\gamma_u}{K_b}} \\ \delta_3 = 7.85476 \rightarrow w_3 = \dfrac{61.69721}{H^2}\sqrt{\dfrac{K_b}{\gamma_u}} \rightarrow T_3 = 0.10184H^2\sqrt{\dfrac{\gamma_u}{K_b}} \end{Bmatrix} \tag{518}$$

- **Mode Shapes**

Considering the first boundary conditions, normalizing to 1 at the top, and writing the resulting linear algebraic system in matrix form:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -\cos\delta & -\sin\delta & \cosh\delta & \sinh\delta \\ \cos\delta & \sin\delta & \cosh\delta & \sinh\delta \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} \tag{519}$$

Clearing the vector of coefficients:

$$\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -\cos\delta & -\sin\delta & \cosh\delta & \sinh\delta \\ \cos\delta & \sin\delta & \cosh\delta & \sinh\delta \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} \tag{520}$$

After some simple manipulations:

$$\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \frac{1}{-\sin\delta + \sinh\delta + \eta(\cos\delta - \cosh\delta)} \begin{Bmatrix} \eta \\ -1 \\ -\eta \\ 1 \end{Bmatrix} \tag{521}$$

Where:

$$\eta = \frac{\sinh\delta + \sin\delta}{\cos\delta + \cosh\delta} \tag{522}$$

Replacing these coefficients, a solution of the following form can be obtained for the mode forms:

$$\emptyset_{(z)} = \frac{-\sin\delta z + \sinh\delta z + \eta(\cos\delta z - \cosh\delta z)}{-\sin\delta + \sinh\beta + \eta(\cos\delta - \cosh\delta)} \qquad (523)$$

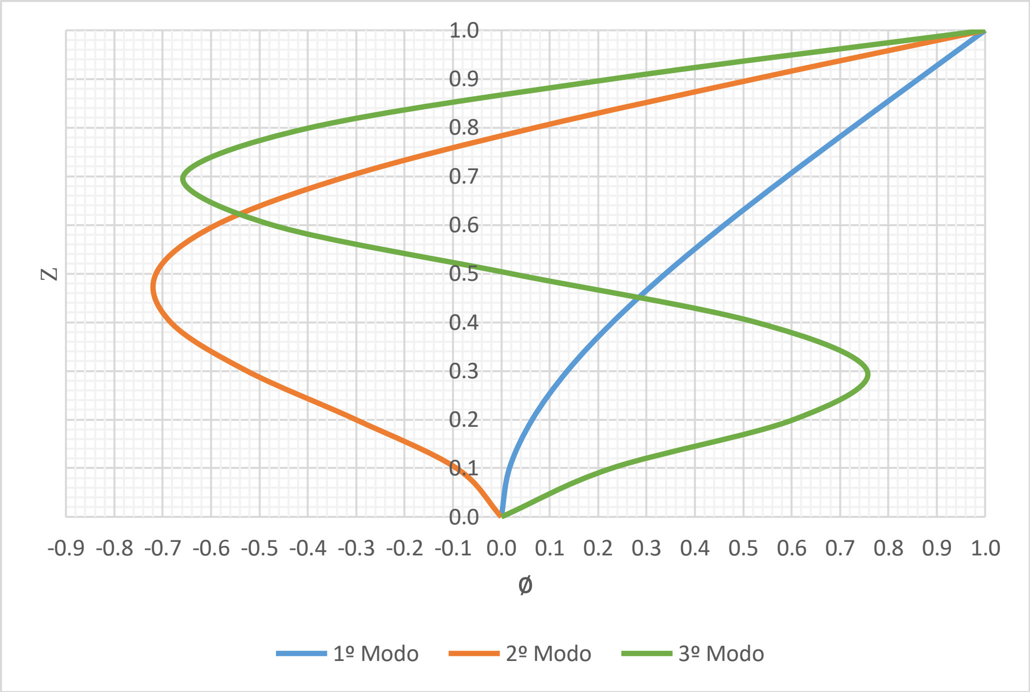


Figure 73. Natural forms of bending vibration for the first three vibration modes.

4.2.1.2 Case 2

Considering the masses concentrated at floor level and analyzing the dynamic equilibrium for the mass m_i, the inertial force is:

$$F_i = m_i \ddot{u}_i = m_i w^2 u_i \qquad (524)$$

For equilibrium:

$$V_{i+1} = V_i + m_i w^2 u_i \qquad (525)$$

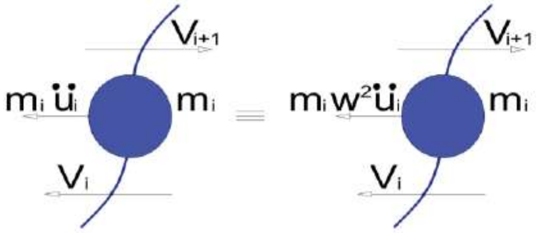


Figure 74. Dynamic forces at the i-th level.

The relationship between forces and displacements between two consecutive floors is obtained by taking into account the transfer matrix and the vector of external point forces. For the j-th floor:

$$\begin{Bmatrix} u_{i+1}(0) \\ u'_{i+1}(0) \\ M_{i+1}(0) \\ V_{i+1}(0) \end{Bmatrix} = T_i(0) \begin{Bmatrix} u_i(0) \\ u'_i(0) \\ M_i(0) \\ V_i(0) \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ m_i w^2 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_i(0) \\ u'_i(0) \\ M_i(0) \\ V_i(0) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m_i w^2 & 0 & 0 & 1 \end{bmatrix} T_i(0) \begin{Bmatrix} u_i(0) \\ u'_i(0) \\ M_i(0) \\ V_i(0) \end{Bmatrix} \quad (526)$$

Rewriting:

$$\begin{Bmatrix} u_{i+1}(0) \\ u'_{i+1}(0) \\ M_{i+1}(0) \\ V_{i+1}(0) \end{Bmatrix} = T_{wi}(0) \begin{Bmatrix} u_i(0) \\ u'_i(0) \\ M_i(0) \\ V_i(0) \end{Bmatrix} \quad (527)$$

Where:

$$T_{wi}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m_i w^2 & 0 & 0 & 1 \end{bmatrix} T_i(0) \quad (528)$$

For the first floor:

$$\begin{Bmatrix} u_1(0) \\ u'_1(0) \\ M_1(0) \\ V_1(0) \end{Bmatrix} = T_{w1}(0) \begin{Bmatrix} u_1(h_1) \\ u'_1(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \quad (529)$$

For the second floor:

$$\begin{Bmatrix} u_2(0) \\ u'_2(0) \\ M_2(0) \\ V_2(0) \end{Bmatrix} = T_{w2}(0) T_{w1}(0) \begin{Bmatrix} u_1(h_1) \\ u'_1(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \quad (530)$$

For the third floor:

$$\begin{Bmatrix} u_3(0) \\ u'_3(0) \\ M_3(0) \\ V_3(0) \end{Bmatrix} = T_{w3}(0) T_{w2}(0) T_{w1}(0) \begin{Bmatrix} u_1(h_1) \\ u'_1(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \quad (531)$$

For the nth floor (top of the beam):

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = T_{wn}(0) \dots T_{w2}(0) T_{w1}(0) \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{532}$$

Expressing the equation between product symbols:

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_{wk}(0) \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{533}$$

Where:

$$\mathrm{t} = \prod_{k=1}^{n} T_{wk}(0) \tag{534}$$

Replacing this parameter:

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{535}$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 4x4 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ u'_{(1)} = 0 \\ u''_{(0)} = 0 \\ u'''_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1(h_1) = 0 \\ u_1'(h_1) = 0 \\ M_n(0) = 0 \\ V_n(0) = 0 \end{Bmatrix} \tag{536}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \quad (537)$$

Solving for the bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \quad (538)$$

Which has a different solution than the trivial one if the determinant is equal to zero (the matrix of coefficients is singular). Solving the determinant finds the angular frequencies of the beam.

4.2.2 Shear Beam of a field (SB)

4.2.2.1 Case 1

The potential energy and kinetic energy of the SB model of a field are:

$$V = \frac{1}{2}\int_0^H K_s {u'_{(x,t)}}^2\,dx, T = \frac{1}{2}\int_0^H \gamma_u {\dot{u}_{(x,t)}}^2\,dx \qquad (539)$$

Where:

$$\gamma_u = \rho A \qquad (540)$$

Consequently, the total potential energy of the SB beam of a field is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H \left[\gamma_u {\dot{u}_{(x,t)}}^2 - K_s {u'_{(x,t)}}^2\right] dx \qquad (541)$$

Closed-form solutions of the model are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\delta\mathcal{U} = \int_0^H \left[\gamma_u \dot{u}_{(x,t)} \delta\dot{u}_{(x,t)} - K_s u'_{(x,t)} \delta u'_{(x,t)}\right] dx \qquad (542)$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\delta\mathcal{U} = \left[\gamma_u \dot{u}_{(x,t)} - K_s u'_{(x,t)}\right] \delta {u_{(x,t)}}_0^H - \int_0^H \left[\gamma_u \ddot{u}_{(x,t)} - K_s u''_{(x,t)}\right] \delta u \qquad (543)$$

Setting the terms equal to zero, the following equation results:

$$\gamma_u \ddot{u}_{(x,t)} - K_s u''_{(x,t)} = 0 \qquad (544)$$

And boundary condition:

$$u'_{(H)} = 0 \qquad (545)$$

The partial differential equation can be solved by separating variables, applying a solution of the following type:

$$u_{(x,t)} = \emptyset_{(x)} q_{(t)} \tag{546}$$

Where $\emptyset_{(x)}$ defines the variation of the displacement along the length of the beam, while $q_{(t)}$ does so with time. Replacing and collecting like terms, we get the following:

$$\frac{\ddot{q}_{(t)}}{q_{(t)}} - \frac{K_s}{\gamma_u} . \frac{1}{\emptyset_{(x)}} \emptyset''_{(x)} = 0 \tag{547}$$

Because the time and height coordinates are independent variables, each of the terms must equal a constant with opposite signs, to ensure that the net result is zero. Consequently, it can be divided into two ordinary differential equations:

$$\begin{Bmatrix} \ddot{q}_{(t)} + w^2 q_{(t)} = 0 \\ K_s \emptyset''_{(x)} + \gamma_u w^2 \emptyset_{(x)} = 0 \end{Bmatrix} \tag{548}$$

Where the first equation is the same that governs the behavior of an SDOF system with vibration frequency w.

A second order differential equation is obtained. Normalizing the differential equation by the variable $z = x/H$:

$$\emptyset''_{(z)} + \left(\frac{\gamma_u w^2 H^2}{K_s} \right) \emptyset_{(z)} = 0 \tag{549}$$

The equation will be rewritten:

$$\emptyset''_{(z)} + \delta^2 \emptyset_{(z)} = 0 \tag{550}$$

Where:

$$\delta = \sqrt{\frac{\gamma_u H^2}{K_s} w^2} \tag{551}$$

A solution can be obtained in the following way for the mode forms:

$$\emptyset_{(z)} = C_1 \cos(\delta z) + C_2 \sin(\delta z) \tag{552}$$

- **Frequency and Periods of Vibration**

The following boundary conditions are considered:

$$\begin{Bmatrix} \emptyset_{(0)} = 0 \\ \emptyset'_{(1)} = 0 \end{Bmatrix} \tag{553}$$

Writing in matrix form the linear algebraic system resulting from expanding the boundary conditions:

$$\begin{bmatrix} 1 & 0 \\ -\sin\delta & \cos\delta \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{554}$$

Which has a different solution than the trivial one if the determinant is equal to zero (the matrix of coefficients is singular).

After some simple manipulations, the modal characteristic equation is obtained, whose roots define a set of particular solutions that satisfy the differential equation of motion and the boundary conditions.

$$Cos(\delta) = 0 \rightarrow \delta = (2n-1)\frac{\pi}{2} \; /n = 1,2,3 \ldots \tag{555}$$

Knowing the value of δ, the frequencies and periods of vibration of the model are obtained.

$$w = \frac{\delta}{H}\sqrt{\frac{K_s}{\gamma_u}} \rightarrow T = \frac{2\pi}{w} = \frac{2\pi H}{\delta}\sqrt{\frac{\gamma_u}{K_s}} \tag{556}$$

- **Eigenvalues**

Solving the value of δ, the vibration frequencies and periods are calculated.

$$\begin{Bmatrix} \delta_1 = 1.57080 \rightarrow w_1 = \dfrac{2.46740}{H}\sqrt{\dfrac{K_s}{\gamma_u}} \rightarrow T_1 = 4.00000H\sqrt{\dfrac{\gamma_u}{K_s}} \\ \delta_2 = 4.71239 \rightarrow w_2 = \dfrac{22.20661}{H}\sqrt{\dfrac{K_s}{\gamma_u}} \rightarrow T_2 = 1.33333H\sqrt{\dfrac{\gamma_u}{K_s}} \\ \delta_3 = 7.85398 \rightarrow w_3 = \dfrac{61.68503}{H}\sqrt{\dfrac{K_s}{\gamma_u}} \rightarrow T_3 = 0.80000H\sqrt{\dfrac{\gamma_u}{K_s}} \end{Bmatrix} \tag{557}$$

- **Mode Shapes**

Considering the first boundary conditions, normalizing to 1 at the top, and writing the resulting linear algebraic system in matrix form:

$$\begin{bmatrix} 1 & 0 \\ \cos\delta & \sin\delta \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \tag{558}$$

Clearing the vector of coefficients:

$$\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ \cos\delta & \sin\delta \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \tag{559}$$

After some simple manipulations:

$$\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \frac{1}{\sin\delta} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \tag{560}$$

Replacing these coefficients, a solution of the following form can be obtained for the mode forms:

$$\emptyset_{(z)} = \frac{\sin\delta z}{\sin\delta} \tag{561}$$

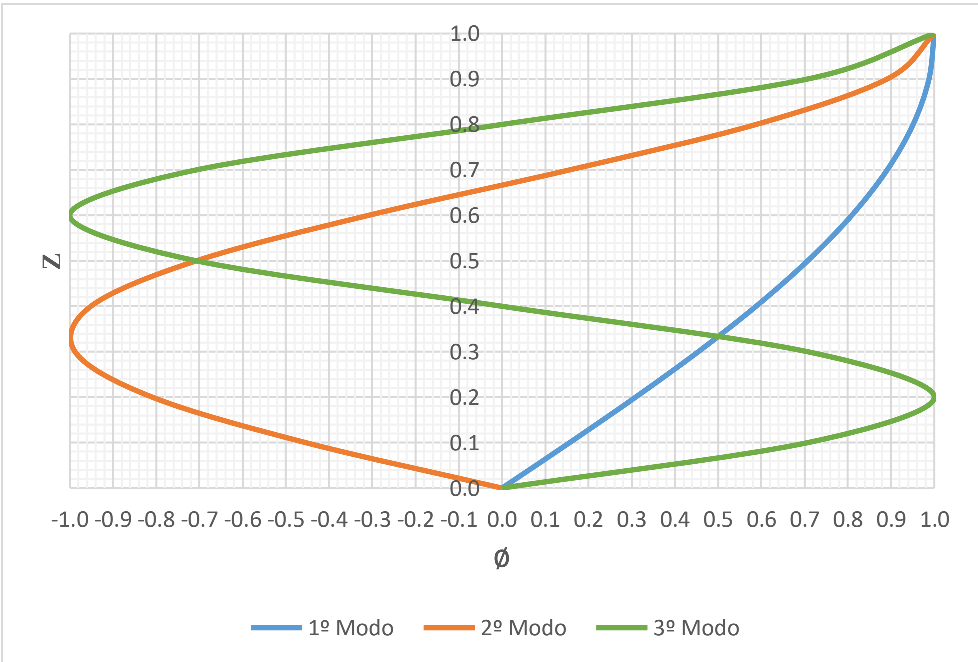


Figure 75. Natural forms of shear vibration for the first three vibration modes.

4.2.2.2 Case 2

The relationship between forces and displacements between two consecutive floors is obtained by taking into account the transfer matrix and the vector of external point forces. For the j-th floor:

$$\begin{Bmatrix} u_{i+1}(0) \\ u'_{i+1}(0) \end{Bmatrix} = T_i(0) \begin{Bmatrix} u_i(0) \\ u'_i(0) \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ m_i w^2 & 0 \end{bmatrix} \begin{Bmatrix} u_i(0) \\ u'_i(0) \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ m_i w^2 & 1 \end{bmatrix} T_i(0) \begin{Bmatrix} u_i(0) \\ u'_i(0) \end{Bmatrix} \tag{562}$$

Rewriting:

$$\begin{Bmatrix} u_{i+1}(0) \\ u'_{i+1}(0) \end{Bmatrix} = T_{wi}(0) \begin{Bmatrix} u_i(0) \\ u'_i(0) \end{Bmatrix} \tag{563}$$

Where:

$$T_{wi}(0) = \begin{bmatrix} 1 & 0 \\ m_i w^2 & 1 \end{bmatrix} T_i(0) \tag{564}$$

Expressing the equation for the nth floor between product symbols:

$$\begin{Bmatrix} u_n(0) \\ u'_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_{wk}(0) \begin{Bmatrix} u_1(h_1) \\ u'_1(h_1) \end{Bmatrix} \tag{565}$$

Where:

$$\mathrm{t} = \prod_{k=1}^{n} T_{wk}(0) \tag{566}$$

Replacing this parameter:

$$\begin{Bmatrix} u_n(0) \\ u'_n(0) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ u'_1(h_1) \end{Bmatrix} \tag{567}$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 2x2 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ u'_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_{(h_1)} = 0 \\ u'_{(0)} = 0 \end{Bmatrix} \tag{568}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} \\ t_{2,1} & t_{2,2} \end{bmatrix} \begin{Bmatrix} 0 \\ u'_1(h_1) \end{Bmatrix} \tag{569}$$

Solving for the bending moment and shear force at the base of the model:

$$\{0\} = t_{2,2} u'_1(h_1) \tag{570}$$

Setting $t_{2,2}$ equal to zero, the angular frequencies of the beam are obtained.

4.2.3 Timoshenko Beam of Two Field (TB)

4.2.3.1 Case 1

The potential energy and kinetic energy of the two-field TB model are:

$$V = \frac{1}{2}\int_0^H \left\{K_b {\theta'_{(x,t)}}^2 + K_s\left[\theta_{(x,t)} - u'_{(x,t)}\right]^2\right\} dx$$
$$T = \frac{1}{2}\int_0^H \left[\gamma_u {\dot{u}_{(x,t)}}^2 + \gamma_\theta {\dot{\theta}_{(x,t)}}^2\right] dx \tag{571}$$

Where:

$$\{\gamma_u = \rho A, \gamma_\theta = \rho I\} \tag{572}$$

Consequently, the total potential energy of the two-field beam TB is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H \left\{\gamma_u {\dot{u}_{(x,t)}}^2 + \gamma_\theta {\dot{\theta}_{(x,t)}}^2 - K_b {\theta'_{(x,t)}}^2 - K_s\left[\theta_{(x,t)} - u'_{(x,t)}\right]^2\right\} \tag{573}$$

Closed-form solutions of the model are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\delta\mathcal{U} = \int_0^H \{\gamma_u \dot{u}_{(x,t)} \delta\dot{u}_{(x,t)} + \gamma_\theta \dot{\theta}_{(x,t)} \delta\dot{\theta}_{(x,t)} - K_b \theta'_{(x,t)} \delta\theta'_{(x,t)}$$
$$- K_s\left[\theta_{(x,t)} - u'_{(x,t)}\right]\left[\delta\theta_{(x,t)} - \delta u'_{(x,t)}\right]\} dx \tag{574}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\delta\mathcal{U} = \{\gamma_u \dot{u}_{(x,t)} + K_s\left[\theta_{(x,t)} - u'_{(x,t)}\right]\}\delta {u_{(x,t)}}_0^H + \{\gamma_\theta \dot{\theta}_{(x,t)} - K_b \theta'_{(x,t)}\}\delta {\theta_{(x,t)}}_0^H$$
$$- \int_0^H \{\gamma_u \ddot{u}_{(x,t)} + K_s\left[\theta'_{(x,t)} - u''_{(x,t)}\right]\}\delta u$$
$$- \int_0^H \{\gamma_\theta \ddot{\theta}_{(x,t)} - K_b \theta''_{(x,t)} + K_s\left[\theta_{(x,t)} - u'_{(x,t)}\right]\}\delta\theta \tag{575}$$

Setting the terms equal to zero, the following equations result:

$$\left\{\begin{matrix} \gamma_u \ddot{u}_{(x,t)} + K_s\left[\theta'_{(x,t)} - u''_{(x,t)}\right] = 0 \\ \gamma_\theta \ddot{\theta}_{(x,t)} - K_b \theta''_{(x,t)} + K_s\left[\theta_{(x,t)} - u'_{(x,t)}\right] = 0 \end{matrix}\right\} \tag{576}$$

And boundary conditions:

$$\left\{\begin{matrix} \theta_{(H)} - u'_{(H)} = 0 \\ K_b \theta'_{(H)} = 0 \end{matrix}\right\} \tag{577}$$

The partial differential equation can be solved by separating variables, applying a solution of the following type:

$$\left\{\begin{matrix} u_{(x,t)} = \emptyset_{(x)} q_{(t)} \\ \theta_{(x,t)} = \lambda_{(x)} q_{(t)} \end{matrix}\right\} \tag{578}$$

Where $\emptyset_{(x)}$ defines the variation of the displacement along the length of the beam, while $q_{(t)}$ does so with time. Replacing and collecting like terms, we get the following:

$$\left\{\begin{matrix} \dfrac{\ddot{q}_{(t)}}{q_{(t)}} + \dfrac{\left[-K_s \emptyset''_{(x)} + K_s \lambda'_{(x)}\right]}{\gamma_u \emptyset_{(x)}} = 0 \\ \dfrac{\ddot{q}_{(t)}}{q_{(t)}} + \dfrac{\left[-K_b \lambda''_{(x)} - K_s \emptyset'_{(x)} + K_s \lambda_{(x)}\right]}{\gamma_\theta \lambda_{(x)}} = 0 \end{matrix}\right\} \tag{579}$$

Because the time and height coordinates are independent variables, each of the terms must equal a constant with opposite signs, to ensure that the net result is zero. Consequently, it can be divided into two ordinary differential equations:

$$\left\{\begin{matrix} \rho \ddot{q}_{(t)} + w^2 q_{(t)} = 0 \\ K_s \emptyset''_{(x)} - K_s \lambda'_{(x)} + w^2 \gamma_u \emptyset_{(x)} = 0 \\ K_b \lambda''_{(x)} + K_s \emptyset'_{(x)} - K_s \lambda_{(x)} + w^2 \gamma_\theta \lambda_{(x)} = 0 \end{matrix}\right\} \tag{580}$$

Where the first equation is the same that governs the behavior of an SDOF system with vibration frequency w.

Solving for $\lambda'_{(x)}$, differentiating twice and replacing:

$$\emptyset''''_{(x)} + w^2\left(\frac{\gamma_u}{K_s} + \frac{\gamma_\theta}{K_b}\right)\emptyset''_{(x)} + \left(\frac{w^4\gamma_u.\gamma_\theta}{K_bK_s} - \frac{\gamma_u w^2}{K_b}\right)\emptyset_{(x)} = 0 \tag{581}$$

A fourth order differential equation is obtained. Normalizing the length by the variable $z = x/H$, the equation can be expressed as:

$$\emptyset''''_{(z)} + \left[H^2w^2\left(\frac{\gamma_u}{K_s} + \frac{\gamma_\theta}{K_b}\right)\right]\emptyset''_{(z)} + \left[\frac{H^4\gamma_u w^2}{K_bK_s}(w^2\gamma_\theta - K_s)\right]\emptyset_{(z)} = 0 \tag{582}$$

The equation is rewritten:

$$\emptyset''''_{(z)} + \left[\delta^2\left(\frac{1}{\alpha^2} + \mu^2\right)\right]\emptyset''_{(z)} + \left[\delta^2\left(\frac{\mu^2}{\alpha^2}\delta^2 - 1\right)\right]\emptyset_{(z)} = 0 \tag{583}$$

Where:

$$\left\{\alpha = H\sqrt{\frac{K_s}{K_b}}, \mu = \frac{1}{H}\sqrt{\frac{\rho I}{\rho A}}, \delta = \sqrt{\frac{\rho A H^4}{K_b}w^2}\right\} \tag{584}$$

To solve the differential equation we consider the characteristic polynomial:

$$P_{(r)} = r^4 + \left[\delta^2\left(\frac{1}{\alpha^2} + \mu^2\right)\right]r^2 + \left[\delta^2\left(\frac{\mu^2}{\alpha^2}\delta^2 - 1\right)\right] = 0 \tag{585}$$

We change the variable and denote:

$$q_i = r_i^2 \rightarrow \left\{\begin{matrix} r_{2i-1} = \sqrt{q_i} \\ r_{2i} = -\sqrt{q_i} \end{matrix}\ ;\ \ i = 1,2\right\} \tag{586}$$

We rewrite the characteristic polynomial:

$$P_{(r)} = q^2 + \left[\delta^2\left(\frac{1}{\alpha^2} + \mu^2\right)\right]q + \left[\delta^2\left(\frac{\mu^2}{\alpha^2}\delta^2 - 1\right)\right] = 0 \tag{587}$$

We define a critical eigenvalue:

$$\delta^2\left(\frac{\mu^2}{\alpha^2}\delta^2-1\right)=0\rightarrow\delta_c^2=\frac{\alpha^2}{\mu^2} \tag{588}$$

Two cases are presented:

- Case 1: When the polynomial has a positive real root and a negative real root.

$$\delta^2\left(\frac{\mu^2}{\alpha^2}\delta^2-1\right)<0\rightarrow\delta^2<\frac{\alpha^2}{\mu^2}\rightarrow\delta<\delta_{cr} \tag{589}$$

- Case 2: When the polynomial has two negative real roots.

$$\delta^2\left(\frac{\mu^2}{\alpha^2}\delta^2-1>\right)0\rightarrow\delta^2>\frac{\alpha^2}{\mu^2}\rightarrow\delta>\delta_{cr} \tag{590}$$

We define q_i in such a way that:

$$q_1<q_2 \tag{591}$$

So that $q_1<0$ and $q_2>0$ for $\delta>\delta_{cr}$ and $q_2<0$ for $\delta<\delta_{cr}$.

The roots of the equation are calculated as:

$$q_{1,2}=\frac{-\left[\delta^2\left(\frac{1}{\alpha^2}+\mu^2\right)\right]\pm\sqrt{\left[\delta^2\left(\frac{1}{\alpha^2}+\mu^2\right)\right]^2-4\left[\delta^2\left(\frac{\mu^2}{\alpha^2}\delta^2-1\right)\right]}}{2} \tag{592}$$

- **Influence of the Relationship ζ=height/thickness of the Shear Wall on the Dynamic Properties**

We will consider the shear wall as a Timoshenko beam with a uniform rectangular cross section and with a Poisson's ratio of 0.20 for concrete.

An adequate shear correction coefficient remains a subject of research. However Cowper (1966); Based on the relationship between the average shear strain in a section and the shear strain at the centroid, he proposed the following value:

$$k=\frac{10(1+\nu)}{12+11\nu} \tag{593}$$

Recently Faghidian (2017) based on two elastostatic approaches for unified formulations, he proposed an innovative shear correction coefficient:

$$k = \frac{10(1+\nu)}{12+\upsilon\left(11-\frac{2}{1+2\zeta^2}\right)} \tag{594}$$

As can be seen in the figure 76, the Faghidian shear correction coefficient is practically identical to that proposed by Cowper for typical shear walls. In this research project, the one proposed by Cowper ($k = 0.845070$) will be used as the correction factor for shear.

The relationship between Young's modulus E and shear modulus G:

$$\frac{G}{E} = \frac{1}{2(1+\nu)} = 0.416667 \tag{595}$$

Taking these two considerations into account, the parameters that control the dynamic behavior turn out to be dependent only on the height/width ratio of the shear wall.

$$\left\{\begin{array}{c} \alpha = H\sqrt{\frac{K_s}{K_b}} = \sqrt{12}\frac{H}{L}\sqrt{\frac{kG}{E}} = 2.055566\frac{H}{L} = 2.055566R \\ \mu = \frac{1}{H}\sqrt{\frac{\rho A}{\rho I}} = \frac{1}{H}\sqrt{\frac{\rho A}{\rho I}} = \frac{1}{\sqrt{12}}.\frac{1}{\frac{H}{L}} = 0.288675\frac{1}{R} \end{array}\right\} \tag{596}$$

Where R is defined as the parameter that relates the height/width of the structural wall.

$$R = \frac{H}{L} \tag{597}$$

As can be seen, the parameters that govern the dynamic behavior are dependent only on the variable R. That is, the behavior in the continuous model only depends on the parameter R.

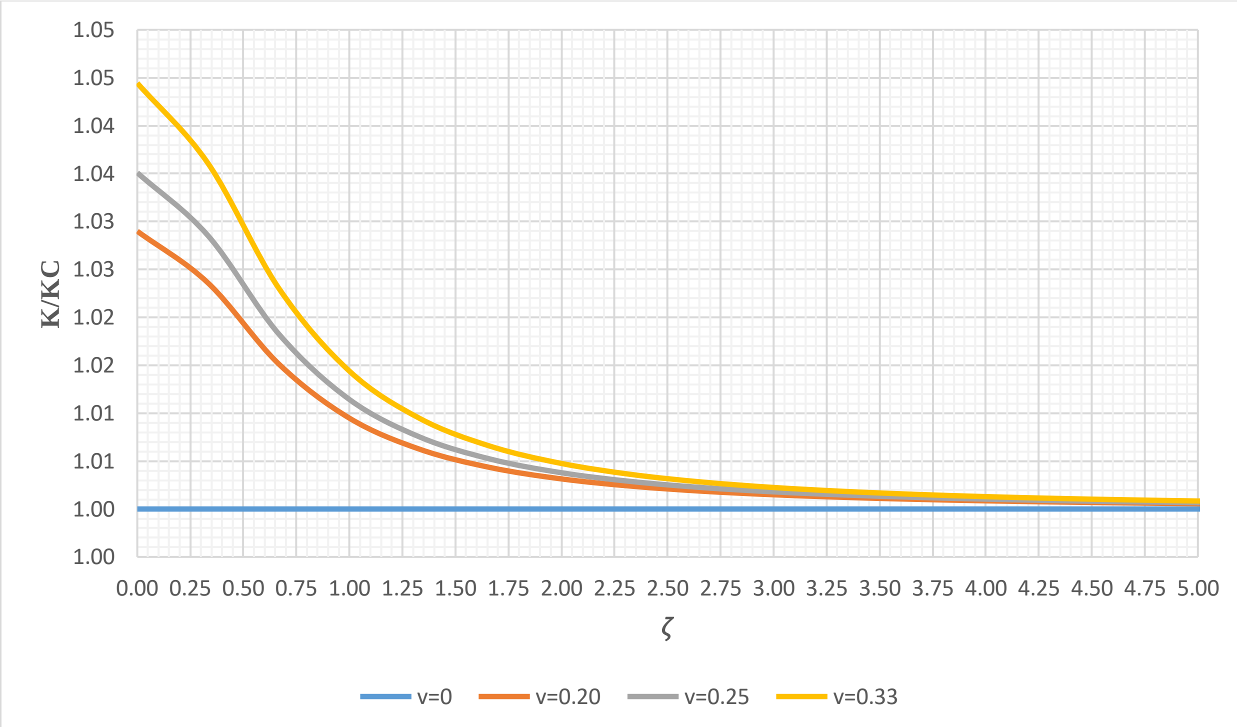


Figure 76. Relationship between the Faghidian and Cowper shear coefficients.

- **Frequency and Periods of Vibration**

Normalizing by the variable $z = x/H$ the two coupled differential equations:

$$\begin{Bmatrix} \emptyset''_{(z)} - H\lambda'_{(z)} + \dfrac{\delta^2}{\alpha^2}\emptyset_{(z)} = 0 \\ H\lambda''_{(z)} + \alpha^2\emptyset'_{(z)} + \left(\delta^2\mu^2 - \alpha^2\right)H\lambda_{(z)} = 0 \end{Bmatrix} \tag{598}$$

The solution will be of the form:

$$W_{(z)} = \begin{Bmatrix} H\lambda_{(z)} \\ \emptyset_{(z)} \end{Bmatrix} = \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} e^{rz} \tag{599}$$

Substituting the equation in the equation, two homogeneous equations are obtained which, written in matrix form, result in:

$$\begin{bmatrix} -r & \left(r^2 + \dfrac{\delta^2}{\alpha^2}\right) \\ r^2 + (\delta^2\mu^2 - \alpha^2) & \alpha^2 r \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{600}$$

To avoid trivial solutions, the determinant must be equal to zero, that is:

$$P_{(r)} = r^4 + \left[\delta^2\left(\frac{1}{\alpha^2} + \mu^2\right)\right] r^2 + \left[\delta^2\left(\frac{\mu^2}{\alpha^2}\delta^2 - 1\right)\right] = 0 \tag{601}$$

For all roots, the equation implies:

$$\begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = \begin{bmatrix} r_i^2 + \dfrac{\delta^2}{\alpha^2} \\ r_i \end{bmatrix} C, i = 1, 2, 3, 4 \tag{602}$$

Where C is an arbitrary constant. We change the variable and denote:

$$q_i = r_i^2 \rightarrow \left\{\begin{matrix} r_{2i-1} = \sqrt{q_i} \\ r_{2i} = -\sqrt{q_i} \end{matrix} \;;\; i = 1, 2\right\} \tag{603}$$

Substituting in the equation:

$$q^2 + \left[\delta^2\left(\frac{1}{\alpha^2} + \mu^2\right)\right] q + \left[\delta^2\left(\frac{\mu^2}{\alpha^2}\delta^2 - 1\right)\right] = 0 \tag{604}$$

It was shown that the roots are always real. Rewriting the complete solution:

$$W_{(z)} = \begin{Bmatrix} H\lambda_{(z)} \\ \emptyset_{(z)} \end{Bmatrix} = \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} e^{rz} = \sum_{i=1}^{4} C_i \begin{bmatrix} r_i^2 + \dfrac{\delta^2}{\alpha^2} \\ r_i \end{bmatrix} e^{r_i z} \tag{605}$$

Substituting this complete equation in the boundary conditions, we obtain:

$$\left\{\begin{matrix} \emptyset_{(0)} = 0 \rightarrow \displaystyle\sum_{i=1}^{4} C_i\, r_i = 0 \\ \emptyset'_{(0)} = 0 \rightarrow \displaystyle\sum_{i=1}^{4} C_i\, r_i^2 = 0 \\ H\lambda_{(1)} - \emptyset'_{(1)} = 0 \rightarrow \displaystyle\sum_{i=1}^{4} C_i\, e^{r_i} = 0 \\ H\lambda'_{(1)} = 0 \rightarrow \displaystyle\sum_{i=1}^{4} C_i\, r_i \left(r_i^2 + \frac{\delta^2}{\alpha^2}\right) e^{r_i} = 0 \end{matrix}\right\}$$

Defining:

$$D_i = r_i^2 + \frac{\delta^2}{\alpha^2}; i = 1, 2 \tag{606}$$

Writing in matrix form the linear algebraic system:

$$\begin{bmatrix} \sqrt{q_1} & -\sqrt{q_1} & \sqrt{q_2} & -\sqrt{q_2} \\ q_1 & q_1 & q_2 & q_2 \\ e^{\sqrt{q_1}} & e^{-\sqrt{q_1}} & e^{\sqrt{q_2}} & e^{-\sqrt{q_2}} \\ \sqrt{q_1}D_1 e^{\sqrt{q_1}} & -\sqrt{q_1}D_1 e^{-\sqrt{q_1}} & \sqrt{q_2}D_2 e^{\sqrt{q_2}} & -\sqrt{q_2}D_2 e^{-\sqrt{q_2}} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{607}$$

Which has a different solution than the trivial one if the determinant is equal to zero (the matrix of coefficients is singular).

By some simple manipulations of the determinant in the equation, it can be written as:

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & q_1 & 0 & q_2 \\ s_1 & c_1 & s_2 & c_2 \\ D_1 c_1 & q_1 D_1 s_1 & D_2 c_2 & q_2 D_2 s_2 \end{vmatrix} = 0 \tag{608}$$

Where:

$$\begin{Bmatrix} s_i(z) = \frac{1}{2\sqrt{q_i}}\left[e^{\sqrt{q_i}z} - e^{-\sqrt{q_i}z}\right] = \begin{Bmatrix} s_1(z) = \begin{Bmatrix} \frac{1}{\sqrt{|q_1|}}\sinh\left(\sqrt{|q_1|}z\right) \ ; \ \lambda < \lambda_c \\ \frac{1}{\sqrt{|q_1|}}\sin\left(\sqrt{|q_1|}z\right) \ ; \ \lambda > \lambda_c \end{Bmatrix} \\ s_2(z) = \frac{1}{\sqrt{|q_2|}}\sin\left(\sqrt{|q_2|}z\right) \end{Bmatrix} \\ c_i(z) = \frac{1}{2}\left[e^{\sqrt{q_i}z} + e^{-\sqrt{q_i}z}\right] = \begin{Bmatrix} c_1(z) = \begin{Bmatrix} \cosh\left(\sqrt{|q_1|}z\right) \ ; \ \lambda < \lambda_c \\ \cos\left(\sqrt{|q_1|}z\right) \ ; \ \lambda > \lambda_c \end{Bmatrix} \\ c_2(z) = \cos\left(\sqrt{|q_2|}z\right) \end{Bmatrix} \\ s_i = s_i(1), c_i = c_i(1) \end{Bmatrix} \tag{609}$$

A further reduction in the equation:

$$\begin{vmatrix} (s_2 - s_1) & (q_1 c_2 - q_2 c_1) \\ (D_2 c_2 - D_1 c_1) & q_1 q_2 (D_2 s_2 - D_1 s_1) \end{vmatrix} = 0 \tag{610}$$

The determinant can be written in its simplest form:

$$F(\delta^2) = q_1 q_2 (s_2 - s_1)(D_2 s_2 - D_1 s_1) - (D_2 c_2 - D_1 c_1)(q_1 c_2 - q_2 c_1) = 0 \tag{611}$$

We solve for both cases:

- Case 1: When the polynomial has a positive real root a negative real root ($\delta < \delta_{cr}$).

We define:

$$\left\{ \begin{array}{c} q_1^* = \dfrac{-\left[\delta^2\left(\frac{1}{\alpha^2} + \mu^2\right)\right] + \sqrt{\left[\delta^2\left(\frac{1}{\alpha^2} + \mu^2\right)\right]^2 - 4\left[\delta^2\left(\frac{\mu^2}{\alpha^2}\delta^2 - 1\right)\right]}}{2} \\ q_2^* = \dfrac{\left[\delta^2\left(\frac{1}{\alpha^2} + \mu^2\right)\right] + \sqrt{\left[\delta^2\left(\frac{1}{\alpha^2} + \mu^2\right)\right]^2 - 4\left[\delta^2\left(\frac{\mu^2}{\alpha^2}\delta^2 - 1\right)\right]}}{2} \end{array} \right\} \tag{612}$$

So that:

$$\left\{ \begin{array}{c} q_1 = q_1^* \\ q_2 = -q_2^* \end{array} \right\} \tag{613}$$

Replacing:

$$\left\{ \begin{array}{c} s_1 = \dfrac{1}{\sqrt{q_1^*}} \sinh\left(\sqrt{q_1^*}\right), s_2 = \dfrac{1}{\sqrt{q_2^*}} \sin\left(\sqrt{q_2^*}\right) \\ c_1 = \cosh\left(\sqrt{q_1^*}\right), c_2 = \cos\left(\sqrt{q_2^*}\right) \\ D_1 = \left(q_1^* + \dfrac{\delta^2}{\alpha^2}\right), D_2 = \left(-q_2^* + \dfrac{\delta^2}{\alpha^2}\right) \end{array} \right\} \tag{614}$$

Rewriting the determinant as $F\left(\delta^2\right)$:

$$F(\delta^2) = q_1 q_2 (s_2 - s_1)(D_2 s_2 - D_1 s_1) - (D_2 c_2 - D_1 c_1)(q_1 c_2 - q_2 c_1) = 0 \tag{615}$$

After some simple manipulations, the modal characteristic equation is obtained, whose roots define a set of particular solutions that satisfy the differential equation of motion and the boundary conditions.

$$\left[2q_1^* q_2^* + \left(\frac{1}{1+\alpha^2\mu^2}\right)(q_2^* - q_1^*)^2\right] + \left[\left(\frac{1-\alpha^2\mu^2}{1+\alpha^2\mu^2}\right)(q_2^* - q_1^*)\right]\sqrt{q_1^*}\sqrt{q_2^*}\sinh\sqrt{q_1^*}\sin\sqrt{q_2^*} + \left[(q_1^{*2} + q_2^{*2}) - \left(\frac{1}{1+\alpha^2\mu^2}\right)(q_2^* - q_1^*)^2\right]\cosh\sqrt{q_1^*}\cos\sqrt{q_1^*} = 0 \tag{616}$$

From the equation, the following equation is obtained in terms of the values q_1^* and q_2^*:

$$\begin{Bmatrix} q_2^* - q_1^* = \delta^2\left(\frac{1}{\alpha^2} + \mu^2\right) \\ q_2^* q_1^* = \delta^2 - \frac{\mu^2}{\alpha^2}\delta^4 \end{Bmatrix} \rightarrow (q_2^* - q_1^*)\left\{\left[-\frac{\alpha^2\mu^2}{(1+\alpha^2\mu^2)^2}\right](q_2^* - q_1^*) + \frac{\alpha^2}{1+\alpha^2\mu^2}\right\} - q_2^* q_1^* = 0 \tag{617}$$

- Case 2: When the polynomial has two negative real roots ($\delta > \delta_{cr}$).

We define:

$$\begin{Bmatrix} q_1^* = \dfrac{\left[\delta^2\left(\frac{1}{\alpha^2} + \mu^2\right)\right] - \sqrt{\left[\delta^2\left(\frac{1}{\alpha^2} + \mu^2\right)\right]^2 - 4\left[\delta^2\left(\frac{\mu^2}{\alpha^2}\delta^2 - 1\right)\right]}}{2} \\ q_2^* = \dfrac{\left[\delta^2\left(\frac{1}{\alpha^2} + \mu^2\right)\right] + \sqrt{\left[\delta^2\left(\frac{1}{\alpha^2} + \mu^2\right)\right]^2 - 4\left[\delta^2\left(\frac{\mu^2}{\alpha^2}\delta^2 - 1\right)\right]}}{2} \end{Bmatrix} \tag{618}$$

So that:

$$\begin{Bmatrix} q_1 = -q_1^* \\ q_2 = -q_2^* \end{Bmatrix} \tag{619}$$

Replacing:

$$\begin{Bmatrix} s_1 = \frac{1}{\sqrt{q_1^*}}\sin\left(\sqrt{q_1^*}\right), s_2 = \frac{1}{\sqrt{q_2^*}}\sin\left(\sqrt{q_2^*}\right) \\ c_1 = \cos\left(\sqrt{q_1^*}\right), c_2 = \cos\left(\sqrt{q_2^*}\right) \\ D_1 = \left(-q_1^* + \frac{\delta^2}{\alpha^2}\right), D_2 = \left(-q_2^* + \frac{\delta^2}{\alpha^2}\right) \end{Bmatrix} \tag{620}$$

Rewriting the determinant as $F(\delta^2)$:

$$F(\delta^2) = q_1 q_2 (s_2 - s_1)(D_2 s_2 - D_1 s_1) - (D_2 c_2 - D_1 c_1)(q_1 c_2 - q_2 c_1) = 0 \tag{621}$$

After some simple manipulations, the modal characteristic equation is obtained, whose roots define a set of particular solutions that satisfy the differential equation of motion and the boundary conditions.

$$\left[2q_1^* q_2^* - \left(\frac{1}{1+\alpha^2\mu^2}\right)(q_1^* + q_2^*)^2\right] + \left[\left(\frac{1-\alpha^2\mu^2}{1+\alpha^2\mu^2}\right)(q_1^* + q_2^*)\right]\sqrt{q_1^*}\sqrt{q_2^*}\sin\sqrt{q_1^*}\sin\sqrt{q_2^*}$$
$$- \left[(q_1^{*2} + q_2^{*2}) - \left(\frac{1}{1+\alpha^2\mu^2}\right)(q_1^* + q_2^*)^2\right]\cos\sqrt{q_1^*}\cos\sqrt{q_2^*} = 0 \tag{622}$$

From the equation, the following equation is obtained in terms of the values q_1^* and q_2^*:

$$\left\{\begin{matrix} q_2^* + q_1^* = \delta^2\left(\frac{1}{\alpha^2} + \mu^2\right) \\ q_2^* q_1^* = -\delta^2 + \frac{\mu^2}{\alpha^2}\delta^4 \end{matrix}\right\} \rightarrow (q_1^* + q_2^*)\left\{\left[\frac{\alpha^2\mu^2}{(1+\alpha^2\mu^2)^2}\right](q_1^* + q_2^*) - \frac{\alpha^2}{1+\alpha^2\mu^2}\right\} - q_1^* q_2^* = 0 \tag{623}$$

For both cases, the eigenvalues q_1^* and q_2^* are derived by simultaneously solving the two equations that are a function of the values adopted for α and μ. Knowing the values of q_1^* and q_2^*, the frequencies and periods are calculated.

$$w = \frac{\delta}{H^2}\sqrt{\frac{K_b}{\rho A}} \rightarrow T = \frac{2\pi}{w} = \frac{2\pi H^2}{\delta}\sqrt{\frac{\rho A}{K_b}} \tag{624}$$

The mode shapes are shown below. As shown above, the value of α→∞ exactly reproduces the mode shape of an EBB (Euler Bernoulli) bending beam.

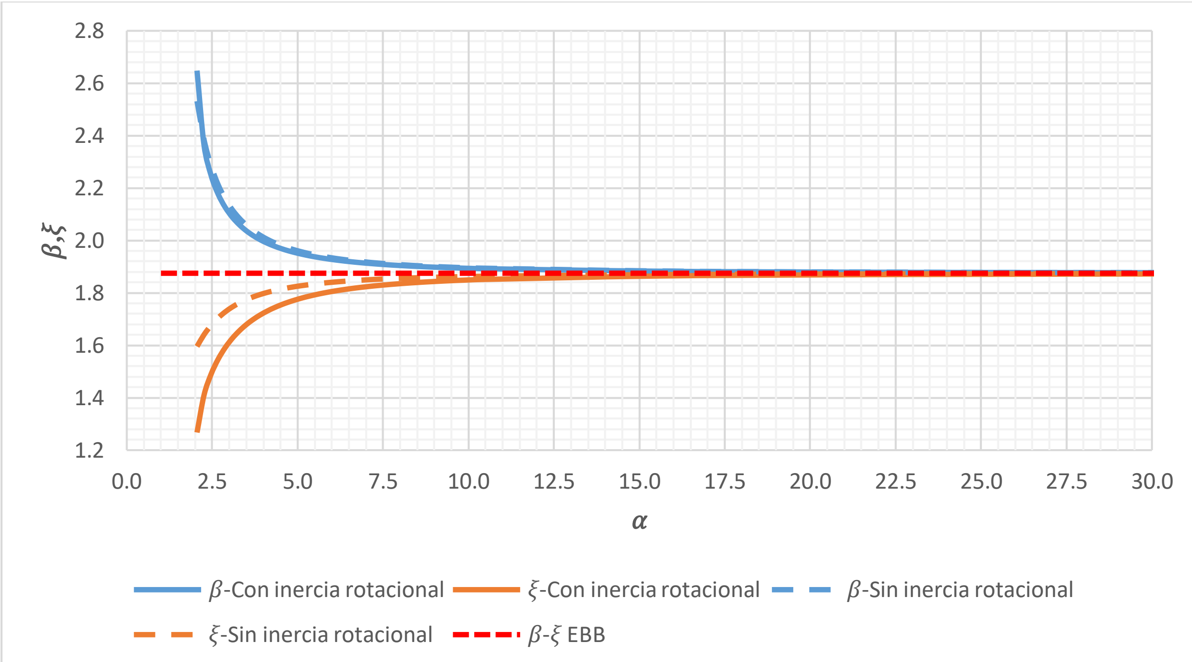


Figure 77. Variation of the parameters δ and β as a function of α for the first vibration mode.

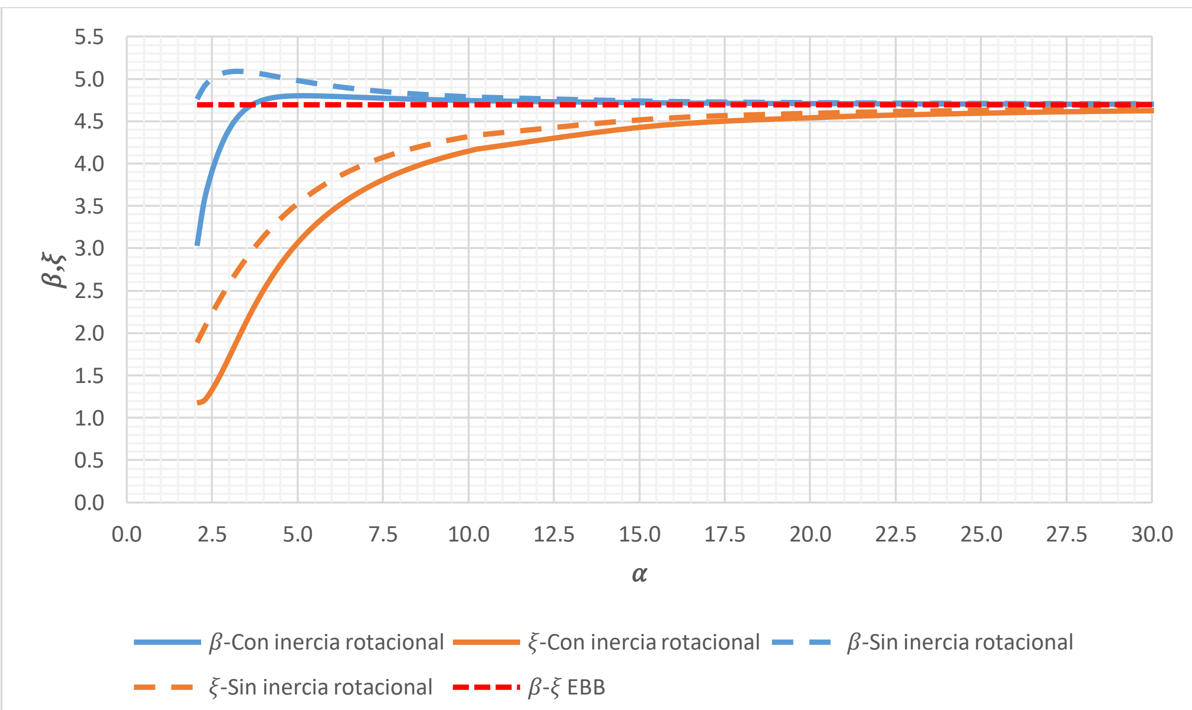


Figure 78. Variation of the parameters δ and β as a function of α for the second vibration mode.

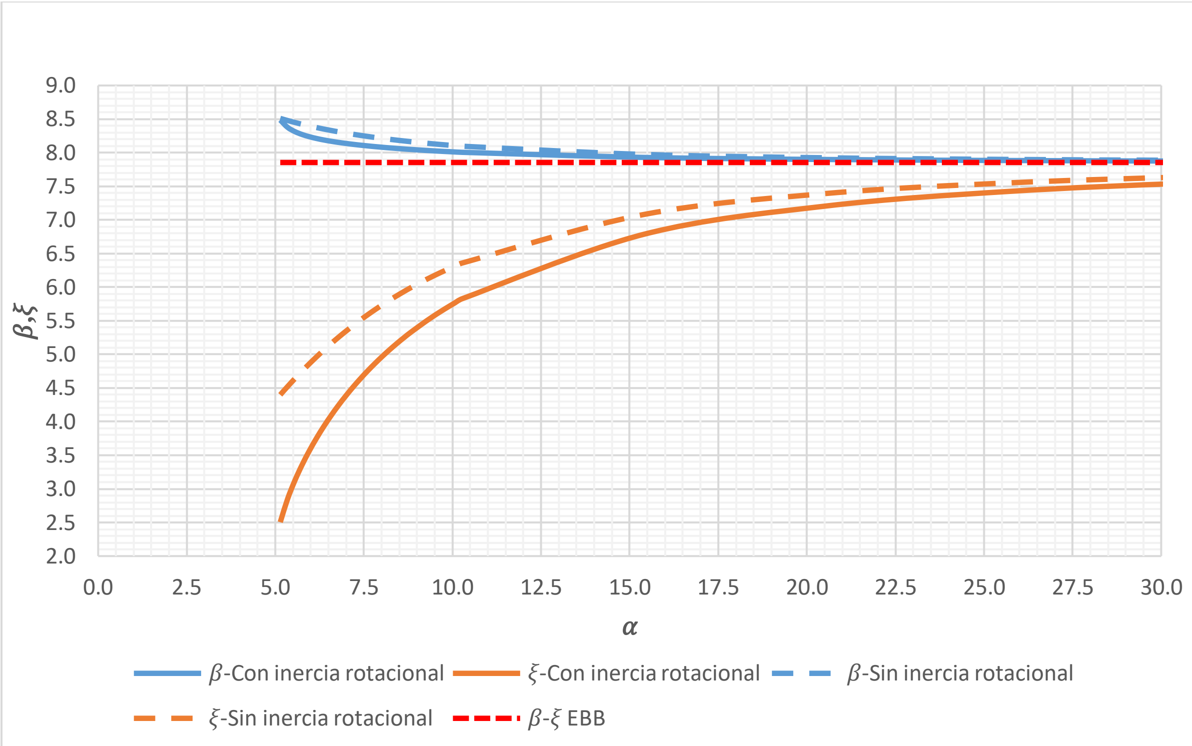


Figure 79. Variation of the parameters δ and β as a function of α for the third vibration mode.

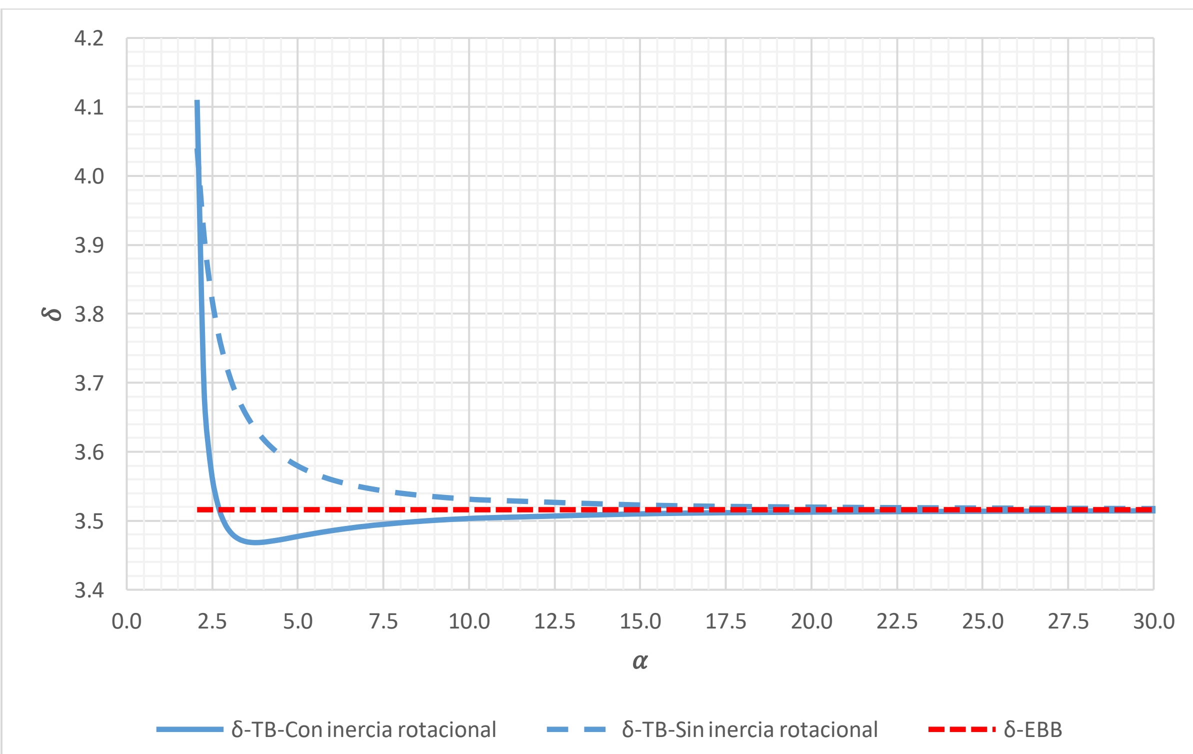


Figure 80. First eigenvalue of beam TB vs. EBB for the case of α variable.

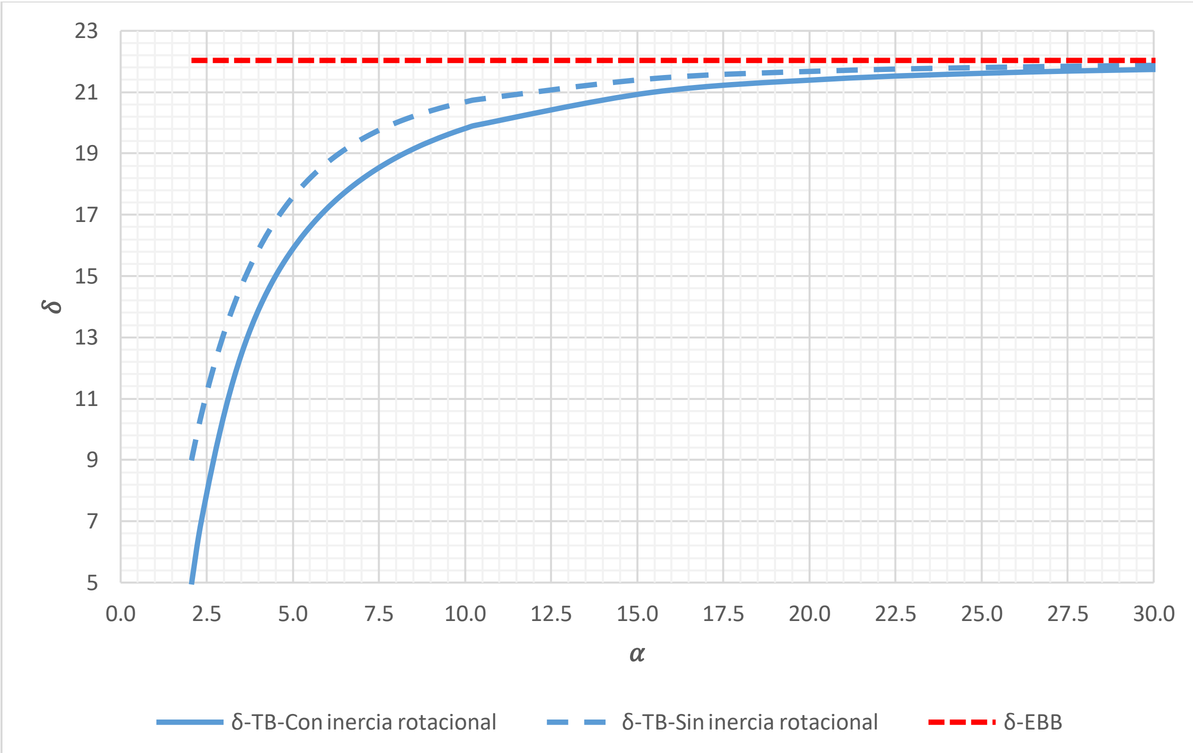


Figure 81. Second eigenvalue of beam TB vs. EBB for the case of α variable.

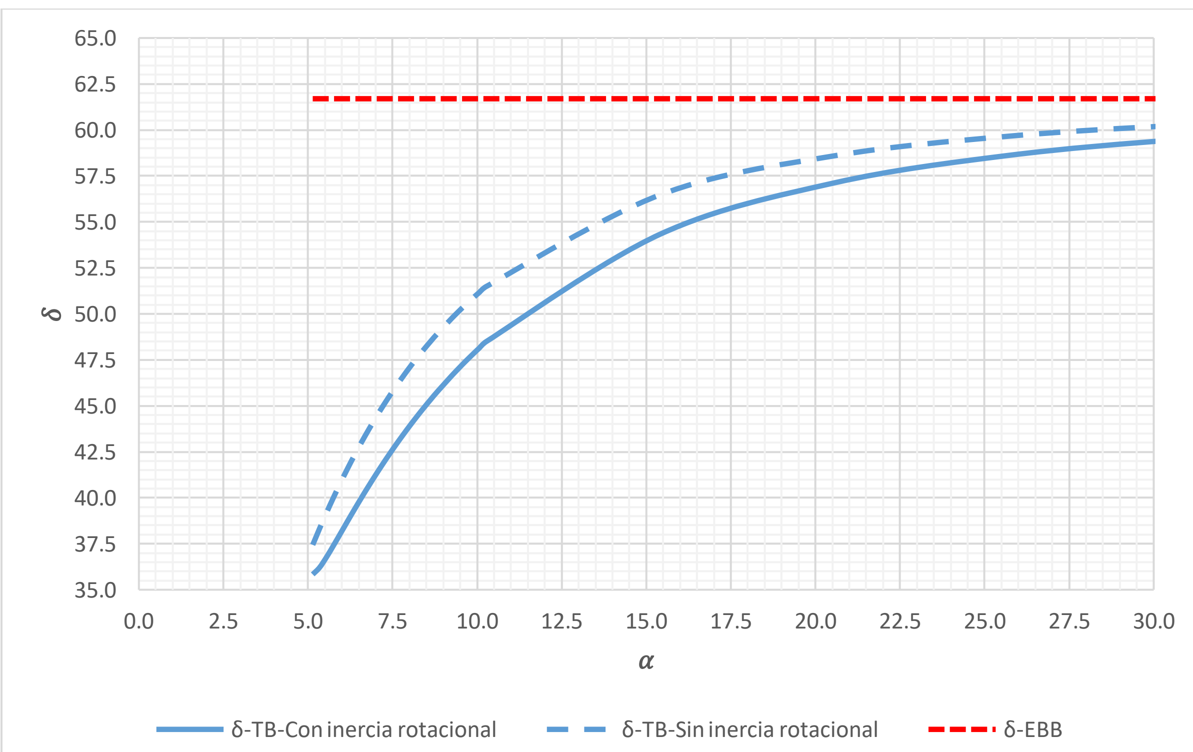


Figure 82. Third eigenvalue of beam TB vs. EBB for the case of α variable.

- **Mode Shapes**

Considering the first boundary conditions, normalizing to 1 at the top, and writing the resulting linear algebraic system in matrix form:

$$\begin{bmatrix} \sqrt{q_1} & -\sqrt{q_1} & \sqrt{q_2} & -\sqrt{q_2} \\ q_1 & q_1 & q_2 & q_2 \\ \sqrt{q_1}D_1e^{\sqrt{q_1}} & -\sqrt{q_1}D_1e^{-\sqrt{q_1}} & \sqrt{q_2}D_2e^{\sqrt{q_2}} & -\sqrt{q_2}D_2e^{-\sqrt{q_2}} \\ \sqrt{q_1}e^{\sqrt{q_1}} & -\sqrt{q_1}e^{-\sqrt{q_1}} & \sqrt{q_2}e^{\sqrt{q_2}} & -\sqrt{q_2}e^{-\sqrt{q_2}} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} \tag{625}$$

The eigenvectors can be obtained by solving the equation for the constants C_i. To simplify the matrix problem, the constants C_i will be transformed into a set of new constants $\bar{C}_i$ using the transformation:

$$\begin{Bmatrix} 2C_{2i-1} = \bar{C}_{2i-1} + \dfrac{1}{\sqrt{q_i}}\bar{C}_{2i} \\ 2C_{2i} = \bar{C}_{2i-1} - \dfrac{1}{\sqrt{q_i}}\bar{C}_{2i} \end{Bmatrix} ; \; i = 1, 2 \tag{626}$$

The equation reduces:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ q_1 & 0 & q_2 & 0 \\ q_1D_1s_1 & D_1c_1 & q_2D_2s_2 & D_2c_2 \\ q_1s_1 & c_1 & q_2s_2 & c_2 \end{bmatrix} \begin{Bmatrix} \bar{C}_1 \\ \bar{C}_2 \\ \bar{C}_3 \\ \bar{C}_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} \tag{627}$$

It is possible to express $\bar{C}_1$, $\bar{C}_2$ and $\bar{C}_3$ in terms of $\bar{C}_4$; the matrix equation reduces to:

$$\begin{bmatrix} 0 & 1 & 0 \\ q_1 & 0 & q_2 \\ q_1D_1s_1 & D_1c_1 & q_2D_2s_2 \end{bmatrix} \begin{Bmatrix} \bar{C}_1 \\ \bar{C}_2 \\ \bar{C}_3 \end{Bmatrix} = -\bar{C}_4 \begin{Bmatrix} 1 \\ 0 \\ D_2c_2 \end{Bmatrix} \tag{628}$$

Which can be solved for the constant $\bar{C}_i$:

$$\begin{Bmatrix} \bar{C}_1 \\ \bar{C}_2 \\ \bar{C}_3 \end{Bmatrix} = -\bar{C}_4 \begin{bmatrix} 0 & 1 & 0 \\ q_1 & 0 & q_2 \\ q_1D_1s_1 & D_1c_1 & q_2D_2s_2 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ 0 \\ D_2c_2 \end{Bmatrix} \tag{629}$$

Operating:

$$\begin{Bmatrix} \bar{C}_1 \\ \bar{C}_2 \\ \bar{C}_3 \end{Bmatrix} = \frac{\bar{C}_4}{D_2 s_2 - D_1 s_1} \begin{Bmatrix} \frac{1}{q_1}(D_2 c_2 - D_1 c_1) \\ -(D_2 s_2 - D_1 s_1) \\ -\frac{1}{q_2}(D_2 c_2 - D_1 c_1) \end{Bmatrix} = \bar{C}_4 \begin{Bmatrix} \frac{\eta}{q_1} \\ -1 \\ -\frac{\eta}{q_2} \end{Bmatrix} \tag{630}$$

Where the parameter η is given by:

$$\eta = \frac{D_2 c_2 - D_1 c_1}{D_2 s_2 - D_1 s_1} \tag{631}$$

Now we can obtain the displacement $H\lambda_{(z)}$ and $\emptyset_{(z)}$ corresponding to the eigenvector. Rewriting:

$$W_{(z)} = \begin{Bmatrix} H\lambda_{(z)} \\ \emptyset_{(z)} \end{Bmatrix} = \begin{Bmatrix} [D_1 c_1(z)]\bar{C}_1 + [D_1 s_1(z)]\bar{C}_2 + [D_2 c_2(z)]\bar{C}_3 + [D_2 s_2(z)]\bar{C}_4 \\ [q_1 s_1(z)]\bar{C}_1 + [c_1(z)]\bar{C}_2 + [q_2 s_2(z)]\bar{C}_3 + [c_2(z)]\bar{C}_4 \end{Bmatrix} \tag{632}$$

We normalize $\emptyset_{(1)} = 1$ on top:

$$\bar{C}_4 = \frac{1}{(c_2 - c_1) - \eta(s_2 - s_1)} \tag{633}$$

After some simple manipulations, we get:

$$\emptyset_{(z)} = \frac{[c_2(z) - c_1(z)] - \eta[s_2(z) - s_1(z)]}{[c_2 - c_1] - \eta[s_2 - s_1]} \tag{634}$$

- Case 1: When the polynomial has a positive real root a negative real root ($\delta < \delta_{cr}$).

$$\begin{Bmatrix} \eta = \dfrac{\left(-q_2^* + \frac{\delta^2}{\alpha^2}\right)\cos\left(\sqrt{q_2^*}\right) - \left(-q_1^* + \frac{\delta^2}{\alpha^2}\right)\cosh\left(\sqrt{q_1^*}\right)}{\left(-q_2^* + \frac{\delta^2}{\alpha^2}\right)\frac{\sin\left(\sqrt{q_2^*}\right)}{\sqrt{q_2^*}} - \left(-q_1^* + \frac{\delta^2}{\alpha^2}\right)\frac{\sinh\left(\sqrt{q_1^*}\right)}{\sqrt{q_1^*}}} \\ \emptyset_{(z)} = \dfrac{\left[\cos\left(\sqrt{q_2^*}z\right) - \cosh\left(\sqrt{q_1^*}z\right)\right] - \eta\left[\frac{\sin\left(\sqrt{q_2^*}z\right)}{\sqrt{q_2^*}} - \frac{\sinh\left(\sqrt{q_1^*}z\right)}{\sqrt{q_1^*}}\right]}{\left[\cos\left(\sqrt{q_2^*}\right) - \cosh\left(\sqrt{q_1^*}\right)\right] - \eta\left[\frac{\sin\left(\sqrt{q_2^*}\right)}{\sqrt{q_2^*}} - \frac{\sinh\left(\sqrt{q_1^*}\right)}{\sqrt{q_1^*}}\right]} \end{Bmatrix} \tag{635}$$

- Case 2: When the polynomial has two negative real roots ($\delta > \delta_{cr}$).

$$\left\{ \begin{aligned} \eta &= \frac{\left(-q_2^* + \frac{\delta^2}{\alpha^2}\right)\cos\left(\sqrt{q_2^*}\right) - \left(-q_1^* + \frac{\delta^2}{\alpha^2}\right)\cos\left(\sqrt{q_1^*}\right)}{\left(-q_2^* + \frac{\delta^2}{\alpha^2}\right)\frac{\sin\left(\sqrt{q_2^*}\right)}{\sqrt{q_2^*}} - \left(-q_1^* + \frac{\delta^2}{\alpha^2}\right)\frac{\sin\left(\sqrt{q_1^*}\right)}{\sqrt{q_1^*}}} \\ \emptyset_{(z)} &= \frac{\left[\cos\left(\sqrt{q_2^*}z\right) - \cos\left(\sqrt{q_1^*}z\right)\right] - \eta\left[\frac{\sin\left(\sqrt{q_2^*}z\right)}{\sqrt{q_2^*}} - \frac{\sin\left(\sqrt{q_1^*}z\right)}{\sqrt{q_1^*}}\right]}{\left[\cos\left(\sqrt{q_2^*}\right) - \cos\left(\sqrt{q_1^*}\right)\right] - \eta\left[\frac{\sin\left(\sqrt{q_2^*}\right)}{\sqrt{q_2^*}} - \frac{\sin\left(\sqrt{q_1^*}\right)}{\sqrt{q_1^*}}\right]} \end{aligned} \right\} \qquad (636)$$

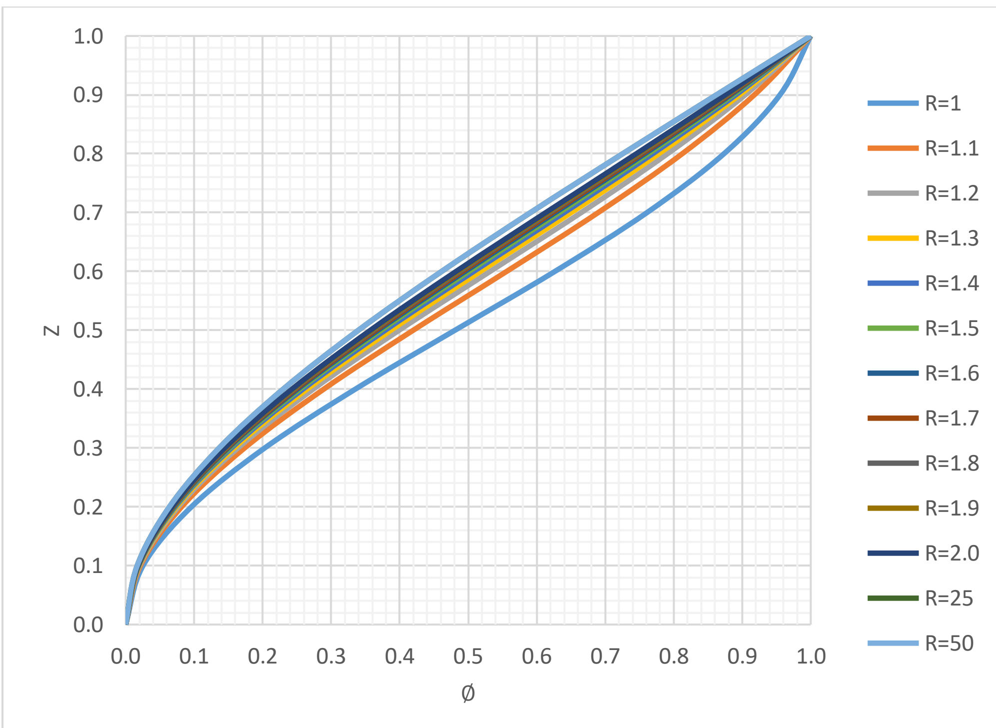


Figure 83. Forms of the first vibration mode as a function of the *R* value.

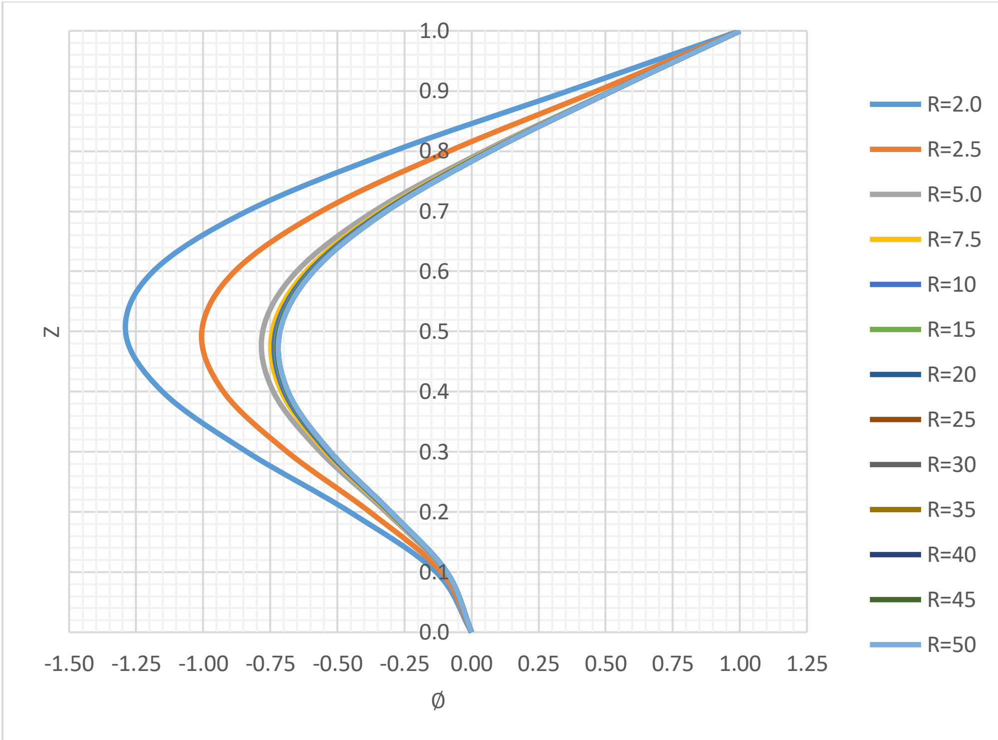


Figure 84. Forms of the second mode of vibration as a function of the *R* value.

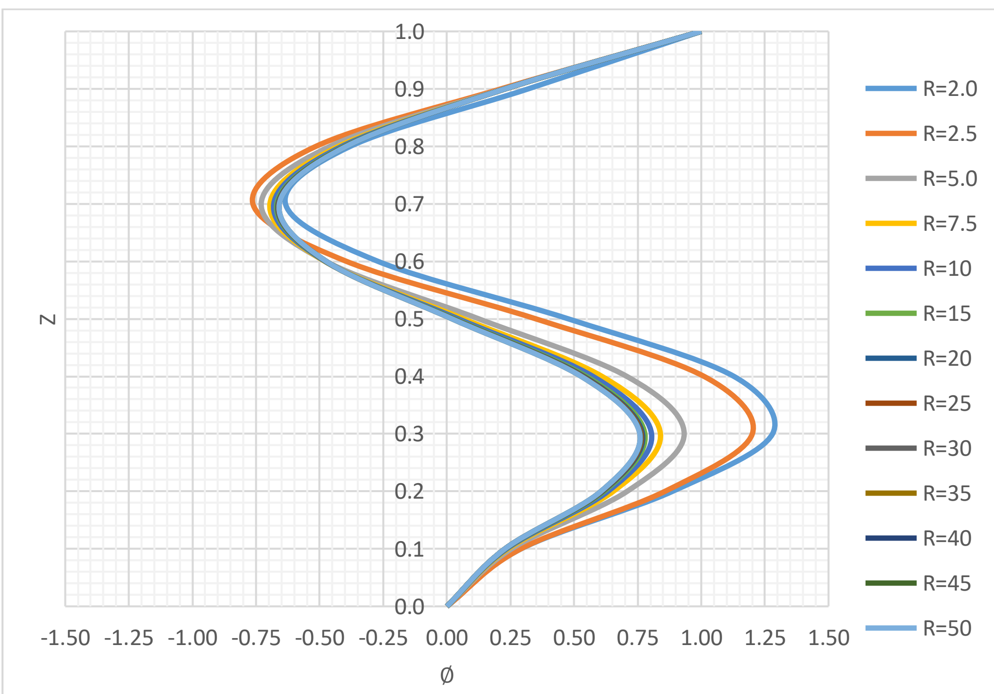


Figure 85. Forms of the third mode of vibration as a function of the *R* value.

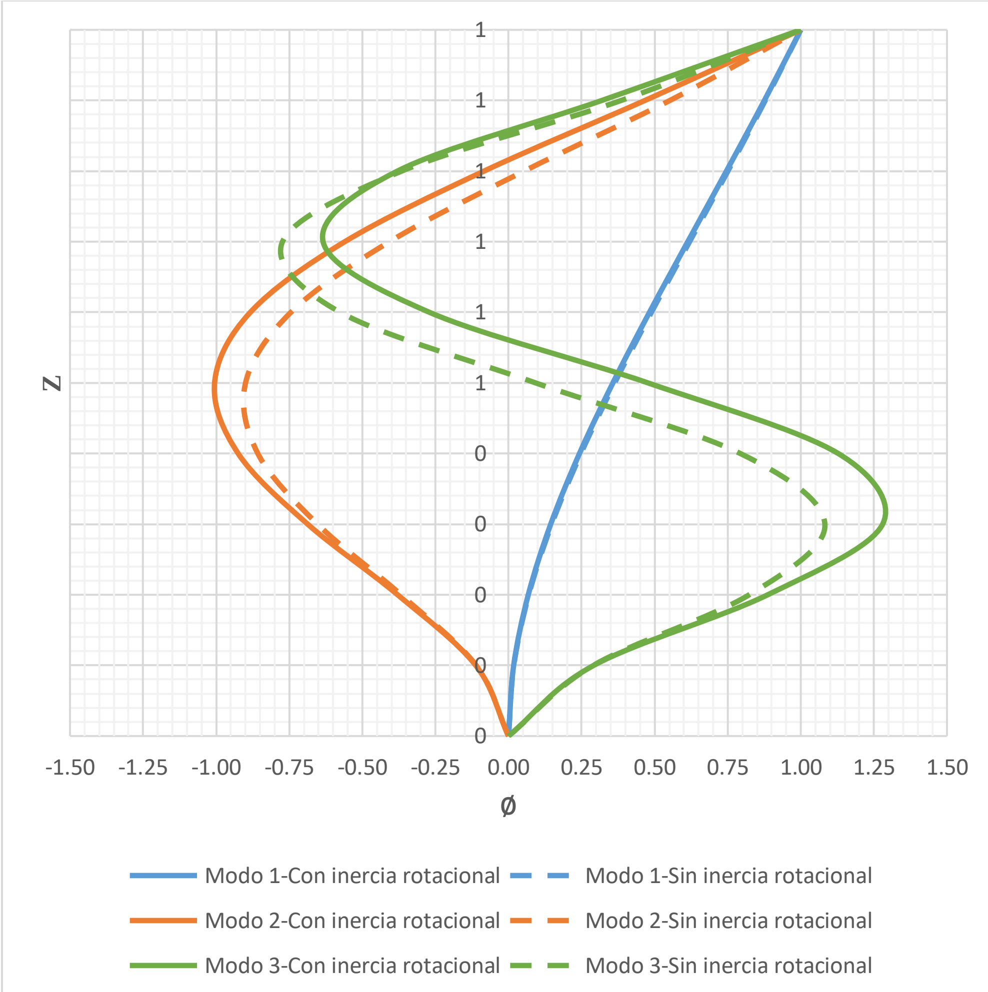


Figure 86. Shapes of vibration modes for $R = 2.5$.

4.2.3.2 Case 2

The relationship between forces and displacements between two consecutive floors is obtained by taking into account the transfer matrix and the vector of external point forces. For the j-th floor:

$$\begin{Bmatrix} u_{i+1}(0) \\ \theta_{i+1}(0) \\ M_{i+1}(0) \\ V_{i+1}(0) \end{Bmatrix} = T_i(0) \begin{Bmatrix} u_i(0) \\ \theta_i(0) \\ M_i(0) \\ V_i(0) \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ m_i w^2 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_i(0) \\ \theta_i(0) \\ M_i(0) \\ V_i(0) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m_i w^2 & 0 & 0 & 1 \end{bmatrix} T_i(0) \begin{Bmatrix} u_i(0) \\ \theta_i(0) \\ M_i(0) \\ V_i(0) \end{Bmatrix} \quad (637)$$

Rewriting:

$$\begin{Bmatrix} u_{i+1}(0) \\ \theta_{i+1}(0) \\ M_{i+1}(0) \\ V_{i+1}(0) \end{Bmatrix} = T_{wi}(0) \begin{Bmatrix} u_i(0) \\ \theta_i(0) \\ M_i(0) \\ V_i(0) \end{Bmatrix} \tag{638}$$

Where:

$$T_{wi}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m_i w^2 & 0 & 0 & 1 \end{bmatrix} T_i(0) \tag{639}$$

Expressing the equation for the nth floor between product symbols:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_{wk}(0) \begin{Bmatrix} u_1(h_1) \\ \theta_1(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{640}$$

Where:

$$\mathrm{t} = \prod_{k=1}^{n} T_{wk}(0) \tag{641}$$

Replacing this parameter:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ \theta_1(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{642}$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 4x4 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ \theta_{(1)} = 0 \\ \theta'_{(0)} = 0 \\ u'_{(0)} - \theta_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1(h_1) = 0 \\ \theta_1(h_1) = 0 \\ M_n(0) = 0 \\ V_n(0) = 0 \end{Bmatrix} \tag{643}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{644}$$

Solving for the bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{645}$$

Which has a different solution than the trivial one if the determinant is equal to zero (the matrix of coefficients is singular). Solving the determinant finds the angular frequencies of the beam.

4.2.4 Parallel Coupling of Bending Beam and Shear Beam of a Field (CTB) - Translational Behavior

4.2.4.1 Case 1

The potential energy and kinetic energy of the CTB model of a field are:

$$V = \frac{1}{2}\int_0^H \left(K_b {u''_{(x,t)}}^2 + K_s {u'_{(x,t)}}^2\right) dx, T = \frac{1}{2}\int_0^H \left(\gamma_u {\dot{u}_{(x,t)}}^2\right) dx \tag{646}$$

Where:

$$\gamma_u = \rho A \tag{647}$$

Consequently, the total potential energy of the CTB beam of a field is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H \left[\gamma_u {\dot{u}_{(x,t)}}^2 - K_b {u''_{(x,t)}}^2 - K_s {u'_{(x,t)}}^2\right] dx \tag{648}$$

Closed-form solutions of the model are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\delta\mathcal{U} = \int_0^H \left[\gamma_u \dot{u}_{(x,t)} \delta\dot{u}_{(x,t)} - K_b u''_{(x,t)} \delta u''_{(x,t)} - K_s u'_{(x,t)} \delta u'_{(x,t)}\right] dx \tag{649}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\begin{aligned} \delta\mathcal{U} = \left[\gamma_u \dot{u}_{(x,t)} + K_b u'''_{(x,t)} - K_s u'_{(x,t)}\right] \delta u_{(x,t)}\Big|_0^H - \left[K_b u''_{(x,t)}\right] \delta u'_{(x,t)}\Big|_0^H \\ - \int_0^H \left[\gamma_u \ddot{u}_{(x,t)} + K_b u''''_{(x,t)} - K_s u''_{(x,t)}\right] \delta u \end{aligned} \tag{650}$$

Setting the terms equal to zero, the following equation results:

$$\gamma_u \ddot{u}_{(x,t)} + K_b u''''_{(x,t)} - K_s u''_{(x,t)} = 0 \tag{651}$$

And boundary conditions:

$$\begin{Bmatrix} K_b u'''_{(H)} - K_s u'_{(H)} = 0 \\ u''_{(H)} = 0 \end{Bmatrix} \tag{652}$$

The partial differential equation can be solved by separating variables, applying a solution of the following type:

$$u_{(x,t)} = \emptyset_{(x)} q_{(t)} \tag{653}$$

Where $\emptyset_{(x)}$ defines the variation of the displacement along the length of the beam, while $q_{(t)}$ does so with time. Replacing and collecting like terms, we get the following:

$$\frac{\ddot{q}_{(t)}}{q_{(t)}} + \frac{K_b}{\gamma_u} \cdot \frac{1}{\emptyset_{(x)}} \emptyset''''_{(x)} - \frac{K_s}{\gamma_u} \cdot \frac{1}{\emptyset_{(x)}} \emptyset''_{(x)} = 0 \tag{654}$$

Because the time and height coordinates are independent variables, each of the terms must equal a constant with opposite signs, to ensure that the net result is zero. Consequently, it can be divided into two ordinary differential equations:

$$\begin{Bmatrix} \ddot{q}_{(t)} + w^2 q_{(t)} = 0 \\ K_b \emptyset''''_{(x)} - K_s \emptyset''_{(x)} - \gamma_u w^2 \emptyset_{(x)} = 0 \end{Bmatrix} \tag{655}$$

Where the first equation is the same that governs the behavior of an SDOF system with vibration frequency w.

A fourth order differential equation is obtained. Normalizing the differential equation by the variable $z = x/H$:

$$\emptyset''''_{(z)} - \left(H^2 \frac{K_s}{K_b}\right) \emptyset''_{(x)} - \left(\frac{\gamma_u w^2 H^4}{K_b}\right) \emptyset_{(x)} = 0 \tag{656}$$

The equation is rewritten:

$$\emptyset''''_{(z)} - \alpha^2 \emptyset''_{(z)} - \delta^2 \emptyset_{(z)} = 0 \tag{657}$$

Where:

$$\left\{\alpha = H\sqrt{\frac{K_s}{K_b}}, \delta = \sqrt{\frac{\gamma_u H^4}{K_b} w^2}\right\} \tag{658}$$

A solution can be obtained in the following way for the mode forms:

$$\emptyset_{(z)} = C_1 \cos(\beta z) + C_2 \sin(\beta z) + C_3 \cosh(\xi z) + C_4 \sinh(\xi z) \tag{659}$$

Where:

$$\left\{\begin{array}{c} \xi = \sqrt{\dfrac{\alpha^2 + \sqrt{\alpha^4 + 4\delta^2}}{2}} \\ \beta = \sqrt{\dfrac{-\alpha^2 + \sqrt{\alpha^4 + 4\delta^2}}{2}} \end{array}\right\} \rightarrow \left\{\begin{array}{c} \xi^2 = \beta^2 + \alpha^2 \\ \beta^2 \xi^2 = \delta^2 \end{array}\right\} \tag{660}$$

- **Frequency and Periods of Vibration**

The following boundary conditions are considered:

$$\left\{\begin{array}{c} \emptyset_{(0)} = 0 \\ \emptyset'_{(0)} = 0 \\ \emptyset'''_{(1)} - \alpha^2 \emptyset'_{(1)} = 0 \\ \emptyset''_{(1)} = 0 \end{array}\right\} \tag{661}$$

Writing in matrix form the linear algebraic system resulting from expanding the boundary conditions:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \beta & 0 & \xi \\ -\beta^2 \cos\beta & -\beta^2 \sin\beta & \xi^2 \cosh\xi & \xi^2 \sinh\xi \\ \beta(\beta^2 + \alpha^2)\sin\beta & -\beta(\beta^2 + \alpha^2)\cos\beta & \xi(\xi^2 - \alpha^2)\sinh\xi & \xi(\xi^2 - \alpha^2)\cosh\xi \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = 0 \tag{662}$$

Which has a different solution than the trivial one if the determinant is equal to zero (the matrix of coefficients is singular).

After some simple manipulations, the modal characteristic equation is obtained, whose roots define a set of particular solutions that satisfy the differential equation of motion and the boundary conditions.

$$[(\beta^4+\xi^4)-\alpha^2(\xi^2-\beta^2)]+[2\alpha^2-(\xi^2-\beta^2)]\xi\beta\sin\beta\sinh\xi \\ +[2\beta^2\xi^2+\alpha^2(\xi^2-\beta^2)]\cos\beta\cosh\xi=0 \qquad (663)$$

And taking into account the relationships between the eigenvalues β, ξ and α; is obtained:

$$2+\left(\frac{\xi^2-\beta^2}{\xi\beta}\right)\sin\beta\sinh\xi+\left(\frac{\xi^4+\beta^4}{\beta^2\xi^2}\right)\cos\beta\cosh\xi=0 \qquad (664)$$

It can be seen that this characteristic equation only depends on the eigenvalues ξ and β, which in turn depend on the parameter α; that is, the dynamic properties of the classical CTB beam of a field depend only on the parameter α.

The eigenvalues ξ and β are derived by numerically solving the characteristic equation. Knowing the values of ξ and β, the value of the parameter δ is calculated, and as a consequence, the frequencies and periods of vibration of the model are obtained.

$$w=\frac{\delta}{H^2}\sqrt{\frac{K_b}{\rho A}}\rightarrow T=\frac{2\pi}{w}=\frac{2\pi H^2}{\delta}\sqrt{\frac{\rho A}{K_b}} \qquad (665)$$

- **Eigenvalues**

Figures 87, 88 and 89 show the first three eigenvalues as a function of the variable α. As can be seen for the three eigenvalues; when $\alpha=0$, eigenvalues are obtained that are identical to those obtained for the bending beam EBB and when $\alpha=100$, eigenvalues are obtained that are identical to those obtained for the shear beam SB.

It is observed that there is a strong and notable variability for the three eigenvalues in the range of α values; that is, the CTB model of a field is very sensitive to the variability of the parameter α. This variability directly affects the dynamic properties of the model such as the period of vibration and the mode shape.

When analyzing the parametric analysis developed to plot the eigenvalues as a function of the parameter α; It is found that the percentage difference between the eigenvalues of the CTB-EBB and CTB-SB models are 3.79% and 23.89% for the first mode, 7.56% and 7.15% for the second mode, and 4.29% and 4.31% for the third mode of vibration.

Based on this observation, it is very important to underline that this variability is accentuated more in the case of the first vibration mode, directly affecting the dynamic properties such as the vibration period and the modal shape. On the other hand, the variability of the second and third modes is less pronounced; also noting that this variability decreases even more as the number of modes to be considered increases.

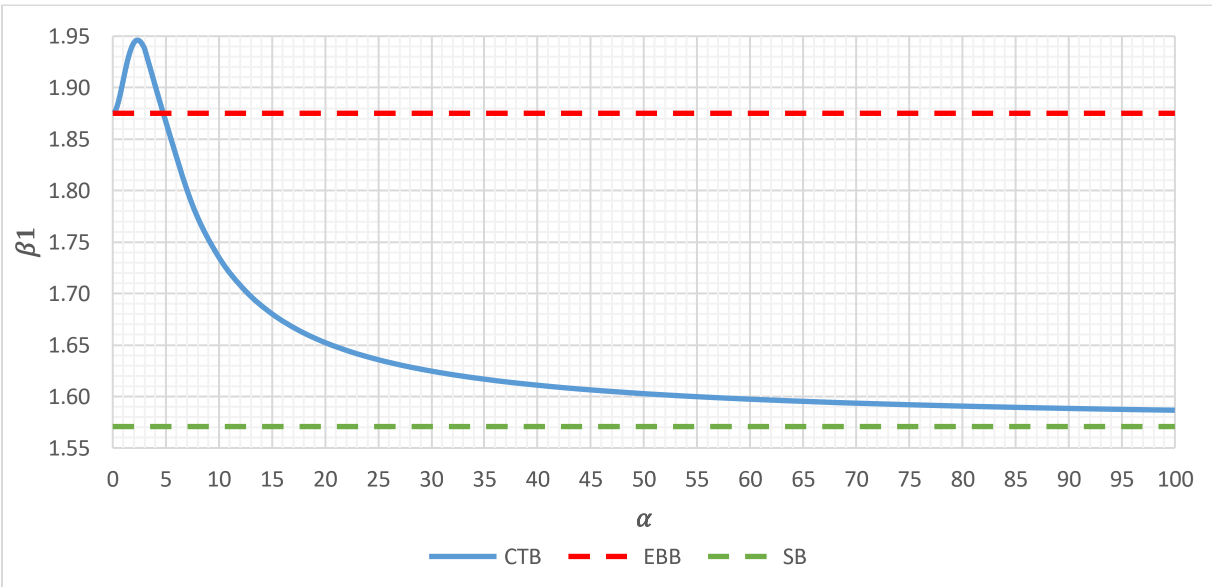


Figure 87. First eigenvalue β_1 for the case of α variable.

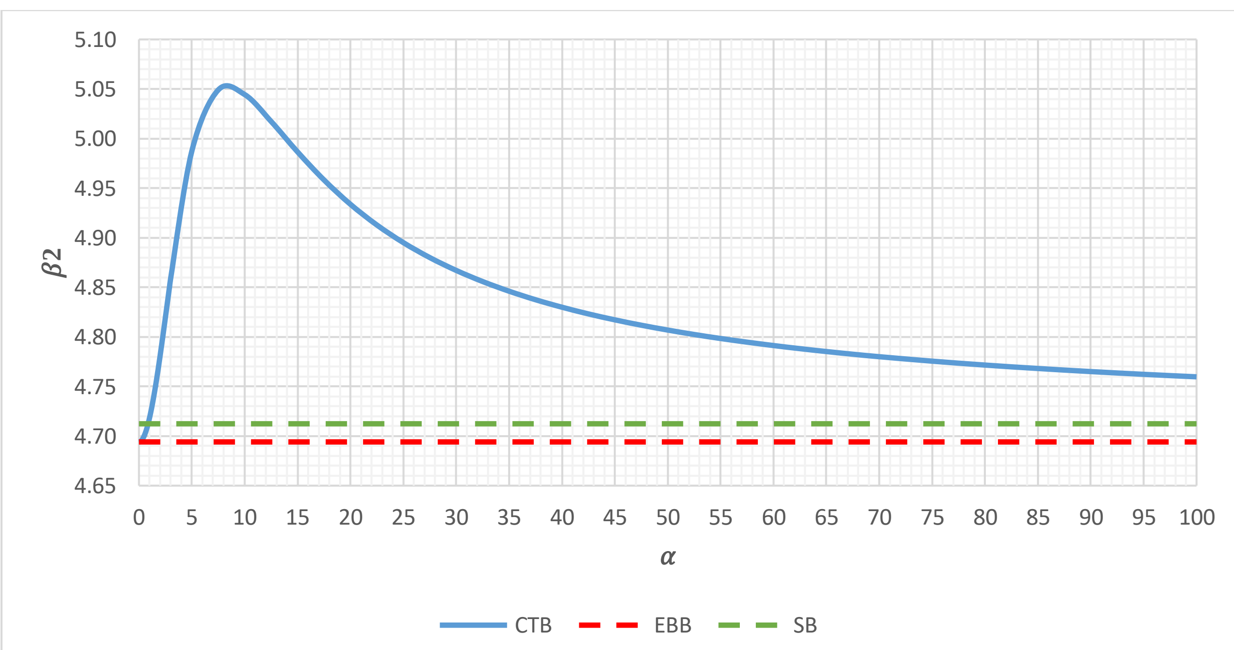


Figure 88. Second eigenvalue β_2 for the case of α variable.

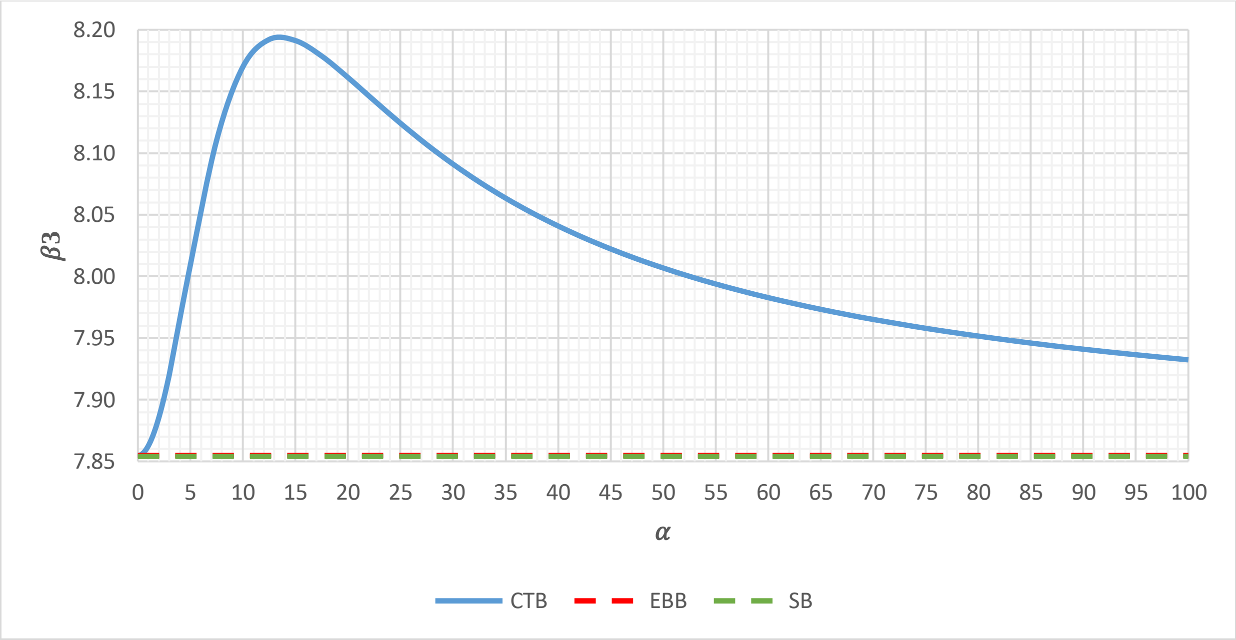


Figure 89. Third eigenvalue β_3 for the case of α variable.

- **Mode Shapes**

Considering the first boundary conditions, normalizing to 1 at the top, and writing the resulting linear algebraic system in matrix form:

Writing in matrix form the linear algebraic system resulting from expanding the boundary conditions:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \beta & 0 & \xi \\ -\beta^2 \cos\beta & -\beta^2 \sin\beta & \xi^2 \cosh\xi & \xi^2 \sinh\xi \\ \cos\beta & \sin\beta & \cosh\xi & \sinh\xi \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} \tag{666}$$

After some simple manipulations:

$$\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \frac{1}{-\frac{\xi}{\beta} Sen\beta + Senh\xi + \eta[Cos\beta - Cosh\xi]} \begin{Bmatrix} \eta \\ -\frac{\xi}{\beta} \\ -\eta \\ 1 \end{Bmatrix} \tag{667}$$

Where:

$$\eta = \frac{\xi^2 Senh\xi + \xi\beta Sen\beta}{\xi^2 Cosh\xi + \beta^2 Cos\beta} \quad (668)$$

Replacing these coefficients, a solution of the following form can be obtained for the mode forms:

$$\emptyset_{(z)} = \frac{-\frac{\xi}{\beta}Sen(\beta z) + Senh(\xi z) + \eta[Cos(\beta z) - Cosh(\xi z)]}{-\frac{\xi}{\beta}Sen\beta + Senh\xi + \eta[Cos\beta - Cosh\xi]} \quad (669)$$

In Figures 90, 91 and 92, the first three forms of vibration mode of the classic CTB model of a field are presented. As mentioned, it has been normalized to one at the top of the model. In addition, modal forms have been represented for different values of α.

As observed; it is easy to notice that the shapes of the corresponding modes for relatively small values of α ($\alpha \to 0$) are identical to that found for the pure bending beam (EBB), while for relatively large values of α ($\alpha \to \infty$) they are identical to what was found for the shear beam (SB). It is also observed that the CTB beam of a field adequately reproduces the mode shapes for dual structures with intermediate values of α ($0 < \alpha < \infty$).

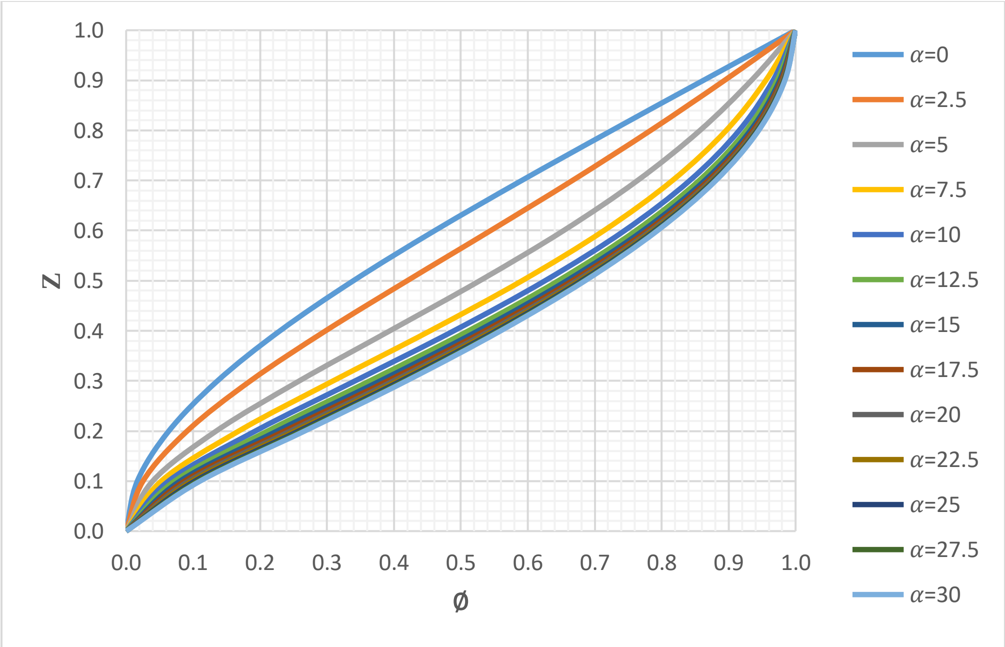


Figure 90. Shapes of the first vibration mode as a function of the α value.

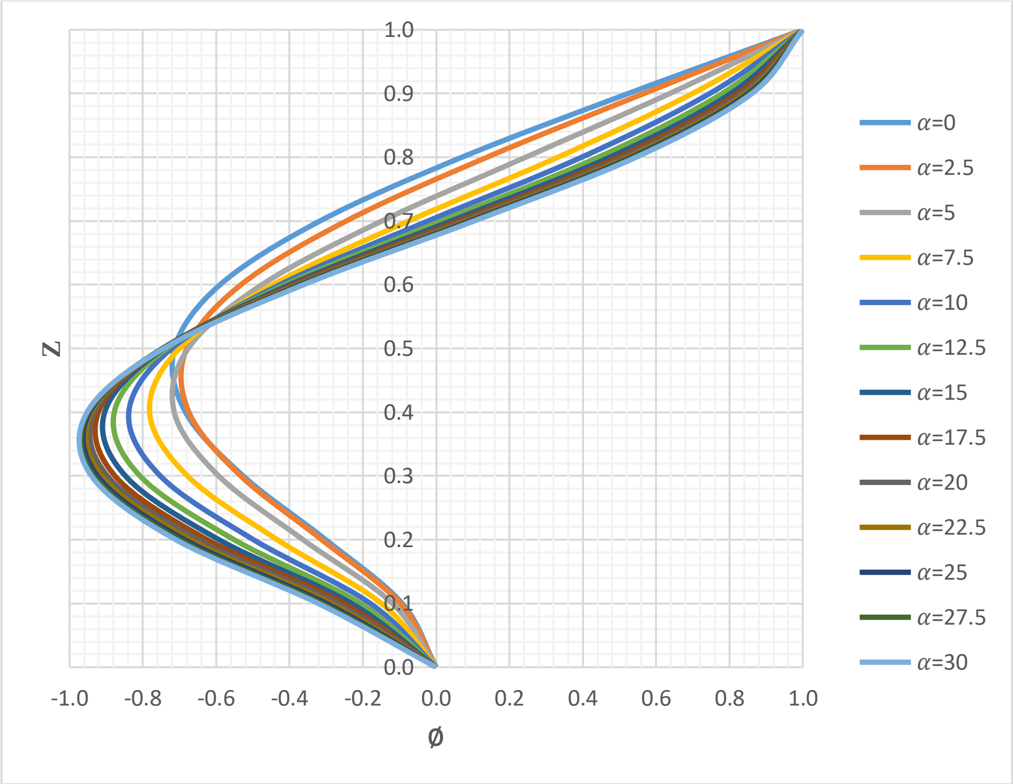


Figure 91. Shapes of the second vibration mode as a function of the α value.

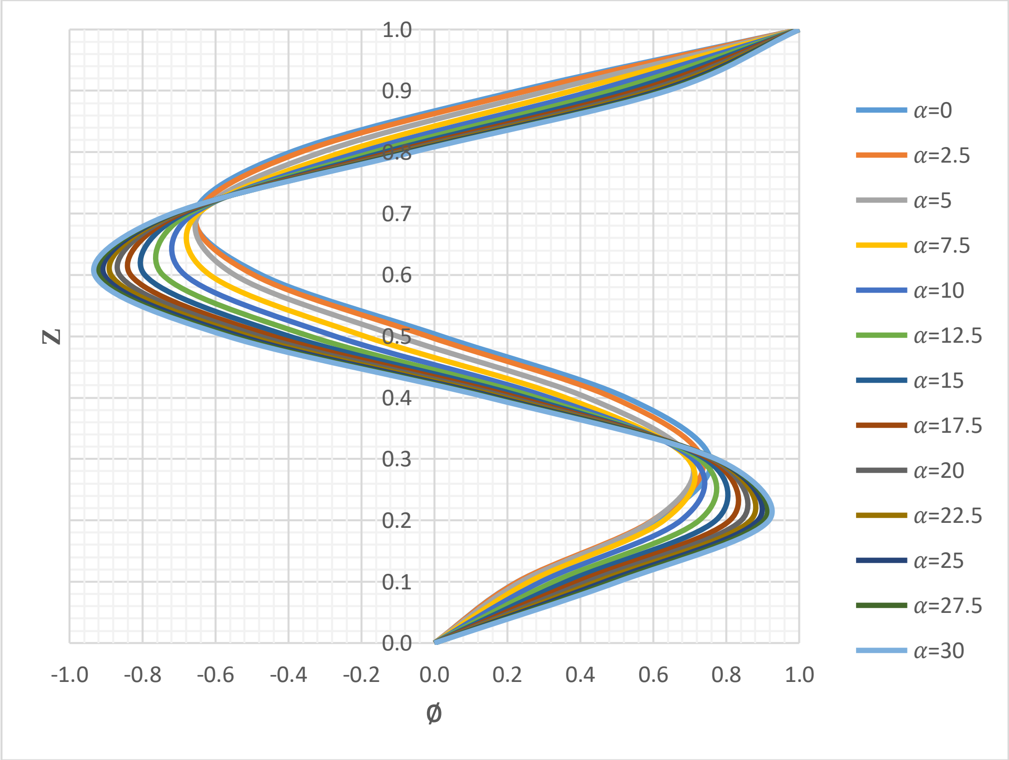


Figure 92. Shapes of the third mode of vibration as a function of the α value.

- **Special Cases**

a) When $\alpha \to 0$. This situation corresponds to a bending beam (Euler Bernoulli).

$$\alpha = 0 \to \begin{Bmatrix} \beta_1 = 1.87510 \to w_1 = \dfrac{3.51600}{H^2}\sqrt{\dfrac{K_b}{\gamma_u}} \to T_1 = 1.78703H^2\sqrt{\dfrac{\gamma_u}{K_b}} \\ \beta_2 = 4.69405 \to w_2 = \dfrac{22.03411}{H^2}\sqrt{\dfrac{K_b}{\gamma_u}} \to T_2 = 0.28516H^2\sqrt{\dfrac{\gamma_u}{K_b}} \\ \beta_3 = 7.85475 \to w_3 = \dfrac{61.69710}{H^2}\sqrt{\dfrac{K_b}{\gamma_u}} \to T_3 = 0.10184H^2\sqrt{\dfrac{\gamma_u}{K_b}} \end{Bmatrix} \tag{670}$$

b) When $\alpha = 2.5$; which corresponds to a dual structure with intermediate α.

$$\alpha = 2.5 \to \begin{Bmatrix} \beta_1 = 1.9455 \to w_1 = \dfrac{6.16296}{H^2}\sqrt{\dfrac{K_b}{\gamma_u}} \to T_1 = 1.01951H^2\sqrt{\dfrac{\gamma_u}{K_b}} \\ \beta_2 = 4.8195 \to w_2 = \dfrac{26.16664}{H^2}\sqrt{\dfrac{K_b}{\gamma_u}} \to T_2 = 0.24012H^2\sqrt{\dfrac{\gamma_u}{K_b}} \\ \beta_3 = 7.90190 \to w_3 = \dfrac{65.49051}{H^2}\sqrt{\dfrac{K_b}{\gamma_u}} \to T_3 = 0.09594H^2\sqrt{\dfrac{\gamma_u}{K_b}} \end{Bmatrix} \tag{671}$$

c) When $\alpha = 30$; which roughly corresponds to a shear beam.

$$\alpha = 30 \to \begin{Bmatrix} \beta_1 = 1.62465 \to w_1 = \dfrac{48.81092}{H^2}\sqrt{\dfrac{K_b}{\gamma_u}} \to T_1 = 0.12872H^2\sqrt{\dfrac{\gamma_u}{K_b}} \\ \beta_2 = 4.86705 \to w_2 = \dfrac{147.92045}{H^2}\sqrt{\dfrac{K_b}{\gamma_u}} \to T_2 = 0.04248H^2\sqrt{\dfrac{\gamma_u}{K_b}} \\ \beta_3 = 8.09105 \to w_3 = \dfrac{251.40457}{H^2}\sqrt{\dfrac{K_b}{\gamma_u}} \to T_3 = 0.02499H^2\sqrt{\dfrac{\gamma_u}{K_b}} \end{Bmatrix} \tag{672}$$

It is observed that the values are almost identical to those obtained when considering a pure shear beam. The error made is 3.43%, 3.28% and 3.02% for the first, second and third modes, respectively. However, these errors are acceptable from an engineering point of view; Therefore, it is practical to set a value of $\alpha = 30$ to characterize buildings whose behavior is pure shear. Considering a value of α=100 reduces the error to 1.01%.

4.2.4.2 Case 2

The relationship between forces and displacements between two consecutive floors is obtained by taking into account the transfer matrix and the vector of external point forces. For the j-th floor:

$$\begin{Bmatrix} u_{i+1}(0) \\ u'_{i+1}(0) \\ M_{i+1}(0) \\ V_{i+1}(0) \end{Bmatrix} = T_i(0)\begin{Bmatrix} u_i(0) \\ u'_i(0) \\ M_i(0) \\ V_i(0) \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ m_i w^2 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_i(0) \\ u'_i(0) \\ M_i(0) \\ V_i(0) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m_i w^2 & 0 & 0 & 1 \end{bmatrix} T_i(0)\begin{Bmatrix} u_i(0) \\ u'_i(0) \\ M_i(0) \\ V_i(0) \end{Bmatrix} \quad (673)$$

Rewriting:

$$\begin{Bmatrix} u_{i+1}(0) \\ u'_{i+1}(0) \\ M_{i+1}(0) \\ V_{i+1}(0) \end{Bmatrix} = T_{wi}(0)\begin{Bmatrix} u_i(0) \\ u'_i(0) \\ M_i(0) \\ V_i(0) \end{Bmatrix} \quad (674)$$

Where:

$$T_{wi}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m_i w^2 & 0 & 0 & 1 \end{bmatrix} T_i(0) \quad (675)$$

Expressing the equation for the nth floor between product symbols:

$$\begin{Bmatrix} u_n(0) \\ u'_n(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_{wk}(0)\begin{Bmatrix} u_1(h_1) \\ u'_n(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \quad (676)$$

Where:

$$\mathrm{t} = \prod_{k=1}^{n} T_{wk}(0) \quad (677)$$

Replacing this parameter:

$$\begin{Bmatrix} u_n(0) \\ u'_n(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ u'_n(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{678}$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 4x4 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ u'_{(1)} = 0 \\ u''_{(0)} = 0 \\ K_b u'''_{(0)} - K_s u'_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1(h_1) = 0 \\ u'_1(h_1) = 0 \\ M_n(0) = 0 \\ V_n(0) = 0 \end{Bmatrix} \tag{679}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ u'_n(0) \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{680}$$

Solving for the bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{681}$$

Which has a different solution than the trivial one if the determinant is equal to zero (the matrix of coefficients is singular). Solving the determinant finds the angular frequencies of the beam.

4.2.5 Parallel Coupling of Bending Beam and Shear Beam of a Field (CTB) - Torsional Behavior

Just as the torsional displacement analysis can be derived based on the analogy that exists between the stresses of thin-walled structures in bending and torsion, the torsional vibration analysis of a structural core can also be extended using this analogy. The model to be used is a thin-walled open cross-section equivalent cantilever having an effective Saint Venant stiffness (GJ_e) and deformation stiffness (EI_w).

4.2.5.1 Case 1

When analyzing the balance of an elementary section of the structural core, its differential equation is:

$$EI_w\varphi'''' - GJ^*\varphi'' + \rho I\ddot{\varphi} = 0 \tag{682}$$

Where:

$$\left\{\begin{array}{c} J^* = J + \bar{J} \\ J = \dfrac{1}{3}\displaystyle\sum_{i=1}^{m} h_i v_i^3\ (Secci\acute{o}n\ abierta\ ), J = \dfrac{4A_0^2}{\sum_{i=1}^{m}\dfrac{h_i}{v_i}}(Secci\acute{o}n\ errada) \\ \bar{J} = \dfrac{4A_0^2}{\dfrac{l^3 sG}{12EI_b} + \dfrac{1.2ls}{A_b}}, A_b = t_b d, I_b = \dfrac{t_b d^3}{12} \end{array}\right\} \tag{683}$$

The partial differential equation can be solved by separating variables, applying a solution of the following type:

$$w_{(x,t)} = \emptyset_{(x)}q_{(t)} \tag{684}$$

Where $\emptyset_{(x)}$ defines the variation of the displacement along the length of the beam, while $q_{(t)}$ does so with time. Replacing and collecting like terms, we get the following:

$$\frac{\ddot{q}_{(t)}}{q_{(t)}} - \frac{EI_w}{\rho I}.\frac{1}{\emptyset_{(x)}}\emptyset''''_{(x)} + \frac{GJ}{\rho I}.\frac{1}{\emptyset_{(x)}}\emptyset''_{(x)} = 0 \tag{685}$$

Because the time and height coordinates are independent variables, each of the terms must equal a constant with opposite signs, to ensure that the net result is zero. Consequently, it can be divided into two ordinary differential equations:

$$\begin{Bmatrix} \ddot{q}_{(t)} + w^2 q_{(t)} = 0 \\ EI_w \emptyset''''_{(x)} - GJ\emptyset''_{(x)} - \rho I w^2 \emptyset_{(x)} = 0 \end{Bmatrix} \tag{686}$$

The first equation is the same one that governs the behavior of an SDOF system with vibration frequency w.

The second differential equation is identical to the equation presented for the case of lateral vibration of a CTB beam, with the difference that only the nomenclature of its stiffnesses changes; furthermore, the same boundary conditions hold, so the solution given in the previous section is completely valid for the pure torsional analysis of a structural core. To solve it, it is necessary to use the equivalent stiffnesses:

$$\begin{Bmatrix} K_b \rightarrow EI_w \\ K_s \rightarrow GJ \\ \rho A \rightarrow \rho I \end{Bmatrix} \tag{687}$$

4.2.5.2 Case 2

The relationship between forces and displacements between two consecutive floors is obtained by taking into account the transfer matrix and the vector of external point forces. For the j-th floor:

$$\begin{Bmatrix} \varphi_{i+1}(0) \\ \varphi'_{i+1}(0) \\ M_{i+1}(0) \\ V_{i+1}(0) \end{Bmatrix} = \left\{ T_i(0) + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ m_{ti}w^2 & 0 & 0 & 0 \end{bmatrix} \right\} \begin{Bmatrix} \varphi_i(0) \\ \varphi'_i(0) \\ M_i(0) \\ V_i(0) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m_{ti}w^2 & 0 & 0 & 1 \end{bmatrix} T_i(0) \begin{Bmatrix} \varphi_i(0) \\ \varphi'_i(0) \\ M_i(0) \\ V_i(0) \end{Bmatrix} \tag{688}$$

Rewriting:

$$\begin{Bmatrix} \varphi_{i+1}(0) \\ \varphi'_{i+1}(0) \\ M_{i+1}(0) \\ V_{i+1}(0) \end{Bmatrix} = T_{wi}(0) \begin{Bmatrix} \varphi_i(0) \\ \varphi'_i(0) \\ M_i(0) \\ V_i(0) \end{Bmatrix} \tag{689}$$

Where:

$$T_{wi}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m_{ti}w^2 & 0 & 0 & 1 \end{bmatrix} T_i(0) \tag{690}$$

Expressing the equation for the nth floor between product symbols:

$$\begin{Bmatrix} \varphi_n(0) \\ \varphi_n^{'}(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_{wk}(0) \begin{Bmatrix} \varphi_1(h_1) \\ \varphi_1^{'}(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{691}$$

Where:

$$\mathrm{t} = \prod_{k=1}^{n} T_{wk}(0) \tag{692}$$

Replacing this parameter:

$$\begin{Bmatrix} \varphi_n(0) \\ \varphi_n^{'}(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} \varphi_1(h_1) \\ \varphi_1^{'}(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{693}$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 4x4 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} \varphi_{(1)} = 0 \\ \varphi'_{(1)} = 0 \\ \varphi''_{(0)} = 0 \\ EI_w\varphi'''_{(0)} - GJ^*\varphi'_{(0)} = 0 \end{Bmatrix} \to \begin{Bmatrix} \varphi_1(h_1) = 0 \\ \varphi'_1(h_1) = 0 \\ M_n(0) = 0 \\ V_n(0) = 0 \end{Bmatrix} \tag{694}$$

Replacing:

$$\begin{Bmatrix} \varphi_n(0) \\ \varphi_n'(0) \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{695}$$

Solving for the bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{696}$$

Which has a different solution than the trivial one if the determinant is equal to zero (the matrix of coefficients is singular). Solving the determinant finds the angular frequencies of the beam.

4.2.6 Sandwich Beam of Two Field (SWB)

4.2.6.1 Case 1

The potential energy and kinetic energy of the two-field SWB model are:

$$V = \frac{1}{2}\int_0^H \left\{K_{b1}{\theta'_{(x)}}^2 + K_{s1}\left[\theta_{(x)} - u'_{(x)}\right]^2\right\} dx + \frac{1}{2}\int_0^H K_{b2}{u''_{(x)}}^2\, dx$$
$$T = \frac{1}{2}\int_0^H \left[\gamma_u(\dot{u})^2 + \gamma_\theta\left(\dot{\theta}\right)^2\right] dx \tag{697}$$

Where:

$$\{\gamma_u = \rho A, \gamma_\theta = \rho I\} \tag{698}$$

Consequently, the total potential energy of the two-field beam SWB is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H \left[\gamma_u(\dot{u})^2 + \gamma_\theta\left(\dot{\theta}\right)^2\right] dx - \frac{1}{2}\int_0^H \left\{K_{b1}{\theta'_{(x)}}^2 + K_{s1}\left[\theta_{(x)} - u'_{(x)}\right]^2 + K_{b2}{u''_{(x)}}^2\right\} dx \tag{699}$$

Closed-form solutions of the model are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\begin{aligned}\delta\mathcal{U} = \int_0^H &\{\gamma_u \dot{u}_{(x,t)}\delta\dot{u}_{(x,t)} + \gamma_\theta\dot{\theta}_{(tx,t)}\delta\dot{\theta}_{(x,t)} - K_{b1}\theta'_{(x,t)}\delta\theta'_{(x,t)} - K_{s1}\left[\theta_{(x,t)} - u'_{(x,t)}\right]\delta\theta_{(x,t)} \\ &+ K_{s1}\left[\theta_{(x,t)} - u'_{(x,t)}\right]\delta u'_{(x,t)} - K_{b2}u''_{(x,t)}\delta u''_{(x,t)}\}dx\end{aligned} \tag{700}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\begin{aligned}\delta\mathcal{U} = &\left\{\{\gamma_u\dot{u}_{(x,t)} + K_{s1}\left[\theta_{(x,t)} - u'_{(x,t)}\right] + K_{b2}u''''_{(x,t)}\}\delta u_{(x,t)}\right\}_0^H - \left[K_{b2}u''_{(x,t)}\delta u'_{(x,t)}\right]_0^H \\ &+ \{\left[\gamma_\theta\dot{\theta}_{(x,t)} - K_{b1}\theta'_{(x,t)}\right]\delta\theta_{(x,t)}\}_0^H \\ &- \int_0^H \{\gamma_u\ddot{u}_{(x,t)} + K_{s1}\left[\theta'_{(x,t)} - u''_{(x,t)}\right] + K_{b2}u''''_{(x,t)}\}\delta u_{(x,t)} \\ &- \int_0^H \left[\gamma_\theta\ddot{\theta}_{(x,t)} - K_{b1}\theta''_{(x,t)} + K_{s1}\left[\theta_{(x,t)} - u'_{(x,t)}\right]\right]\delta\theta_{(x,t)}\end{aligned} \tag{701}$$

Setting the terms equal to zero, the following equations result:

$$\begin{Bmatrix} \gamma_u \ddot{u}_{(x,t)} + K_s\left[\theta'_{(x,t)} - u''_{(x,t)}\right] + K_{b2} u''''_{(x,t)} = 0 \\ \gamma_\theta \ddot{\theta}_{(x,t)} - K_{b1}\theta''_{(x,t)} + K_s\left[\theta_{(x,t)} - u'_{(x,t)}\right] = 0 \end{Bmatrix} \quad (702)$$

And boundary conditions:

$$\begin{Bmatrix} \theta'_{(H)} = 0 \\ u''_{(H)} = 0 \\ K_s\left[\theta_{(H)} - u'_{(H)}\right] + K_{b2} u'''_{(H)} = 0 \end{Bmatrix} \quad (703)$$

The partial differential equation can be solved by separating variables, applying a solution of the following type:

$$\begin{Bmatrix} u_{(x,t)} = \emptyset_{(x)} q_{(t)} \\ \theta_{(x,t)} = \lambda_{(x)} q_{(t)} \end{Bmatrix} \quad (704)$$

Where $\emptyset_{(x)}$ defines the variation of the displacement along the length of the beam, while $q_{(t)}$ does so with time. Replacing and collecting like terms, we get the following:

$$\begin{Bmatrix} \dfrac{\ddot{q}_{(t)}}{q_{(t)}} + \dfrac{\left[K_s\lambda'_{(x)} - K_s\emptyset''_{(x)} + K_{b2}\emptyset''''_{(x)}\right]}{\gamma_u \emptyset_{(x)}} = 0 \\ \dfrac{\ddot{q}_{(t)}}{q_{(t)}} + \dfrac{\left[-K_{b1}\lambda''_{(x)} + K_s\lambda_{(x)} - K_s\emptyset'_{(x)}\right]}{\gamma_\theta \lambda_{(x)}} = 0 \end{Bmatrix} \quad (705)$$

Because the time and height coordinates are independent variables, each of the terms must equal a constant with opposite signs, to ensure that the net result is zero. Consequently, it can be divided into two ordinary differential equations:

$$\begin{Bmatrix} \ddot{q}_{(t)} + w^2 q_{(t)} = 0 \\ K_{b2}\emptyset''''_{(x)} - K_s\emptyset''_{(x)} + K_s\lambda'_{(x)} - w^2\gamma_u\emptyset_{(x)} = 0 \\ K_{b1}\lambda''_{(x)} + K_s\emptyset'_{(x)} - K_s\lambda_{(x)} + w^2\gamma_\theta\lambda_{(x)} = 0 \end{Bmatrix} \quad (706)$$

The first equation is the same one that governs the behavior of an SDOF system with vibration frequency w.

Solving for $\lambda'_{(x)}$, differentiating twice and replacing:

$$\emptyset_{(x)}'''''' - \left[\frac{K_{s1}}{K_{b2}} + \frac{K_{s1}}{K_{b1}} - \frac{\gamma_\theta w^2}{K_{b1}}\right]\emptyset_{(x)}'''' - \left[\frac{\gamma_u w^2}{K_{b2}}\left(1 + \frac{\gamma_\theta}{\gamma_u}\frac{K_{s1}}{K_{b1}}\right)\right]\emptyset_{(x)}'' + \frac{\gamma_u w^2}{K_{b2}}\left(\frac{K_{s1}}{K_{b1}} - \frac{\gamma_\theta w^2}{K_{b1}}\right)\emptyset_{(x)} = 0 \qquad (707)$$

A sixth order differential equation is obtained. Normalizing the differential equation by the variable $z = x/H$:

$$\emptyset_{(z)}'''''' - \left[\frac{K_{s1}}{K_{b2}} + \frac{K_{s1}}{K_{b1}} - \frac{\gamma_\theta w^2}{K_{b1}}\right]H^2\emptyset_{(z)}'''' - \left[\frac{\gamma_u w^2}{K_{b2}}\left(1 + \frac{\gamma_\theta}{\gamma_u}\frac{K_{s1}}{K_{b1}}\right)\right]H^4\emptyset_{(z)}'' + \frac{\gamma_u w^2}{K_{b2}}\left(\frac{K_{s1}}{K_{b1}} - \frac{\gamma_\theta w^2}{K_{b1}}\right)H^6\emptyset_{(z)} = 0 \qquad (708)$$

The equation can be rewritten as:

$$\emptyset_{(z)}'''''' - [\alpha^2\kappa^2 - \delta^2\mu^2]\emptyset_{(z)}'''' - \delta^2[1 + \alpha^2\mu^2]\emptyset_{(z)}'' + \delta^2[\alpha^2(\kappa^2 - 1) - \delta^2\mu^2]\emptyset_{(z)} = 0 \qquad (709)$$

Where:

$$\left\{\alpha = H\sqrt{\frac{K_{s1}}{K_{b2}}}, \kappa = \sqrt{1 + \frac{K_{b2}}{K_{b1}}}, \mu = \frac{1}{H}\sqrt{\frac{\gamma_\theta}{\gamma_u}\frac{K_{b2}}{K_{b1}}}, \delta = \sqrt{\frac{\gamma_u H^4}{K_{b2}}w^2}\right\} \qquad (710)$$

We rewrite the equation again:

$$\emptyset_{(z)}'''''' - [\pi_1 + \pi_1\pi_2 - \delta^2\pi_3]\emptyset_{(z)}'''' - [\delta^2(1 + \pi_1\pi_3)]\emptyset_{(z)}'' + [\delta^2(\pi_1\pi_2 - \delta^2\pi_3)]\emptyset_{(z)} = 0 \qquad (711)$$

Where:

$$\{\pi_1 = \alpha^2, \pi_2 = \kappa^2 - 1, \pi_3 = \mu^2\} \qquad (712)$$

To solve the differential equation we consider the characteristic polynomial:

$$P_{(r)} = r^6 - [\pi_1 + \pi_1\pi_2 - \delta^2\pi_3]r^4 - [\delta^2(1 + \pi_1\pi_3)]r^2 + [\delta^2(\pi_1\pi_2 - \delta^2\pi_3)] = 0 \qquad (713)$$

We change the variable and denote:

$$q_i = r_i^2 \rightarrow \left\{\begin{matrix} r_{2i-1} = \sqrt{q_i} \\ r_{2i} = -\sqrt{q_i} \end{matrix} \; ; \; i = 1, 2, 3\right\} \qquad (714)$$

We rewrite the characteristic polynomial:

$$P_{(r)} = q^3 - (\pi_1 + \pi_1\pi_2 - \delta^2\pi_3)q^2 - [\delta^2(1+\pi_1\pi_3)]q + [\delta^2(\pi_1\pi_2 - \delta^2\pi_3)] = 0 \quad (715)$$

This equation will have three real and unequal roots in q, if:

$$\frac{a^3}{27} + \frac{b^2}{4} < 0 \quad (716)$$

Where:

$$\left\{\begin{matrix} a = -\frac{1}{3}\left[3(1+\pi_1\pi_3)\delta^2 + \left(\pi_1 + \pi_1\pi_2 - \delta^2\pi_3\right)^2\right] \\ b = -\frac{1}{27}\left\{2\left(\pi_1 + \pi_1\pi_2 - \delta^2\pi_3\right)^3 + 9\left(\pi_1 + \pi_1\pi_2 - \delta^2\pi_3\right)(1+\pi_1\pi_3)\delta^2 - 27\left(\pi_1 + \pi_1\pi_2 - \delta^2\pi_3\right)\delta^2\right\} \end{matrix}\right\} \quad (717)$$

Replacing:

$$\begin{aligned} \frac{a^3}{27} + \frac{b^2}{4} = -\frac{\delta^2}{108}\{&\pi_1^4[4\pi_2(1+\pi_2)^3] \\ &+ \pi_1^2[(1+20\pi_2 - 8\pi_2^2) - 2(1+\pi_2)(1+8\pi_2^2)\pi_1\pi_3 + (1+\pi_2)^2\pi_1^2\pi_3^2]\delta^2 \\ &+ [4 - 8(1-2\pi_1)\pi_1\pi_3 + 2(1 - 2\pi_1 + 12\pi_2^2)\pi_1^2\pi_3^2 + 2(1-\pi_1)\pi_1^3\pi_3^3]\delta^4 \\ &+ \pi_3^2[-8 - 8(1-2\pi_2)\pi_1\pi_3 + \pi_1^2\pi_3^2]\delta^6 + (4\pi_3^4)\delta^8\} \end{aligned} \quad (718)$$

Numerically, it can be shown that the equation is always negative when:

$$\begin{Bmatrix} 0 < \pi_1 < 10^6 \\ 0 < \pi_2 < 0.5 \\ 0 < \pi_3 < 0.5 \end{Bmatrix} \rightarrow \begin{Bmatrix} 0 < \alpha < 10^3 \\ 0 < \kappa < 1.2247 \\ 0 < \mu < 0.7071 \end{Bmatrix} \quad (719)$$

We define a critical eigenvalue:

$$\pi_1\pi_2 - \delta^2\pi_3 = 0 \rightarrow \delta_c^2 = \frac{\pi_1\pi_2}{\pi_3} \quad (720)$$

Two cases are presented:

- Case 1: When the polynomial has two positive real roots and one negative real root.

$$\pi_1\pi_2 - \delta^2\pi_3 > 0 \rightarrow \delta^2 < \frac{\pi_1\pi_2}{\pi_3} \rightarrow \delta < \delta_{cr} \tag{721}$$

- Case 2: When the polynomial has a positive real root and two negative real roots.

$$\pi_1\pi_2 - \delta^2\pi_3 < 0 \rightarrow \delta^2 > \frac{\pi_1\pi_2}{\pi_3} \rightarrow \delta > \delta_{cr} \tag{722}$$

We define q_i in such a way that:

$$q_1 < q_2 < q_3 \tag{723}$$

In such a way that $q_1 < 0$, $q_3 > 0$ and $q_2 > 0$ for $\delta < \delta_{cr}$ and $q_2 < 0$ for $\delta > \delta_{cr}$.

The roots of the equation are calculated as:

$$\left\{\begin{array}{c} q_i = 2\sqrt{-\frac{a}{3}}\cos\left(\frac{\emptyset}{3} + \frac{2\pi i}{3}\right) + \frac{\pi_1 + \pi_1\pi_2 - \delta^2\pi_3}{3} \quad ; \quad i = 1,2,3 \\ \emptyset = \arccos\left(\frac{3b}{2a}\sqrt{-\frac{3}{a}}\right) \end{array}\right\} \tag{724}$$

- **Frequency and Periods of Vibration**

Normalizing by the variable $z = x/H$ the two coupled differential equations:

$$\left\{\begin{array}{c} \emptyset''''_{(z)} - \alpha^2\emptyset''_{(z)} + \alpha^2 H\lambda'_{(z)} - \delta^2\emptyset_{(z)} = 0 \\ H\lambda''_{(z)} + [\alpha^2(\kappa^2 - 1)]\emptyset'_{(z)} - [\alpha^2(\kappa^2 - 1) - \delta^2\mu^2]H\lambda_{(z)} = 0 \end{array}\right\} \tag{725}$$

Expressing it in terms of π_1, π_2 and π_3:

$$\left\{\begin{array}{c} \emptyset''''_{(z)} - \pi_1\emptyset''_{(z)} + \pi_1 H\lambda'_{(z)} - \delta^2\emptyset_{(z)} = 0 \\ H\lambda''_{(z)} + \pi_1\pi_2\emptyset'_{(z)} - (\pi_1\pi_2 - \delta^2\pi_3)H\lambda_{(z)} = 0 \end{array}\right\} \tag{726}$$

The solution will be of the form:

$$W_{(z)} = \begin{Bmatrix} H\lambda_{(z)} \\ \emptyset_{(z)} \end{Bmatrix} = \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} e^{rz} \tag{727}$$

Substituting the equation in the equation, two homogeneous equations are obtained which, written in matrix form, result in:

$$\begin{bmatrix} \pi_1 r & r^4 - \pi_1 r^2 - \delta^2 \\ r^2 - (\pi_1\pi_2 - \delta^2\pi_3) & \pi_1\pi_2 r \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{728}$$

To avoid trivial solutions, the determinant must be equal to zero, that is:

$$r^6 - [\pi_1 + \pi_1\pi_2 - \delta^2\pi_3]r^4 - [\delta^2(1 + \pi_1\pi_3)]r^2 + [\delta^2(\pi_1\pi_2 - \delta^2\pi_3)] = 0 \tag{729}$$

For all roots, the equation implies:

$$\begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = \begin{bmatrix} r_i^4 - \pi_1 r_i^2 - \delta^2 \\ -\pi_1 r_i \end{bmatrix} C \;\; ; \;\; i = 1, 2, \ldots, 6 \tag{730}$$

Where C is an arbitrary constant. We change the variable and denote:

$$q_i = r_i^2 \rightarrow \begin{Bmatrix} r_{2i-1} = \sqrt{q_i} \\ r_{2i} = -\sqrt{q_i} \end{Bmatrix} \;\; ; \;\; i = 1, 2, 3 \Bigg\} \tag{731}$$

Substituting in the equation:

$$q^3 - (\pi_1 + \pi_1\pi_2 - \delta^2\pi_3)q^2 - [\delta^2(1 + \pi_1\pi_3)]q + [\delta^2(\pi_1\pi_2 - \delta^2\pi_3)] = 0 \tag{732}$$

It was shown that the roots are always real for the given intervals in the equation.

Rewriting the complete solution:

$$W_{(z)} = \begin{Bmatrix} H\lambda_{(z)} \\ \emptyset_{(z)} \end{Bmatrix} = \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} e^{rz} = \sum_{i=1}^{6} C_i \begin{bmatrix} r_i^4 - \pi_1 r_i^2 - \delta^2 \\ -\pi_1 r_i \end{bmatrix} e^{r_i z} \tag{733}$$

Substituting this complete equation in the boundary conditions, we obtain:

- At the base ($z = 0$):

$$\left\{\begin{array}{c} \emptyset_{(0)} = 0 \rightarrow \sum_{i=1}^{6} C_i\, r_i = 0 \\ \emptyset'_{(0)} = 0 \rightarrow \sum_{i=1}^{6} C_i\, r_i^2 = 0 \\ H\lambda_{(0)} = 0 \rightarrow \sum_{i=1}^{6} C_i\left(r_i^4 - \pi_1 r_i^2 - \delta^2\right) = 0 \end{array}\right\} \qquad (734)$$

- At the top ($z = 1$):

$$\left\{\begin{array}{c} \emptyset''_{(1)} = 0 \rightarrow \sum_{i=1}^{6} C_i\, r_i^3 e^{r_i} = 0 \\ \emptyset'''_{(1)} - \pi_1\left[\emptyset'_{(1)} - H\lambda_{(1)}\right] = 0 \rightarrow \sum_{i=1}^{6} C_i\, e^{r_i} = 0 \\ H\lambda'_{(1)} = 0 \rightarrow \sum_{i=1}^{6} C_i\, r_i\left(r_i^4 - \pi_1 r_i^2 - \delta^2\right) e^{r_i} = 0 \end{array}\right\} \qquad (735)$$

Defining:

$$D_i = r_i^4 - \pi_1 r_i^2 - \delta^2 = q_i^2 - \pi_1 q_i - \delta^2 \quad ; \quad i = 1, 2, 3 \qquad (736)$$

The linear algebraic system resulting from developing the boundary conditions is written in the form of a matrix:

$$\begin{bmatrix} \sqrt{q_1} & -\sqrt{q_1} & \sqrt{q_2} & -\sqrt{q_2} & \sqrt{q_3} & -\sqrt{q_3} \\ q_1 & q_1 & q_2 & q_2 & q_3 & q_3 \\ D_1 & D_1 & D_2 & D_2 & D_3 & D_3 \\ q_1^{3/2} e^{\sqrt{q_1}} & -q_1^{3/2} e^{-\sqrt{q_1}} & q_2^{3/2} e^{\sqrt{q_2}} & -q_2^{3/2} e^{-\sqrt{q_2}} & q_3^{3/2} e^{\sqrt{q_3}} & -q_3^{3/2} e^{-\sqrt{q_3}} \\ e^{\sqrt{q_1}} & e^{-\sqrt{q_1}} & e^{\sqrt{q_2}} & e^{-\sqrt{q_2}} & e^{\sqrt{q_3}} & e^{-\sqrt{q_3}} \\ \sqrt{q_1} D_1 e^{\sqrt{q_1}} & -\sqrt{q_1} D_1 e^{-\sqrt{q_1}} & \sqrt{q_2} D_2 e^{\sqrt{q_2}} & -\sqrt{q_2} D_2 e^{-\sqrt{q_2}} & \sqrt{q_3} D_3 e^{\sqrt{q_3}} & -\sqrt{q_3} D_3 e^{-\sqrt{q_3}} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (737)$$

Which has a different solution than the trivial one if the determinant is equal to zero (the matrix of coefficients is singular).

By some simple manipulations of the determinant in the equation, it can be written as:

$$\begin{vmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & q_1 & 0 & q_2 & 0 & q_3 \\ 0 & D_1 & 0 & D_2 & 0 & D_3 \\ q_1 c_1 & q_1^2 s_1 & q_2 c_2 & q_2^2 s_2 & q_3 c_3 & q_3^2 s_3 \\ s_1 & c_1 & s_2 & c_2 & s_3 & c_3 \\ D_1 c_1 & q_1 D_1 s_1 & D_2 c_2 & q_2 D_2 s_2 & D_3 c_3 & q_3 D_3 s_3 \end{vmatrix} = 0 \tag{738}$$

Where:

$$\left\{\begin{array}{c} s_i(z) = \dfrac{1}{2\sqrt{q_i}}\left[e^{\sqrt{q_1}z} - e^{-\sqrt{q_1}z}\right] = \left\{\begin{array}{c} s_1(z) = \dfrac{1}{|q_1|}\sin\left(\sqrt{|q_1|}z\right) \\ s_2(z) = \left\{\begin{array}{c} \dfrac{1}{\sqrt{|q_2|}}\sinh\left(\sqrt{|q_2|}z\right)\ ;\ \lambda < \lambda_c \\ \dfrac{1}{\sqrt{|q_2|}}\sin\left(\sqrt{|q_2|}z\right)\ ;\ \lambda > \lambda_c \end{array}\right\} \\ s_3(z) = \dfrac{1}{\sqrt{|q_3|}}\sinh\left(\sqrt{|q_3|}z\right) \end{array}\right\} \\ c_i(z) = \dfrac{1}{2}\left[e^{\sqrt{q_1}z} + e^{-\sqrt{q_1}z}\right] = \left\{\begin{array}{c} c_1(z) = \cos\left(\sqrt{|q_1|}z\right) \\ c_2(z) = \left\{\begin{array}{c} \cosh\left(\sqrt{|q_2|}z\right)\ ;\ \lambda < \lambda_c \\ \cos\left(\sqrt{|q_2|}z\right)\ ;\ \lambda > \lambda_c \end{array}\right\} \\ c_3(z) = \cosh\left(\sqrt{|q_3|}z\right) \end{array}\right\} \\ s_i = s_i(1)\ ;\ c_i = c_i(1) \end{array}\right\} \tag{739}$$

A further reduction in the equation:

$$\begin{vmatrix} (q_1^2 Q_1 s_1 + q_2^2 Q_2 s_2 + q_3^2 Q_3 s_3) & (q_2 c_2 - q_1 c_1) & (q_3 c_3 - q_1 c_1) \\ (Q_1 c_1 + Q_2 c_2 + Q_3 c_3) & (s_2 - s_1) & (s_3 - s_1) \\ (q_1 D_1 Q_1 s_1 + q_2 D_2 Q_2 s_2 + q_3 D_3 Q_3 s_3) & (D_2 c_2 - D_1 c_1) & (D_3 c_3 - D_1 c_1) \end{vmatrix} = 0 \tag{740}$$

Taking into account that:

$$c_i^2 - q_i s_i^2 = 1; i = 1, 2, 3 \tag{741}$$

The determinant can be written in its simplest form:

$$\begin{gathered} Q_2 Q_3[2 + (q_2 + q_3)s_2 s_3]c_1 + Q_3 Q_1[2 + (q_3 + q_1)s_3 s_1]c_2 + Q_1 Q_2[2 + (q_1 + q_2)s_1 s_2]c_3 \\ + (Q_1^2 + Q_2^2 + Q_3^2)c_1 c_2 c_3 = 0 \end{gathered} \tag{742}$$

Where:

$$\begin{Bmatrix} Q_1 = q_3 D_2 - q_2 D_3 = (q_2 - q_3)(q_2 q_3 + \delta^2) \\ Q_2 = q_1 D_3 - q_3 D_1 = (q_3 - q_1)(q_3 q_1 + \delta^2) \\ Q_3 = q_2 D_1 - q_1 D_2 = (q_1 - q_2)(q_1 q_2 + \delta^2) \end{Bmatrix} \quad (743)$$

Rewriting the determinant as $F(\delta^2)$:

$$\begin{aligned} F(\delta^2) = Q_2 Q_3 [2 + (q_2 + q_3) s_2 s_3] c_1 + Q_3 Q_1 [2 + (q_3 + q_1) s_3 s_1] c_2 \\ + Q_1 Q_2 [2 + (q_1 + q_2) s_1 s_2] c_3 + (Q_1^2 + Q_2^2 + Q_3^2) c_1 c_2 c_3 = 0 \end{aligned} \quad (744)$$

Solving this characteristic equation, solutions for δ are obtained by numerical methods and consequently the vibration periods are obtained. It is important to mention that in general q_3 tends to a large numerical value, so care must be taken to avoid numerical problems.

$$w = \frac{\delta}{H^2}\sqrt{\frac{K_{b2}}{\gamma_u}} \to T = \frac{2\pi}{w} = \frac{2\pi H^2}{\delta}\sqrt{\frac{\gamma_u}{K_{b2}}} \quad (745)$$

- **Mode Shapes**

The eigenvectors can be obtained by solving the equation for the constants C_i. To simplify the matrix problem, the constants C_i will be transformed into a set of new constants $\overline{C_i}$ using the transformation:

$$\begin{Bmatrix} 2C_{2i-1} = \bar{C}_{2i-1} + \dfrac{1}{\sqrt{q_i}}\bar{C}_{2i} \\ 2C_{2i} = \bar{C}_{2i-1} - \dfrac{1}{\sqrt{q_i}}\bar{C}_{2i} \end{Bmatrix} ; i = 1, 2, 3 \quad (746)$$

The equation reduces:

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ q_1 & 0 & q_2 & 0 & q_3 & 0 \\ D_1 & 0_1 & D_2 & 0 & D_3 & 0 \\ q_1^2 s_1 & q_1 c_1 & q_2^2 s_2 & q_2 c_2 & q_3^2 s_3 & q_3 c_3 \\ c_1 & s_1 & c_2 & s_2 & c_3 & s_3 \\ q_1 D_1 s_1 & D_1 c_1 & q_2 D_2 s_2 & D_2 c_2 & q_3 D_3 s_3 & D_3 c_3 \end{bmatrix} \begin{Bmatrix} \bar{C}_1 \\ \bar{C}_2 \\ \bar{C}_3 \\ \bar{C}_4 \\ \bar{C}_5 \\ \bar{C}_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (747)$$

Since the determinant of this matrix is zero, the constants $\bar{C}_1$ to $\bar{C}_5$ can be expressed as a function of $\bar{C}_6$. The matrix equation reduces to:

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ q_1 & 0 & q_2 & 0 & q_3 \\ D_1 & 0 & D_2 & 0 & D_3 \\ q_1^2 s_1 & q_1 c_1 & q_2^2 s_2 & q_2 c_2 & q_3^2 s_3 \\ q_1 D_1 s_1 & D_1 c_1 & q_2 D_2 s_2 & D_2 c_2 & q_3 D_3 s_3 \end{bmatrix} \begin{Bmatrix} \bar{C}_1 \\ \bar{C}_2 \\ \bar{C}_3 \\ \bar{C}_4 \\ \bar{C}_5 \end{Bmatrix} = -\bar{C}_6 \begin{Bmatrix} 1 \\ 0 \\ 0 \\ q_3 c_3 \\ D_3 c_3 \end{Bmatrix} \tag{748}$$

Which can be solved for the constant $\bar{C}_i$:

$$\begin{Bmatrix} \bar{C}_1 \\ \bar{C}_2 \\ \bar{C}_3 \\ \bar{C}_4 \\ \bar{C}_5 \end{Bmatrix} = -\bar{C}_6 \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ q_1 & 0 & q_2 & 0 & q_3 \\ D_1 & 0 & D_2 & 0 & D_3 \\ q_1^2 s_1 & q_1 c_1 & q_2^2 s_2 & q_2 c_2 & q_3^2 s_3 \\ q_1 D_1 s_1 & D_1 c_1 & q_2 D_2 s_2 & D_2 c_2 & q_3 D_3 s_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ 0 \\ 0 \\ q_3 c_3 \\ D_3 c_3 \end{Bmatrix} \tag{749}$$

Operating:

$$\begin{Bmatrix} \bar{C}_1 \\ \bar{C}_2 \\ \bar{C}_3 \\ \bar{C}_4 \\ \bar{C}_5 \end{Bmatrix} = \frac{\bar{C}_6}{Q_2 Q_3 (q_2 s_2 - q_3 s_3) - Q_3 Q_1 (q_3 s_3 - q_1 s_1)} \begin{Bmatrix} Q_1 (Q_1 c_2 c_3 + Q_2 c_3 c_1 + Q_3 c_1 c_2) \\ Q_3 Q_1 (q_3 s_3 - q_1 s_1) c_2 - Q_1 Q_2 (q_1 s_1 - q_2 s_2) c_3 \\ Q_2 (Q_1 c_2 c_3 + Q_2 c_3 c_1 + Q_3 c_1 c_2) \\ Q_1 Q_2 (q_1 s_1 - q_2 s_2) c_3 - Q_2 Q_3 (q_2 s_2 - q_3 s_3) c_1 \\ Q_3 (Q_1 c_2 c_3 + Q_2 c_3 c_1 + Q_3 c_1 c_2) \end{Bmatrix} \tag{750}$$

Finally:

$$\begin{Bmatrix} \bar{C}_1 \\ \bar{C}_2 \\ \bar{C}_3 \\ \bar{C}_4 \\ \bar{C}_5 \\ \bar{C}_6 \end{Bmatrix} = \mathrm{C} \begin{Bmatrix} Q_1 \chi_4 \\ \chi_2 - \chi_3 \\ Q_2 \chi_4 \\ \chi_3 - \chi_1 \\ Q_3 \chi_4 \\ \chi_1 - \chi_2 \end{Bmatrix} \tag{751}$$

Where C is a constant and

$$\begin{Bmatrix} \chi_1 = Q_2 Q_3 (q_2 s_2 - q_3 s_3) c_1 \\ \chi_2 = Q_3 Q_1 (q_3 s_3 - q_1 s_1) c_2 \\ \chi_3 = Q_1 Q_2 (q_1 s_1 - q_2 s_2) c_3 \\ \chi_4 = Q_1 c_2 c_3 + Q_2 c_3 c_1 + Q_3 c_1 c_2 \end{Bmatrix} \tag{752}$$

Now we can obtain the displacement $\emptyset_{(z)}$ corresponding to the eigenvector:

$$\emptyset_{(z)} = -\pi_1[\bar{C}_1 q_1 s_1(z) + \bar{C}_2 c_1(z) + \bar{C}_3 q_2 s_2(z) + \bar{C}_4 c_2(z) + \bar{C}_5 q_3 s_3(z) + \bar{C}_6 c_3(z)] \tag{753}$$

And substituting the values of $\bar{C}_i$ given in the equation

$$\begin{aligned}\emptyset_{(z)} = -\pi_1 C\{\chi_4[Q_1 q_1 s_1(z) + Q_2 q_2 s_2(z) + Q_3 q_3 s_3(z)] + (\chi_2 - \chi_3)c_1(z) \\ + (\chi_3 - \chi_1)c_2(z) + (\chi_1 - \chi_2)c_3(z)\}\end{aligned} \tag{754}$$

We normalize in such a way that $\emptyset_{(1)} = 1$:

$$\bar{\emptyset}_{(z)} = \frac{\emptyset_{(z)}}{\emptyset_{(1)}} \tag{755}$$

4.2.6.2 Case 2

The relationship between forces and displacements between two consecutive floors is obtained by taking into account the transfer matrix and the vector of external point forces. For the j-th floor:

$$\left\{\begin{array}{l}\begin{Bmatrix} u_{i+1}(0) \\ u'_{i+1}(0) \\ \theta_{i+1}(0) \\ M_{\mathrm{l}i+1}(0) \\ M_{\mathrm{r}i+1}(0) \\ V_{i+1}(0) \end{Bmatrix} = \left\{ T_i(0) + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ m_i w^2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\} \begin{Bmatrix} u_i(0) \\ u'_i(0) \\ \theta_i(0) \\ M_{\mathrm{l}i}(0) \\ M_{\mathrm{r}i}(0) \\ V_i(0) \end{Bmatrix} \\ \begin{Bmatrix} u_{i+1}(0) \\ u'_{i+1}(0) \\ \theta_{i+1}(0) \\ M_{\mathrm{l}i+1}(0) \\ M_{\mathrm{r}i+1}(0) \\ V_{i+1}(0) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ m_i w^2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} T_i(0) \begin{Bmatrix} u_i(0) \\ u'_i(0) \\ \theta_i(0) \\ M_{\mathrm{l}i}(0) \\ M_{\mathrm{r}i}(0) \\ V_i(0) \end{Bmatrix}\end{array}\right\} \tag{756}$$

Rewriting:

$$\begin{Bmatrix} u_{i+1}(0) \\ u'_{i+1}(0) \\ \theta_{i+1}(0) \\ M_{\mathrm{l}i+1}(0) \\ M_{\mathrm{r}i+1}(0) \\ V_{i+1}(0) \end{Bmatrix} = T_{wi}(0) \begin{Bmatrix} u_i(0) \\ u'_i(0) \\ \theta_i(0) \\ M_{\mathrm{l}i}(0) \\ M_{\mathrm{r}i}(0) \\ V_i(0) \end{Bmatrix} \tag{757}$$

Where:

$$T_{wi}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ m_i w^2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} T_i(0) \tag{758}$$

Expressing the equation for the nth floor between product symbols:

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ \theta_n(0) \\ M_{\mathrm{l}n}(0) \\ M_{\mathrm{r}n}(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_{wk}(0) \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ \theta_1(h_1) \\ M_{\mathrm{l}1}(h_1) \\ M_{\mathrm{r}1}(h_1) \\ V_1(h_1) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ \theta_1(h_1) \\ M_{\mathrm{l}1}(h_1) \\ M_{\mathrm{r}1}(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{759}$$

Where:

$$\mathrm{t} = \prod_{k=1}^{n} T_{wk}(0) \tag{760}$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 6x6 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ u'_{(1)} = 0 \\ \theta_{(1)} = 0 \\ \theta'_{(0)} = 0 \\ u''_{(0)} = 0 \\ K_{s1}\left[\theta_{(0)} - u'_{(0)}\right] + K_{b2} u'''_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1(h_1) = 0 \\ u_1'(h_1) = 0 \\ \theta_1(h_1) = 0 \\ M_{ln}(0) = 0 \\ M_{rn}(0) = 0 \\ V_n(0) = 0 \end{Bmatrix} \tag{761}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ \theta_n(0) \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & t_{1,5} & t_{1,6} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} & t_{2,5} & t_{2,6} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} & t_{3,5} & t_{3,6} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,1} & t_{5,2} & t_{5,3} & t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,1} & t_{6,2} & t_{6,3} & t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ M_{\mathrm{l}1}(h_1) \\ M_{\mathrm{r}1}(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{762}$$

Solving for the bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} M_{\mathrm{l}1}(h_1) \\ M_{\mathrm{r}1}(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{763}$$

Which has a different solution than the trivial one if the determinant is equal to zero (the matrix of coefficients is singular). Solving the determinant finds the angular frequencies of the beam.

4.2.7 Generalized Sandwich Beam of Three Field (GSB1)

4.2.7.1 Case 1

The potential energy and kinetic energy of the three-field GSB1 model are:

$$V = \frac{1}{2}\int_0^H \left\{K_{b1}{\psi'_{(x,t)}}^2 + K_{s1}\left[u'_{(x,t)} - \psi_{(x,t)}\right]^2 + K_{b2}{\theta'_{(x,t)}}^2 + K_{s2}\left[u'_{(x,t)} - \theta_{(x,t)}\right]^2\right\} dx$$
$$T = \frac{1}{2}\int_0^H \left[\gamma_u {\dot{u}_{(x,t)}}^2 + \gamma_\psi {\dot{\psi}_{(x,t)}}^2 + \gamma_\theta {\dot{\theta}_{(x,t)}}^2\right] dx \tag{764}$$

Where:

$$\{\gamma_u = \rho(A_1 + A_2);\ \ \gamma_\psi = \rho I_1;\ \ \gamma_\theta = \rho I_2\} \tag{765}$$

Consequently, the total potential energy of the three-field beam GSB1 is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H \left[\gamma_u {\dot{u}_{(x,t)}}^2 + \gamma_\psi {\dot{\psi}_{(x,t)}}^2 + \gamma_\theta {\dot{\theta}_{(x,t)}}^2\right] dx$$
$$- \frac{1}{2}\int_0^H \left\{K_{b1}{\psi'_{(x,t)}}^2 + K_{s1}\left[u'_{(x,t)} - \psi_{(x,t)}\right]^2 + K_{b2}{\theta'_{(x,t)}}^2 + K_{s2}\left[u'_{(x,t)} - \theta_{(x,t)}\right]^2\right\} dx \tag{766}$$

Closed-form solutions of the model are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\begin{aligned}\delta\mathcal{U} = \int_0^H \{&\gamma_u \dot{u}_{(x,t)}\delta\dot{u}_{(x,t)} + \gamma_\psi \dot{\psi}_{(x,t)}\delta\dot{\varphi}_{(x,t)} + \gamma_\theta \dot{\theta}_{(x,t)}\delta\dot{\theta}_{(x,t)} - K_{b1}\psi'_{(x,t)}\delta\psi'_{(x,t)} \\ &- K_{s1}\left[u'_{(x,t)} - \psi_{(x,t)}\right]\delta u'_{(x,t)} + K_{s1}\left[u'_{(x,t)} - \psi_{(x,t)}\right]\delta\psi_{(x,t)} - K_{b2}\theta'_{(x,t)}\delta\theta'_{(x,t)} \\ &- K_{s2}\left[u'_{(x,t)} - \theta_{(x,t)}\right]\delta u'_{(x,t)} + K_{s2}\left[u'_{(x,t)} - \theta_{(x,t)}\right]\delta\theta_{(x,t)}\}dx\end{aligned} \tag{767}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\delta\mathcal{U} = \left\{\left[\gamma_u \dot{u}_{(x,t)} - (K_{s1} + K_{s2})u'_{(x,t)} + K_{s1}\psi_{(x,t)} + K_{s2}\theta_{(x,t)}\right]\delta u_{(x,t)}\right\}_0^H$$
$$+ \left[\gamma_\psi \dot{\varphi}_{(x,t)} - K_{b1}\psi'_{(x,t)}\right]\delta\psi_{(x,t)}\Big|_0^H + \left[\gamma_\theta \theta_{(x,t)} - K_{b2}\theta'_{(x,t)}\right]\delta\theta_{(x,t)}\Big|_0^H$$
$$- \int_0^H \left\{\gamma_u \ddot{u}_{(x,t)} - (K_{s1} + K_{s2})u''_{(x,t)} + K_{s1}\psi'_{(x,t)} + K_{s2}\theta'_{(x,t)}\right\}\delta u_{(x,t)}$$
$$- \int_0^H \left\{\gamma_\psi \ddot{\psi}_{(x,t)} - K_{b1}\psi''_{(x,t)} - K_{s1}\left[u'_{(x,t)} - \psi_{(x,t)}\right]\right\}\delta\psi_{(x,t)}$$
$$- \int_0^H \left\{\gamma_\theta \ddot{\theta}_{(x,t)} - K_{b2}\theta''_{(x,t)} - K_{s2}\left[u'_{(x,t)} - \theta_{(x,t)}\right]\right\}\delta\theta_{(x,t)} \tag{768}$$

Setting the terms equal to zero, the following equations result:

$$\left\{\begin{matrix} \delta u_{(x,t)}:\ \gamma_u \ddot{u}_{(x,t)} - (K_{s1} + K_{s2})u''_{(x,t)} + K_{s1}\psi'_{(x,t)} + K_{s2}\theta'_{(x,t)} = 0 \\ \delta\psi_{(x,t)}:\ \gamma_\psi \ddot{\psi}_{(x,t)} - K_{b1}\psi''_{(x,t)} - K_{s1}\left[u'_{(x,t)} - \psi_{(x,t)}\right] = 0 \\ \delta\theta_{(x,t)}:\ \gamma_\theta \ddot{\theta}_{(x,t)} - K_{b2}\theta''_{(x,t)} - K_{s2}\left[u'_{(x,t)} - \theta_{(x,t)}\right] = 0 \end{matrix}\right\} \tag{769}$$

And boundary conditions:

$$\left\{\begin{matrix} \delta u_{(x,t)}:\ (K_{s1} + K_{s2})u'_{(H)} - K_{s1}\psi_{(H)} - K_{s2}\theta_{(H)} = 0 \\ \delta\psi_{(x,t)}:\ \psi'_{(H)} = 0 \\ \delta\theta_{(x,t)}:\ \theta'_{(H)} = 0 \end{matrix}\right\} \tag{770}$$

The partial differential equation can be solved by separating variables, applying a solution of the following type:

$$\left\{\begin{matrix} u_{(x,t)} = \emptyset_{(x)} q_{(t)} \\ \psi_{(x,t)} = \lambda_{1(x)} q_{(t)} \\ \theta_{(x,t)} = \lambda_{2(x)} q_{(t)} \end{matrix}\right\} \tag{771}$$

Where $\emptyset_{(x)}$ defines the variation of the displacement along the length of the beam, while $q_{(t)}$ does so with time. Replacing and collecting like terms, we get the following:

$$\begin{Bmatrix} \dfrac{\ddot{q}_{(t)}}{q_{(t)}} + \left[\dfrac{-(K_{s1} + K_{s2})\emptyset''_{(x)} + K_{s1}\lambda'_{1(x)} + K_{s2}\lambda'_{2(x)}}{\gamma_u \emptyset_{(x)}} \right] = 0 \\ \dfrac{\ddot{q}_{(t)}}{q_{(t)}} + \left[\dfrac{-K_{b1}\lambda''_{1(x)} - K_{s1}\emptyset'_{(x)} + K_{s1}\lambda_{1(x)}}{\gamma_\psi \lambda_{1(x)}} \right] = 0 \\ \dfrac{\ddot{q}_{(t)}}{q_{(t)}} + \left[\dfrac{-K_{b2}\lambda''_{2(x)} - K_{s2}\emptyset'_{(x)} + K_{s2}\lambda_{2(x)}}{\gamma_\theta \lambda_{2(x)}} \right] = 0 \end{Bmatrix} \tag{772}$$

Because the time and height coordinates are independent variables, each of the terms must equal a constant with opposite signs, to ensure that the net result is zero. Consequently, it can be divided into two ordinary differential equations:

$$\begin{Bmatrix} \ddot{q}_{(t)} + w^2 q_{(t)} = 0 \\ (K_{s1} + K_{s2})\emptyset''_{(x)} - K_{s1}\lambda'_{1(x)} - K_{s2}\lambda'_{2(x)} + w^2\gamma_u \emptyset_{(x)} = 0 \\ K_{b1}\lambda''_{1(x)} + K_{s1}\emptyset'_{(x)} - K_{s1}\lambda_{1(x)} + w^2\gamma_\psi \lambda_{1(x)} = 0 \\ K_{b2}\lambda''_{2(x)} + K_{s2}\emptyset'_{(x)} - K_{s2}\lambda_{2(x)} + w^2\gamma_\theta \lambda_{2(x)} = 0 \end{Bmatrix} \tag{773}$$

The first equation is the same one that governs the behavior of an SDOF system with vibration frequency w.

Using the method of differential operators for the solution of the system of equations:

$$\begin{bmatrix} (K_{s1} + K_{s2})D^2 + w^2\gamma_u & -K_{s1}D & -K_{s2}D \\ K_{s1}D & K_{b1}D^2 + (w^2\gamma_\psi - K_{s1}) & 0 \\ K_{s2}D & 0 & K_{b2}D^2 + (w^2\gamma_\theta - K_{s2}) \end{bmatrix} \begin{Bmatrix} \emptyset_{(x)} \\ \lambda_{1(x)} \\ \lambda_{2(x)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{774}$$

i.e.,

$$\begin{aligned} \emptyset''''''_{(x)} - &\left[\frac{K_{s1}K_{s2}(K_{b1} + K_{b2})}{K_{b1}K_{b2}(K_{s1} + K_{s2})} - \frac{K_{b1}K_{b2}\gamma_u + (K_{b1}\gamma_\theta + K_{b2}\gamma_\psi)(K_{s1} + K_{s2})}{K_{b1}K_{b2}(K_{s1} + K_{s2})} w^2 \right] \emptyset''''_{(x)} \\ &- w^2 \left\{ \frac{K_{s1}K_{s2}(\gamma_\theta + \gamma_\psi) + (K_{s1}K_{b2} + K_{s2}K_{b1})\gamma_u}{K_{b1}K_{b2}(K_{s1} + K_{s2})} \right. \\ &\left. - \frac{(K_{s1} + K_{s2})\gamma_\theta\gamma_\psi + (K_{b1}\gamma_\theta + K_{b2}\gamma_\psi)\gamma_u}{K_{b1}K_{b2}(K_{s1} + K_{s2})} w^2 \right\} \emptyset''_{(x)} \\ &+ \frac{\gamma_u w^2 (w^2\gamma_\psi - K_{s1})(w^2\gamma_\theta - K_{s2})}{K_{b1}K_{b2}(K_{s1} + K_{s2})} \emptyset_{(z)} = 0 \end{aligned} \tag{775}$$

A sixth order differential equation is obtained. Normalizing the differential equation by the variable $z = x/H$:

$$\begin{aligned}
&\emptyset_{(z)}'''''' - \left[\frac{K_{s1}K_{s2}(K_{b1}+K_{b2})}{K_{b1}K_{b2}(K_{s1}+K_{s2})} - \frac{K_{b1}K_{b2}\gamma_u + (K_{b1}\gamma_\theta + K_{b2}\gamma_\psi)(K_{s1}+K_{s2})}{K_{b1}K_{b2}(K_{s1}+K_{s2})}w^2\right]H^2\emptyset_{(z)}'''' \\
&\qquad - w^2\left\{\frac{K_{s1}K_{s2}(\gamma_\theta+\gamma_\psi) + (K_{s1}K_{b2}+K_{s2}K_{b1})\gamma_u}{K_{b1}K_{b2}(K_{s1}+K_{s2})}\right. \\
&\qquad \left. - \frac{(K_{s1}+K_{s2})\gamma_\theta\gamma_\psi + (K_{b1}\gamma_\theta + K_{b2}\gamma_\psi)\gamma_u}{K_{b1}K_{b2}(K_{s1}+K_{s2})}w^2\right\}H^4\emptyset_{(z)}'' \\
&\qquad + \frac{\gamma_u w^2(w^2\gamma_\psi - K_{s1})(w^2\gamma_\theta - K_{s2})H^6}{K_{b1}K_{b2}(K_{s1}+K_{s2})} = 0
\end{aligned} \tag{776}$$

The equation can be rewritten as:

$$\begin{aligned}
&\emptyset_{(z)}'''''' - \left[\alpha^2\kappa^2 - \left(1+\mu_\psi^2+\mu_\theta^2\right)\delta^2\right]\emptyset_{(z)}'''' \\
&\qquad - \delta^2\left\{\alpha^2\left[(\kappa^2-1)\mu_\theta^2+\mu_\psi^2\right] + \left(\eta_\theta^2+\eta_\varphi^2\right) - \left(\mu_\psi^2\mu_\theta^2+\mu_\psi^2+\mu_\theta^2\right)\delta^2\right\}\emptyset_{(z)}'' \\
&\qquad + \delta^2\left(\delta^2\mu_\psi^2-\eta_\psi^2\right)\left(\delta^2\mu_\theta^2-\eta_\theta^2\right)\emptyset_{(z)} = 0
\end{aligned} \tag{777}$$

Where:

$$\left\{\begin{matrix}
\alpha = H\sqrt{\dfrac{K_{s1}K_{s2}}{K_{b2}(K_{s1}+K_{s2})}}, \kappa = \sqrt{1+\dfrac{K_{b2}}{K_{b1}}}, \delta = \sqrt{\dfrac{\gamma_u H^2}{K_{s1}+K_{s2}}w^2} \\
\mu_\varphi = \sqrt{\dfrac{K_{s1}+K_{s2}}{K_{b1}}\dfrac{\gamma_\psi}{\gamma_u}}, \mu_\theta = \sqrt{\dfrac{K_{s1}+K_{s2}}{K_{b2}}\dfrac{\gamma_\theta}{\gamma_u}}, \eta_\varphi = H\sqrt{\dfrac{K_{s1}}{K_{b1}}}, \eta_\theta = H\sqrt{\dfrac{K_{s2}}{K_{b2}}}
\end{matrix}\right\} \tag{778}$$

To solve the differential equation we consider the characteristic polynomial:

$$\begin{aligned}
&P_{(r)} = r^6 - \left[\alpha^2\kappa^2 - \left(1+\mu_\psi^2+\mu_\theta^2\right)\delta^2\right]r^4 \\
&\qquad - \delta^2\left\{\alpha^2\left[(\kappa^2-1)\mu_\theta^2+\mu_\psi^2\right] + \left(\eta_\theta^2+\eta_\varphi^2\right) - \left(\mu_\psi^2\mu_\theta^2+\mu_\psi^2+\mu_\theta^2\right)\delta^2\right\}r^2 \\
&\qquad + \delta^2\left(\delta^2\mu_\psi^2-\eta_\psi^2\right)\left(\delta^2\mu_\theta^2-\eta_\theta^2\right) = 0
\end{aligned} \tag{779}$$

We change the variable and denote:

$$q_i = r_i^2 \rightarrow \left\{\begin{matrix} r_{2i-1} = \sqrt{q_i} \\ r_{2i} = -\sqrt{q_i} \end{matrix} \; ; \; i = 1,2,3\right\} \tag{780}$$

We rewrite the characteristic polynomial:

$$
\begin{aligned}
P_{(r)} = q^3 - \left[\alpha^2\kappa^2 - \left(1 + \mu_\psi^2 + \mu_\theta^2\right)\delta^2\right]q^2 \\
- \delta^2\left\{\alpha^2\left[(\kappa^2 - 1)\mu_\theta^2 + \mu_\psi^2\right] + \left(\eta_\theta^2 + \eta_\varphi^2\right) - \left(\mu_\psi^2\mu_\theta^2 + \mu_\psi^2 + \mu_\theta^2\right)\delta^2\right\}q \\
+ \delta^2\left(\delta^2\mu_\psi^2 - \eta_\psi^2\right)\left(\delta^2\mu_\theta^2 - \eta_\theta^2\right) = 0
\end{aligned} \tag{781}
$$

This equation will have three real and unequal roots in q, if:

$$
\frac{a^3}{27} + \frac{b^2}{4} < 0 \tag{782}
$$

Where:

$$
\begin{aligned}
a = -\frac{1}{3}\Big[3\delta^2\left\{\alpha^2\left[(\kappa^2 - 1)\mu_\theta^2 + \mu_\psi^2\right] + \left(\eta_\theta^2 + \eta_\varphi^2\right) - \left(\mu_\psi^2\mu_\theta^2 + \mu_\psi^2 + \mu_\theta^2\right)\delta^2\right\} \\
+ \left[\alpha^2\kappa^2 - \left(1 + \mu_\psi^2 + \mu_\theta^2\right)\delta^2\right]^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
b = -\frac{1}{27}\Big\{2\left[\alpha^2\kappa^2 - \left(1 + \mu_\psi^2 + \mu_\theta^2\right)\delta^2\right]^3 \\
+ 9\left[\alpha^2\kappa^2 - \left(1 + \mu_\psi^2 + \mu_\theta^2\right)\delta^2\right]\delta^2\left\{\alpha^2\left[(\kappa^2 - 1)\mu_\theta^2 + \mu_\psi^2\right] + \left(\eta_\theta^2 + \eta_\varphi^2\right)\right. \\
\left. - \left(\mu_\psi^2\mu_\theta^2 + \mu_\psi^2 + \mu_\theta^2\right)\delta^2\right\} - 27\delta^2\left(\delta^2\mu_\psi^2 - \eta_\psi^2\right)\left(\delta^2\mu_\theta^2 - \eta_\theta^2\right)\Big\}
\end{aligned} \tag{783}
$$

Two cases are presented:

- Case 1: When the polynomial has two positive real roots and one negative real root.
- Case 2: When the polynomial has a positive real root and two negative real roots.

We define q_i in such a way that:

$$
q_1 < q_2 < q_3 \tag{784}
$$

Las raíces de la ecuación se calculan como:

$$
\left\{
\begin{gathered}
q_i = 2\sqrt{-\frac{a}{3}}\cos\left(\frac{\emptyset}{3} + \frac{2\pi i}{3}\right) + \frac{\pi_1 + \pi_1\pi_2 - \delta^2\pi_3}{3} \quad ; \quad i = 1, 2, 3 \\
\emptyset = \arccos\left(\frac{3b}{2a}\sqrt{-\frac{3}{a}}\right)
\end{gathered}
\right\} \tag{785}
$$

- **Frequency and Periods of Vibration**

Normalizing by the variable $z = x/H$ the two coupled differential equations:

$$\begin{Bmatrix} (K_{s1}+K_{s2})\emptyset''_{(z)} - K_{s1}H\lambda'_{1(x)} - K_{s2}H\lambda'_{2(x)} + H^2w^2\gamma_u\emptyset_{(x)} = 0 \\ K_{b1}H\lambda''_{1(z)} + H^2K_{s1}\emptyset'_{(z)} - H^2K_{s1}H\lambda_{1(z)} + H^2w^2\gamma_\psi H\lambda_{1(z)} = 0 \\ K_{b2}H\lambda''_{2(z)} + H^2K_{s2}\emptyset'_{(z)} - H^2K_{s2}H\lambda_{2(z)} + H^2w^2\gamma_\theta H\lambda_{2(z)} = 0 \end{Bmatrix} \tag{786}$$

The solution will be of the form:

$$W_{(z)} = \begin{Bmatrix} \emptyset_{(z)} \\ H\lambda_{1(z)} \\ H\lambda_{2(z)} \end{Bmatrix} = \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix} e^{rz} \tag{787}$$

Substituting the equation in the equation, two homogeneous equations are obtained which, written in matrix form, result in:

$$\begin{bmatrix} (K_{s1}+K_{s2})r^2 + H^2w^2\gamma_u & -K_{s1}r & -K_{s2}r \\ H^2K_{s1}r & K_{b1}r^2 + H^2(w^2\gamma_\psi - K_{s1}) & 0 \\ H^2K_{s2}r & 0 & K_{b2}r^2 + H^2(w^2\gamma_\theta - K_{s2}) \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{788}$$

To avoid trivial solutions, the determinant must be equal to zero, that is:

$$\begin{aligned} P_{(r)} = r^6 &- \left[\alpha^2\kappa^2 - \left(1+\mu_\psi^2+\mu_\theta^2\right)\delta^2\right]r^4 \\ &- \delta^2\left\{\alpha^2\left[(\kappa^2-1)\mu_\theta^2+\mu_\psi^2\right] + \left(\eta_\theta^2+\eta_\varphi^2\right) - \left(\mu_\psi^2\mu_\theta^2+\mu_\psi^2+\mu_\theta^2\right)\delta^2\right\}r^2 \\ &+ \delta^2\left(\delta^2\mu_\psi^2-\eta_\psi^2\right)\left(\delta^2\mu_\theta^2-\eta_\theta^2\right) = 0 \end{aligned} \tag{789}$$

For all roots, the equation implies:

$$\begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix} = \begin{Bmatrix} K_{b1}K_{b2}\left\{r_i^4 + \left[\left(\mu_\psi^2+\mu_\theta^2\right)\delta^2 - \left(\eta_\theta^2+\eta_\varphi^2\right)\right]r_i^2 + \left(\delta^2\mu_\psi^2-\eta_\psi^2\right)\left(\delta^2\mu_\theta^2-\eta_\theta^2\right)\right\} \\ -H^2K_{s1}K_{b2}r_i\left[r_i^2 + \left(\delta^2\mu_\theta^2-\eta_\theta^2\right)\right] \\ -H^2K_{s2}K_{b1}r_i\left[r_i^2 + \left(\delta^2\mu_\psi^2-\eta_\psi^2\right)\right] \end{Bmatrix} C \tag{790}$$

Where i=1, 2, 3,.., 6 and C is an arbitrary constant. We change the variable and denote:

$$q_i = r_i^2 \to \begin{Bmatrix} r_{2i-1} = \sqrt{q_i} \\ r_{2i} = -\sqrt{q_i} \end{Bmatrix} ; \; i = 1, 2, 3 \tag{791}$$

Substituting in the equation:

$$\begin{aligned} q^3 - \left[\alpha^2\kappa^2 - \left(1 + \mu_\psi^2 + \mu_\theta^2\right)\delta^2\right]q^2 \\ - \delta^2\left\{\alpha^2\left[(\kappa^2-1)\mu_\theta^2 + \mu_\psi^2\right] + \left(\eta_\theta^2 + \eta_\varphi^2\right) - \left(\mu_\psi^2\mu_\theta^2 + \mu_\psi^2 + \mu_\theta^2\right)\delta^2\right\}q \\ + \delta^2\left(\delta^2\mu_\psi^2 - \eta_\psi^2\right)\left(\delta^2\mu_\theta^2 - \eta_\theta^2\right) = 0 \end{aligned} \tag{792}$$

Rewriting the complete solution:

$$W_{(z)} = \sum_{i=1}^{6} C_i \begin{bmatrix} K_{b1}K_{b2}\left\{r_i^4 + \left[\left(\mu_\psi^2 + \mu_\theta^2\right)\delta^2 - \left(\eta_\varphi^2 + \eta_\theta^2\right)\right]r_i^2 + \left(\delta^2\mu_\psi^2 - \eta_\psi^2\right)\left(\delta^2\mu_\theta^2 - \eta_\theta^2\right)\right\} \\ -H^2K_{s1}K_{b2}r_i\left[r_i^2 + \left(\delta^2\mu_\theta^2 - \eta_\theta^2\right)\right] \\ -H^2K_{s2}K_{b1}r_i\left[r_i^2 + \left(\delta^2\mu_\psi^2 - \eta_\psi^2\right)\right] \end{bmatrix} e^{r_i z} \tag{793}$$

Substituting this complete equation in the boundary conditions, we obtain:

- At the base (z=0):

$$\begin{Bmatrix} \emptyset_{(0)} = 0 \to \sum_{i=1}^{6} C_i\left\{r_i^4 + \left[\left(\mu_\psi^2 + \mu_\theta^2\right)\delta^2 - \left(\eta_\varphi^2 + \eta_\theta^2\right)\right]r_i^2 + \left(\delta^2\mu_\psi^2 - \eta_\psi^2\right)\left(\delta^2\mu_\theta^2 - \eta_\theta^2\right)\right\} = 0 \\ H\lambda_{1(0)} = 0 \to \sum_{i=1}^{6} C_i\, r_i\left[r_i^2 + \left(\delta^2\mu_\theta^2 - \eta_\theta^2\right)\right] = 0 \\ H\lambda_{2(0)} = 0 \to \sum_{i=1}^{6} r_i\left[r_i^2 + \left(\delta^2\mu_\psi^2 - \eta_\psi^2\right)\right] = 0 \end{Bmatrix} \tag{794}$$

- At the top (z=1):

$$\begin{Bmatrix} \begin{Bmatrix} (K_{s1} + K_{s2})\emptyset'_{(1)} - K_{s1}H\lambda_{1(1)} - K_{s2}H\lambda_{2(1)} = 0 \\ \to \sum_{i=1}^{6} C_i\, r_i\left\{r_i^4 - \left[\alpha^2\kappa^2 - \left(\mu_\psi^2 + \mu_\theta^2\right)\delta^2\right]r_i^2 - \delta^2\left\{\alpha^2\left[(\kappa^2-1)\mu_\theta^2 + \mu_\psi^2\right] - \mu_\psi^2\mu_\theta^2\delta^2\right\}\right\}e^{r_i} = 0 \end{Bmatrix} \\ H\lambda'_{(1)} = 0 \to \sum_{i=1}^{6} C_i\, r_i^2[r_i^2 + (\delta^2\mu_\theta^2 - \eta_\theta^2)]e^{r_i} = 0 \\ H\lambda'_{(2)} = 0 \to \sum_{i=1}^{6} C_i\, r_i^2\left[r_i^2 + \left(\delta^2\mu_\psi^2 - \eta_\psi^2\right)\right]e^{r_i} = 0 \end{Bmatrix} \tag{795}$$

Defining:

$$\left\{\begin{array}{c} Z_{1i} = q_i^2 + \left[\left(\mu_\psi^2 + \mu_\theta^2\right)\delta^2 - \left(\eta_\varphi^2 + \eta_\theta^2\right)\right]q_i + \left(\delta^2\mu_\psi^2 - \eta_\psi^2\right)\left(\delta^2\mu_\theta^2 - \eta_\theta^2\right) \\ Z_{2i} = q_i^2 - \left[\alpha^2\kappa^2 - \left(\mu_\psi^2 + \mu_\theta^2\right)\delta^2\right]q_i - \delta^2\left\{\alpha^2\left[(\kappa^2 - 1)\mu_\theta^2 + \mu_\psi^2\right] - \mu_\psi^2\mu_\theta^2\delta^2\right\} \\ O_i = r_i^2 + \left(\delta^2\mu_\theta^2 - \eta_\theta^2\right) = q_i + \left(\delta^2\mu_\theta^2 - \eta_\theta^2\right) \\ E_i = r_i^2 + \left(\delta^2\mu_\psi^2 - \eta_\psi^2\right) = q_i + \left(\delta^2\mu_\psi^2 - \eta_\psi^2\right) \end{array}; i = 1, 2, 3 \right\} \tag{796}$$

The linear algebraic system resulting from developing the boundary conditions is written in the form of a matrix:

$$\begin{bmatrix} Z_{11} & -Z_{11} & Z_{12} & -Z_{12} & Z_{13} & -Z_{13} \\ \sqrt{q_1}O_1 & -\sqrt{q_1}O_1 & \sqrt{q_2}O_2 & -\sqrt{q_2}O_2 & \sqrt{q_3}O_3 & -\sqrt{q_3}O_3 \\ \sqrt{q_1}E_1 & -\sqrt{q_1}E_1 & \sqrt{q_2}E_2 & -\sqrt{q_2}E_2 & \sqrt{q_3}E_3 & -\sqrt{q_3}E_3 \\ \sqrt{q_1}Z_{21}e^{\sqrt{q_1}} & -\sqrt{q_1}Z_{21}e^{-\sqrt{q_1}} & \sqrt{q_2}Z_{22}e^{\sqrt{q_2}} & -\sqrt{q_2}Z_{22}e^{-\sqrt{q_2}} & \sqrt{q_3}Z_{23}e^{\sqrt{q_3}} & -\sqrt{q_3}Z_{23}e^{-\sqrt{q_3}} \\ q_1O_1e^{\sqrt{q_1}} & q_1O_1e^{-\sqrt{q_1}} & q_2O_2e^{\sqrt{q_2}} & q_2O_2e^{-\sqrt{q_2}} & q_3O_3e^{\sqrt{q_3}} & q_3O_3e^{-\sqrt{q_3}} \\ q_1E_1e^{\sqrt{q_1}} & q_1E_1e^{-\sqrt{q_1}} & q_2E_2e^{\sqrt{q_2}} & q_2E_2e^{-\sqrt{q_2}} & q_3E_3e^{\sqrt{q_3}} & q_3E_3e^{-\sqrt{q_3}} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{797}$$

Which has a different solution than the trivial one if the determinant is equal to zero (the matrix of coefficients is singular).

By some simple manipulations of the determinant in the equation, it can be written as:

$$\begin{vmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ \sqrt{q_1}O_1/Z_{11} & 0 & \sqrt{q_2}O_2/Z_{12} & 0 & \sqrt{q_3}O_3/Z_{13} & 0 \\ \sqrt{q_1}E_1/Z_{11} & 0 & \sqrt{q_2}E_2/Z_{12} & 0 & \sqrt{q_3}E_3/Z_{13} & 0 \\ \sqrt{q_1}C_1Z_{21}/Z_{11} & \sqrt{q_1}S_1Z_{21} & \sqrt{q_2}C_2Z_{22}/Z_{12} & \sqrt{q_2}S_2Z_{22} & \sqrt{q_3}C_3Z_{23}/Z_{13} & \sqrt{q_3}S_3Z_{23} \\ q_1C_1O_1/Z_{11} & q_1S_1O_1 & q_2C_2O_2/Z_{12} & q_2S_2O_2 & q_3C_3O_3/Z_{13} & q_3S_3O_3 \\ q_1C_1E_1/Z_{11} & q_1S_1E_1 & q_2C_2E_2/Z_{12} & q_2S_2E_2 & q_3C_3E_3/Z_{13} & q_3S_3E_3 \end{vmatrix} = 0 \tag{798}$$

Where:

$$\left\{\begin{array}{c} s_i(z) = \frac{1}{2}\left[e^{\sqrt{q_1}z} - e^{-\sqrt{q_1}z}\right] = \left\{ s_i(z) = \begin{Bmatrix} \sinh\left(\sqrt{|q_i|}z\right); q_i > 0 \\ \sin\left(\sqrt{|q_i|}z\right); q_i < 0 \end{Bmatrix} \right\} \\ c_i(z) = \frac{1}{2}\left[e^{\sqrt{q_1}z} + e^{-\sqrt{q_1}z}\right] = \left\{ c_i(z) = \begin{Bmatrix} \cosh\left(\sqrt{|q_i|}z\right); q_i > 0 \\ \cos\left(\sqrt{|q_i|}z\right); q_i < 0 \end{Bmatrix} \right\} \\ s_i = s_i(1)\ ;\ c_i = c_i(1) \end{array}\right\} \tag{799}$$

A further reduction in the equation:

$$\begin{vmatrix} \frac{O_2}{Z_{12}} - \frac{O_1}{Z_{12}}\frac{E_2}{E_1} & \frac{O_3}{Z_{13}} - \frac{O_1}{Z_{13}}\frac{E_3}{E_1} \\ \frac{E_2}{Z_{12}} - \frac{E_1}{Z_{12}}\frac{O_2}{O_1} & \frac{E_3}{Z_{13}} - \frac{E_1}{Z_{13}}\frac{O_3}{O_1} \end{vmatrix} = 0 \tag{800}$$

The determinant can be written in its simplest form:

$$\left(\frac{O_2}{Z_{12}} - \frac{O_1}{Z_{12}}\frac{E_2}{E_1}\right)\left(\frac{E_3}{Z_{13}} - \frac{E_1}{Z_{13}}\frac{O_3}{O_1}\right) - \left(\frac{E_2}{Z_{12}} - \frac{E_1}{Z_{12}}\frac{O_2}{O_1}\right)\left(\frac{O_3}{Z_{13}} - \frac{O_1}{Z_{13}}\frac{E_3}{E_1}\right) = 0 \tag{801}$$

Rewriting the determinant as $F(\delta^2)$:

$$F(\delta^2) = \frac{O_2E_3}{Z_{12}Z_{13}}\left(1 - \frac{O_1}{O_2}\frac{E_2}{E_1}\right)\left(1 - \frac{E_1}{E_3}\frac{O_3}{O_1}\right) - \frac{O_3E_2}{Z_{12}Z_{13}}\left(1 - \frac{O_1}{O_3}\frac{E_3}{E_1}\right)\left(1 - \frac{E_1}{E_2}\frac{O_2}{O_1}\right) = 0 \tag{802}$$

Solving this characteristic equation, solutions for δ are obtained by numerical methods and consequently the vibration periods are obtained. It is important to mention that in general $q_3$tends to a large numerical value, so care must be taken to avoid numerical problems.

$$w = \frac{\delta}{H}\sqrt{\frac{K_{s2}}{\gamma_u}} \rightarrow T = \frac{2\pi}{w} = \frac{2\pi H}{\delta}\sqrt{\frac{\gamma_u}{K_{s1} + K_{s2}}} \tag{803}$$

4.2.7.2 Case 2

The relationship between forces and displacements between two consecutive floors is obtained by taking into account the transfer matrix and the vector of external point forces. For the j-th floor:

$$\left\{\begin{array}{c} \begin{Bmatrix} u_{i+1}(0) \\ \psi_{i+1}(0) \\ \theta_{i+1}(0) \\ M_{\mathrm{l}i+1}(0) \\ M_{\mathrm{r}i+1}(0) \\ V_{i+1}(0) \end{Bmatrix} = \left\{ T_i(0) + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ m_i w^2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\} \begin{Bmatrix} u_i(0) \\ \psi_i(0) \\ \theta_i(0) \\ M_{\mathrm{l}i}(0) \\ M_{\mathrm{r}i}(0) \\ V_i(0) \end{Bmatrix} \\ \begin{Bmatrix} u_{i+1}(0) \\ \psi_{i+1}(0) \\ \theta_{i+1}(0) \\ M_{\mathrm{l}i+1}(0) \\ M_{\mathrm{r}i+1}(0) \\ V_{i+1}(0) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ m_i w^2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} T_i(0) \begin{Bmatrix} u_i(0) \\ \psi_i(0) \\ \theta_i(0) \\ M_{\mathrm{l}i}(0) \\ M_{\mathrm{r}i}(0) \\ V_i(0) \end{Bmatrix} \end{array}\right\} \tag{804}$$

Rewriting:

$$\begin{Bmatrix} u_{i+1}(0) \\ \psi_{i+1}(0) \\ \theta_{i+1}(0) \\ M_{\mathrm{l}i+1}(0) \\ M_{\mathrm{r}i+1}(0) \\ V_{i+1}(0) \end{Bmatrix} = T_{wi}(0) \begin{Bmatrix} u_i(0) \\ \psi_i(0) \\ \theta_i(0) \\ M_{\mathrm{l}i}(0) \\ M_{\mathrm{r}i}(0) \\ V_i(0) \end{Bmatrix} \tag{805}$$

Where:

$$T_{wi}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ m_i w^2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} T_i(0) \tag{806}$$

Expressing the equation for the nth floor between product symbols:

$$\begin{Bmatrix} u_n(0) \\ \psi_n(0) \\ \theta_n(0) \\ M_{\mathrm{l}n}(0) \\ M_{\mathrm{r}n}(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_{wk}(0) \begin{Bmatrix} u_1(h_1) \\ \psi_1(h_1) \\ \theta_1(h_1) \\ M_{\mathrm{l}1}(h_1) \\ M_{\mathrm{r}1}(h_1) \\ V_1(h_1) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ \psi_1(h_1) \\ \theta_1(h_1) \\ M_{\mathrm{l}1}(h_1) \\ M_{\mathrm{r}1}(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{807}$$

Where:

$$\mathrm{t} = \prod_{k=1}^{n} T_{wk}(0) \tag{808}$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 6x6 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\left\{\begin{matrix} u_{(1)} = 0 \\ \psi_{(1)} = 0 \\ \theta_{(1)} = 0 \\ \psi'_{(0)} = 0 \\ \theta'_{(0)} = 0 \\ (K_{s1} + K_{s2})u'_{(0)} - K_{s1}\psi_{(0)} - K_{s2}\theta_{(0)} = 0 \end{matrix}\right\} \rightarrow \left\{\begin{matrix} u_1(h_1) = 0 \\ \psi_1(h_1) = 0 \\ \theta_1(h_1) = 0 \\ M_{ln\,(0)} = 0 \\ M_{rn\,(0)} = 0 \\ V_{n\,(0)} = 0 \end{matrix}\right\} \tag{809}$$

Replacing:

$$\left\{\begin{matrix} u_n(0) \\ \psi_n(0) \\ \theta_n(0) \\ 0 \\ 0 \\ 0 \end{matrix}\right\} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & t_{1,5} & t_{1,6} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} & t_{2,5} & t_{2,6} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} & t_{3,5} & t_{3,6} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,1} & t_{5,2} & t_{5,3} & t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,1} & t_{6,2} & t_{6,3} & t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \left\{\begin{matrix} 0 \\ 0 \\ 0 \\ M_{\mathrm{l}1}(h_1) \\ M_{\mathrm{r}1}(h_1) \\ V_1(h_1) \end{matrix}\right\} \tag{810}$$

Solving for the bending moment and shear force at the base of the model:

$$\left\{\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right\} = \begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \left\{\begin{matrix} M_{\mathrm{l}1}(h_1) \\ M_{\mathrm{r}1}(h_1) \\ V_1(h_1) \end{matrix}\right\} \tag{811}$$

Which has a different solution than the trivial one if the determinant is equal to zero (the matrix of coefficients is singular). Solving the determinant finds the angular frequencies of the beam.

4.2.8 Generalized Sandwich Beam of Three Field (GSB2)

4.2.8.1 Case 1

The potential energy and kinetic energy of the three-field GSB2 model are:

$$V = \frac{1}{2}\int_0^H \left\{K_{b1}{\theta'_{(x,t)}}^2 + K_{s1}\left[\theta_{(x,t)} - \psi_{(x,t)}\right]^2 + K_{b2}{\psi'_{(x,t)}}^2\right\} dx + \frac{1}{2}\int_0^H K_{s2}\left[\psi_{(x,t)} - u'_{(x,t)}\right]^2 dx$$
$$T = \frac{1}{2}\int_0^H \left[\gamma_u {\dot{u}_{(x,t)}}^2 + \gamma_\psi {\dot{\psi}_{(x,t)}}^2 + \gamma_\theta {\dot{\theta}_{(x,t)}}^2\right] dx \tag{812}$$

Where:

$$\left\{\gamma_u = \rho(A_1 + A_2);\ \ \gamma_\psi = \rho I_1;\ \ \gamma_\theta = \rho I_2\right\} \tag{813}$$

Consequently, the total potential energy of the three-field beam GSB2 is expressed as:

$$\begin{aligned}\mathcal{U} = &\frac{1}{2}\int_0^H \left[\gamma_u {\dot{u}_{(x,t)}}^2 + \gamma_\psi {\dot{\psi}_{(x,t)}}^2 + \gamma_\theta {\dot{\theta}_{(x,t)}}^2\right] dx \\ &- \frac{1}{2}\int_0^H \left\{K_{b1}{\theta'_{(x,t)}}^2 + K_{s1}\left[\theta_{(x,t)} - \psi_{(x,t)}\right]^2 + K_{b2}{\psi'_{(x,t)}}^2 \right.\\ &\left. + K_{s2}\left[\psi_{(x,t)} - u'_{(x,t)}\right]^2\right\} dx\end{aligned} \tag{814}$$

Closed-form solutions of the model are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\begin{aligned}\delta\mathcal{U} = \int_0^H \{&\gamma_u \dot{u}_{(x,t)}\delta\dot{u}_{(x,t)} + \gamma_\psi \dot{\psi}_{(x,t)}\delta\dot{\varphi}_{(x,t)} + \gamma_\theta \dot{\theta}_{(x,t)}\delta\dot{\theta}_{(x,t)} - K_{b1}\theta'_{(x,t)}\delta\theta'_{(x,t)} \\ &- K_{s1}\left[\theta_{(x,t)} - \psi_{(x,t)}\right]\delta\theta_{(x,t)} + K_{s1}\left[\theta_{(x,t)} - \psi_{(x,t)}\right]\delta\psi_{(x,t)} - K_{b2}\psi'_{(x,t)}\delta\psi'_{(x,t)} \\ &- K_{s2}\left[\psi_{(x,t)} - u'_{(x,t)}\right]\delta\psi_{(x,t)} + K_{s2}\left[\psi_{(x,t)} - u'_{(x,t)}\right]\delta u'_{(x,t)}\}dx\end{aligned} \tag{815}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\delta\mathcal{U} = \left\{\gamma_u \dot{u}_{(x,t)} + K_{s2}\left[\psi_{(x,t)} - u'_{(x,t)}\right]\right\}\delta u_{(x,t)}\Big|_0^H + \left[\gamma_\psi \dot{\psi}_{(x,t)} - K_{b2}\psi'_{(x,t)}\right]\delta\psi_{(x,t)}\Big|_0^H$$
$$+ \left[\gamma_\theta \theta_{(x,t)} - K_{b1}\theta'_{(x,t)}\right]\delta\theta_{(x,t)}\Big|_0^H - \int_0^H \left\{\gamma_u \ddot{u}_{(x,t)} + K_{s2}\left[\psi'_{(x,t)} - u''_{(x,t)}\right]\right\}\delta u_{(x,t)}$$
$$- \int_0^H \left\{\gamma_\psi \ddot{\psi}_{(x,t)} - K_{b2}\psi''_{(x,t)} + K_{s2}\left[\psi_{(x,t)} - u'_{(x,t)}\right] - K_{s1}\left[\theta_{(x,t)} - \psi_{(x,t)}\right]\right\}\delta\psi_{(x,t)}$$
$$- \int_0^H \left\{\gamma_\theta \ddot{\theta}_{(x,t)} - K_{b1}\theta''_{(x,t)} + K_{s1}\left[\theta_{(x,t)} - \psi_{(x,t)}\right]\right\}\delta\theta_{(x,t)} \tag{816}$$

Setting the terms equal to zero, the following equations result:

$$\begin{Bmatrix} \delta u_{(x,t)}:\ \gamma_u \ddot{u}_{(x,t)} + K_{s2}\left[\psi'_{(x,t)} - u''_{(x,t)}\right] = 0 \\ \delta\theta_{(x,t)}:\ \gamma_\theta \ddot{\theta}_{(x,t)} - K_{b1}\theta''_{(x,t)} + K_{s1}\left[\theta_{(x,t)} - \psi_{(x,t)}\right] = 0 \\ \delta\psi_{(x,t)}:\ \gamma_\psi \ddot{\psi}_{(x,t)} - K_{b2}\psi''_{(x,t)} + K_{s2}\left[\psi_{(x,t)} - u'_{(x,t)}\right] - K_{s1}\left[\theta_{(x,t)} - \psi_{(x,t)}\right] = 0 \end{Bmatrix} \tag{817}$$

And boundary conditions:

$$\begin{Bmatrix} \delta u_{(x,t)}: \psi_{(H)} - u'_{(H)} = 0 \\ \delta\theta_{(x,t)}:\ \theta'_{(H)} = 0 \\ \delta\psi_{(x,t)}:\ \psi'_{(H)} = 0 \end{Bmatrix} \tag{818}$$

The partial differential equation can be solved by separating variables, applying a solution of the following type:

$$\begin{Bmatrix} u_{(x,t)} = \emptyset_{(x)} q_{(t)} \\ \theta_{(x,t)} = \lambda_{1(x)} q_{(t)} \\ \psi_{(x,t)} = \lambda_{2(x)} q_{(t)} \end{Bmatrix} \tag{819}$$

Where $\emptyset_{(x)}$ defines the variation of the displacement along the length of the beam, while $q_{(t)}$ does so with time. Replacing and collecting like terms, we get the following:

$$\begin{Bmatrix} \dfrac{\ddot{q}_{(t)}}{q_{(t)}} + \left\{\dfrac{K_{s2}\left[\lambda'_{2(x)} - \emptyset''_{(x)}\right]}{\gamma_u \emptyset_{(x)}}\right\} = 0 \\ \dfrac{\ddot{q}_{(t)}}{q_{(t)}} + \left\{\dfrac{-K_{b1}\lambda''_{1(x)} + K_{s1}\left[\lambda_{1(x)} - \lambda_{2(x)}\right]}{\gamma_\theta \lambda_{1(x)}}\right\} = 0 \\ \dfrac{\ddot{q}_{(t)}}{q_{(t)}} + \left[\dfrac{-K_{b2}\lambda''_{2(x)} - K_{s2}\emptyset'_{(x)} + (K_{s1} + K_{s2})\lambda_{2(x)} - K_{s1}\lambda_{1(x)}}{\gamma_\psi \lambda_{2(x)}}\right] = 0 \end{Bmatrix} \tag{820}$$

Because the time and height coordinates are independent variables, each of the terms must equal a constant with opposite signs, to ensure that the net result is zero. Consequently, it can be divided into two ordinary differential equations:

$$\begin{Bmatrix} \ddot{q}_{(t)} + w^2 q_{(t)} = 0 \\ K_{s2}\left[\lambda'_{2(x)} - \emptyset''_{(x)}\right] - w^2\gamma_u\emptyset_{(x)} = 0 \\ -K_{b1}\lambda''_{1(x)} + K_{s1}\left[\lambda_{1(x)} - \lambda_{2(x)}\right] - w^2\gamma_\theta\lambda_{1(x)} = 0 \\ -K_{b2}\lambda''_{2(x)} - K_{s2}\emptyset'_{(x)} + (K_{s1}+K_{s2})\lambda_{2(x)} - K_{s1}\lambda_{1(x)} - w^2\gamma_\psi\lambda_{2(x)} = 0 \end{Bmatrix} \tag{821}$$

The first equation is the same one that governs the behavior of an SDOF system with vibration frequency w.

Using the method of differential operators for the solution of the system of equations:

$$\begin{bmatrix} K_{s2}D^2 + w^2\gamma_u & 0 & -K_{s2}D \\ 0 & K_{b1}D^2 + [w^2\gamma_\theta - K_{s1}] & K_{s1} \\ K_{s2}D & K_{s1} & K_{b2}D^2 + \left[w^2\gamma_\psi - (K_{s1}+K_{s2})\right] \end{bmatrix} \begin{Bmatrix} \emptyset_{(x)} \\ \lambda_{1(x)} \\ \lambda_{2(x)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{822}$$

i.e.,

$$\begin{aligned} \emptyset''''''_{(x)} &- \left[\frac{K_{s1}(K_{b1}+K_{b2})}{K_{b1}K_{b2}} - \frac{K_{b1}K_{b2}\gamma_u + K_{s2}(K_{b1}\gamma_\psi + K_{b2}\gamma_\theta)}{K_{b1}K_{b2}K_{s2}}w^2\right]\emptyset''''_{(x)} \\ &- w^2\left\{\frac{K_{s1}K_{s2}(\gamma_\psi+\gamma_\theta) + (K_{s1}K_{b2} + K_{s2}K_{b1} + K_{s1}K_{b1})\gamma_u}{K_{b1}K_{b2}K_{s2}}\right. \\ &\left. - \frac{K_{b1}\gamma_u\gamma_\psi + K_{b2}\gamma_u\gamma_\theta + K_{S2}\gamma_\psi\gamma_\theta}{K_{b1}K_{b2}K_{s2}}w^2\right\}\emptyset''_{(x)} \\ &+ \frac{\gamma_u w^2\left[\gamma_\psi\gamma_\theta w^4 - \left[(K_{s1}+K_{s2})\gamma_\theta + K_{s1}\gamma_\psi\right]w^2 + K_{s1}K_{s2}\right]}{K_{b1}K_{b2}K_{s2}}\emptyset_{(z)} = 0 \end{aligned} \tag{823}$$

A sixth order differential equation is obtained. Normalizing the differential equation by the variable $z = x/H$:

$$\emptyset_{(z)}'''''' - \left[\frac{K_{s1}(K_{b1}+K_{b2})}{K_{b1}K_{b2}} - \frac{K_{b1}K_{b2}\gamma_u + K_{s2}(K_{b1}\gamma_\psi + K_{b2}\gamma_\theta)}{K_{b1}K_{b2}K_{s2}} w^2\right] H^2 \emptyset_{(z)}''''$$
$$- w^2 \left\{\frac{K_{s1}K_{s2}(\gamma_\psi + \gamma_\theta) + (K_{s1}K_{b2} + K_{s2}K_{b1} + K_{s1}K_{b1})\gamma_u}{K_{b1}K_{b2}K_{s2}}\right.$$
$$\left. - \frac{K_{b1}\gamma_u\gamma_\psi + K_{b2}\gamma_u\gamma_\theta + K_{S2}\gamma_\psi\gamma_\theta}{K_{b1}K_{b2}K_{s2}} w^2\right\} H^4 \emptyset_{(z)}''$$
$$+ \frac{\gamma_u w^2\{\gamma_\psi\gamma_\theta w^4 - [(K_{s1}+K_{s2})\gamma_\theta + K_{s1}\gamma_\psi]w^2 + K_{s1}K_{s2}\}H^6}{K_{b1}K_{b2}K_{s2}} \emptyset_{(z)} = 0 \qquad (824)$$

When the rotational inertias are ignored, it results:

$$\emptyset_{(z)}'''''' - \left[\frac{K_{s1}(K_{b1}+K_{b2})}{K_{b1}K_{b2}} - \frac{1}{K_{s2}}\gamma_u w^2\right] H^2 \emptyset_{(z)}'''' - \left(\frac{K_{s1}K_{b2} + K_{s2}K_{b1} + K_{s1}K_{b1}}{K_{b1}K_{b2}K_{s2}}\right)\gamma_u w^2 H^4 \emptyset_{(z)}''$$
$$+ \frac{K_{s1}}{K_{b1}K_{b2}}\gamma_u w^2 H^6 \emptyset_{(z)} = 0 \qquad (825)$$

Or its equivalent:

$$\frac{K_{b1}K_{b2}}{K_{s1}}\emptyset_{(z)}'''''' - \left[(K_{b1}+K_{b2}) - \frac{K_{b1}K_{b2}}{K_{s1}K_{s2}}\gamma_u w^2\right] H^2 \emptyset_{(z)}'''' - \left[\frac{K_{b2}}{K_{s2}} + K_{b1}\left(\frac{1}{K_{s1}} + \frac{1}{K_{s2}}\right)\right]\gamma_u w^2 H^4 \emptyset_{(z)}'' + \gamma_u w^2 H^6 \emptyset_{(z)} = 0$$
$$(826)$$

This differential equation is identical to the differential equation proposed by Chesnais (2010) in his doctoral thesis.

To solve the differential equation we consider the characteristic polynomial:

$$P_{(r)} = r^6 - \left\{\left[\frac{K_{s1}(K_{b1}+K_{b2})}{K_{b1}K_{b2}} - \frac{K_{b1}K_{b2}\gamma_u + K_{s2}(K_{b1}\gamma_\psi + K_{b2}\gamma_\theta)}{K_{b1}K_{b2}K_{s2}} w^2\right] H^2\right\} r^4$$
$$- \left\{\left\{\frac{K_{s1}K_{s2}(\gamma_\psi + \gamma_\theta) + (K_{s1}K_{b2} + K_{s2}K_{b1} + K_{s1}K_{b1})\gamma_u}{K_{b1}K_{b2}K_{s2}}\right.\right.$$
$$\left.\left. - \frac{K_{b1}\gamma_u\gamma_\psi + K_{b2}\gamma_u\gamma_\theta + K_{S2}\gamma_\psi\gamma_\theta}{K_{b1}K_{b2}K_{s2}} w^2\right\} w^2 H^4\right\} r^2$$
$$+ \left\{\frac{\gamma_u w^2\{\gamma_\psi\gamma_\theta w^4 - [(K_{s1}+K_{s2})\gamma_\theta + K_{s1}\gamma_\psi]w^2 + K_{s1}K_{s2}\}H^6}{K_{b1}K_{b2}K_{s2}}\right\} = 0 \qquad (827)$$

We change the variable and denote:

$$q_i = r_i^2 \rightarrow \begin{Bmatrix} r_{2i-1} = \sqrt{q_i} \\ r_{2i} = -\sqrt{q_i} \end{Bmatrix} ;\ \ i = 1, 2, 3 \Bigg\} \tag{828}$$

We rewrite the characteristic polynomial:

$$\begin{aligned} P_{(r)} = q^3 &- \left\{\left[\frac{K_{s1}(K_{b1}+K_{b2})}{K_{b1}K_{b2}} - \frac{K_{b1}K_{b2}\gamma_u + K_{s2}(K_{b1}\gamma_\psi + K_{b2}\gamma_\theta)}{K_{b1}K_{b2}K_{s2}} w^2\right] H^2\right\} q^2 \\ &- \left\{\left\{\frac{K_{s1}K_{s2}(\gamma_\psi+\gamma_\theta) + (K_{s1}K_{b2}+K_{s2}K_{b1}+K_{s1}K_{b1})\gamma_u}{K_{b1}K_{b2}K_{s2}} \right.\right. \\ &\left.\left. - \frac{K_{b1}\gamma_u\gamma_\psi + K_{b2}\gamma_u\gamma_\theta + K_{S2}\gamma_\psi\gamma_\theta}{K_{b1}K_{b2}K_{s2}} w^2\right\} w^2 H^4\right\} q \\ &+ \left\{\frac{\gamma_u w^2\{\gamma_\psi\gamma_\theta w^4 - [(K_{s1}+K_{s2})\gamma_\theta + K_{s1}\gamma_\psi]w^2 + K_{s1}K_{s2}\}H^6}{K_{b1}K_{b2}K_{s2}}\right\} = 0 \end{aligned} \tag{829}$$

This equation will have three real and unequal roots in q, if:

$$\frac{a^3}{27} + \frac{b^2}{4} < 0 \tag{830}$$

Where:

$$\begin{aligned} a = -\frac{1}{3}&\left[3\left\{\left\{\frac{K_{s1}K_{s2}(\gamma_\psi+\gamma_\theta) + (K_{s1}K_{b2}+K_{s2}K_{b1}+K_{s1}K_{b1})\gamma_u}{K_{b1}K_{b2}K_{s2}} - \frac{K_{b1}\gamma_u\gamma_\psi + K_{b2}\gamma_u\gamma_\theta + K_{S2}\gamma_\psi\gamma_\theta}{K_{b1}K_{b2}K_{s2}} w^2\right\} w^2 H^4\right\}\right. \\ &\left. + \left\{\left[\frac{K_{s1}(K_{b1}+K_{b2})}{K_{b1}K_{b2}} - \frac{K_{b1}K_{b2}\gamma_u + K_{s2}(K_{b1}\gamma_\psi + K_{b2}\gamma_\theta)}{K_{b1}K_{b2}K_{s2}} w^2\right] H^2\right\}^2\right] \end{aligned}$$

$$\begin{aligned} b &= -\frac{1}{27}\left\{2\left\{\left[\frac{K_{s1}(K_{b1}+K_{b2})}{K_{b1}K_{b2}} - \frac{K_{b1}K_{b2}\gamma_u + K_{s2}(K_{b1}\gamma_\psi + K_{b2}\gamma_\theta)}{K_{b1}K_{b2}K_{s2}} w^2\right] H^2\right\}^3\right. \\ &+ 9\left\{\left[\frac{K_{s1}(K_{b1}+K_{b2})}{K_{b1}K_{b2}}\right.\right. \\ &\left.\left. - \frac{K_{b1}K_{b2}\gamma_u + K_{s2}(K_{b1}\gamma_\psi + K_{b2}\gamma_\theta)}{K_{b1}K_{b2}K_{s2}} w^2\right] H^2\right\}\left\{\left\{\frac{K_{s1}K_{s2}(\gamma_\psi+\gamma_\theta) + (K_{s1}K_{b2}+K_{s2}K_{b1}+K_{s1}K_{b1})\gamma_u}{K_{b1}K_{b2}K_{s2}}\right.\right. \\ &\left.\left. - \frac{K_{b1}\gamma_u\gamma_\psi + K_{b2}\gamma_u\gamma_\theta + K_{S2}\gamma_\psi\gamma_\theta}{K_{b1}K_{b2}K_{s2}} w^2\right\} w^2 H^4\right\} \\ &\left. - 27\left\{\frac{\gamma_u w^2\{\gamma_\psi\gamma_\theta w^4 - [(K_{s1}+K_{s2})\gamma_\theta + K_{s1}\gamma_\psi]w^2 + K_{s1}K_{s2}\}H^6}{K_{b1}K_{b2}K_{s2}}\right\}\right\} \end{aligned} \tag{831}$$

Two cases are presented:

- Case 1: When the polynomial has two positive real roots and one negative real root.
- Case 2: When the polynomial has a positive real root and two negative real roots.

We define q_i in such a way that:

$$q_1 < q_2 < q_3 \tag{832}$$

In such a way that $q_1 < 0$, $q_3 > 0$ and $q_2 > 0$.

The roots of the equation are calculated as:

$$\left\{ \begin{array}{c} q_i = 2\sqrt{-\frac{a}{3}}\cos\left(\frac{\emptyset}{3} + \frac{2\pi i}{3}\right) + \frac{\pi_1 + \pi_1\pi_2 - \delta^2\pi_3}{3} \;\; ; \;\; i = 1, 2, 3 \\ \emptyset = \arccos\left(\frac{3b}{2a}\sqrt{-\frac{3}{a}}\right) \end{array} \right\} \tag{833}$$

- **Frequency and Periods of Vibration**

Normalizing by the variable $z = x/H$ the two coupled differential equations:

$$\left\{ \begin{array}{c} K_{s2}\left[H\lambda'_{2(z)} - \emptyset''_{(z)}\right] - H^2 w^2 \gamma_u \emptyset_{(z)} = 0 \\ -K_{b1}\lambda''_{1(z)} + H^2 K_{s1}\left[\lambda_{1(z)} - \lambda_{2(z)}\right] - H^2 w^2 \gamma_\theta \lambda_{1(z)} = 0 \\ -K_{b2}H\lambda''_{2(z)} - H^2 K_{s2}\emptyset'_{(z)} + H^2(K_{s1} + K_{s2})H\lambda_{2(z)} - H^2 K_{s1} H\lambda_{1(z)} - H^2 w^2 \gamma_\psi H\lambda_{2(z)} = 0 \end{array} \right\} \tag{834}$$

The solution will be of the form:

$$W_{(z)} = \begin{Bmatrix} \emptyset_{(z)} \\ H\lambda_{1(z)} \\ H\lambda_{2(z)} \end{Bmatrix} = \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix} e^{rz} \tag{835}$$

Substituting the equation in the equation, two homogeneous equations are obtained which, written in matrix form, result in:

$$\begin{bmatrix} K_{s2}r^2 + H^2 w^2 \gamma_u & 0 & -K_{s2}r \\ 0 & K_{b1}r^2 + H^2\left[w^2\gamma_\theta - K_{s1}\right] & H^2 K_{s1} \\ H^2 K_{s2} r & H^2 K_{s1} & K_{b2}r^2 + \left[w^2\gamma_\psi - H^2(K_{s1} + K_{s2})\right] \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$(836)$$

To avoid trivial solutions, the determinant must be equal to zero, that is:

$$\begin{aligned}P_{(r)} = r^6 &- \left\{\left[\frac{K_{s1}(K_{b1}+K_{b2})}{K_{b1}K_{b2}} - \frac{K_{b1}K_{b2}\gamma_u + K_{s2}(K_{b1}\gamma_\psi + K_{b2}\gamma_\theta)}{K_{b1}K_{b2}K_{s2}}w^2\right]H^2\right\}r^4 \\ &- \left\{\left\{\frac{K_{s1}K_{s2}(\gamma_\psi+\gamma_\theta) + (K_{s1}K_{b2} + K_{s2}K_{b1} + K_{s1}K_{b1})\gamma_u}{K_{b1}K_{b2}K_{s2}}\right.\right. \\ &\left.\left. - \frac{K_{b1}\gamma_u\gamma_\psi + K_{b2}\gamma_u\gamma_\theta + K_{S2}\gamma_\psi\gamma_\theta}{K_{b1}K_{b2}K_{s2}}w^2\right\}w^2H^4\right\}r^2 \\ &+ \left\{\frac{\gamma_u w^2\left\{\gamma_\psi\gamma_\theta w^4 - \left[(K_{s1}+K_{s2})\gamma_\theta + K_{s1}\gamma_\psi\right]w^2 + K_{s1}K_{s2}\right\}H^6}{K_{b1}K_{b2}K_{s2}}\right\} = 0\end{aligned} \quad (837)$$

For all roots, the equation implies:

$$\begin{Bmatrix}\eta_1\\ \eta_2\\ \eta_3\end{Bmatrix} = \begin{bmatrix} r_i\left(r_i^2 - \dfrac{K_{s1}}{K_{b1}}H^2\right) \\ r_i^2 + \dfrac{H^2w^2\gamma_u}{K_{s2}} \\ r_i^4 + \left(\dfrac{H^2w^2\gamma_u}{K_{s2}} - \dfrac{K_{s1}H^2}{K_{b1}}\right)r_i^2 - \dfrac{K_{s1}H^2w^2\gamma_u}{K_{b1}K_{s2}}\end{bmatrix} C\ ;\ i = 1, 2, \dots, 6 \quad (838)$$

Where C is an arbitrary constant. We change the variable and denote:

$$q_i = r_i^2 \to \left\{\begin{matrix} r_{2i-1} = \sqrt{q_i} \\ r_{2i} = -\sqrt{q_i}\end{matrix}\ ;\ i = 1, 2, 3\right\} \quad (839)$$

Substituting in the equation:

$$q^3 - \left\{\left[\frac{K_{s1}(K_{b1}+K_{b2})}{K_{b1}K_{b2}} - \frac{K_{b1}K_{b2}\gamma_u + K_{s2}(K_{b1}\gamma_\psi + K_{b2}\gamma_\theta)}{K_{b1}K_{b2}K_{s2}} w^2\right] H^2\right\} q^2$$

$$- \left\{\left\{\frac{K_{s1}K_{s2}(\gamma_\psi+\gamma_\theta) + (K_{s1}K_{b2} + K_{s2}K_{b1} + K_{s1}K_{b1})\gamma_u}{K_{b1}K_{b2}K_{s2}}\right.\right.$$

$$\left.\left. - \frac{K_{b1}\gamma_u\gamma_\psi + K_{b2}\gamma_u\gamma_\theta + K_{S2}\gamma_\psi\gamma_\theta}{K_{b1}K_{b2}K_{s2}} w^2\right\} w^2 H^4\right\} q$$

$$+ \left\{\frac{\gamma_u w^2\{\gamma_\psi\gamma_\theta w^4 - [(K_{s1}+K_{s2})\gamma_\theta + K_{s1}\gamma_\psi]w^2 + K_{s1}K_{s2}\}H^6}{K_{b1}K_{b2}K_{s2}}\right\} = 0 \tag{840}$$

It was shown that the roots are always real for the given intervals in the equation.

Rewriting the complete solution:

$$W_{(z)} = \begin{Bmatrix} \emptyset_{(z)} \\ H\lambda_{1(z)} \\ H\lambda_{2(z)} \end{Bmatrix} = \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix} e^{rz} = \sum_{i=1}^{6} C_i \begin{bmatrix} r_i\left(r_i^2 - \dfrac{K_{s1}}{K_{b1}}H^2\right) \\ r_i^2 + \dfrac{H^2w^2\gamma_u}{K_{s2}} \\ r_i^4 + \left(\dfrac{H^2w^2\gamma_u}{K_{s2}} - \dfrac{K_{s1}H^2}{K_{b1}}\right)r_i^2 - \dfrac{K_{s1}H^2w^2\gamma_u}{K_{b1}K_{s2}} \end{bmatrix} e^{r_i z} \tag{841}$$

Substituting this complete equation in the boundary conditions, we obtain:

- At the base (z=0):

$$\left\{\begin{matrix} \emptyset_{(0)} = 0 \rightarrow \displaystyle\sum_{i=1}^{6} C_i\, r_i\left(r_i^2 - \eta_\theta^2\right) = 0 \\ H\lambda_{1(0)} = 0 \rightarrow \displaystyle\sum_{i=1}^{6} C_i\left(r_i^2 + \delta^2\right) = 0 \\ H\lambda_{2(0)} = 0 \rightarrow \displaystyle\sum_{i=1}^{6} C_i\left[r_i^4 - \left(\delta^2 - \eta_\theta^2\right)r_i^2 - \eta_\theta^2\delta^2\right] = 0 \end{matrix}\right\} \tag{842}$$

- At the top (z=1):

$$\left\{\begin{array}{c} H\lambda_{2(1)} - \emptyset'_{(1)} = 0 \rightarrow \sum_{i=1}^{6} C_i \left(r_i^2 - \eta_\theta^2\right) e^{r_i} = 0 \\ H\lambda'_{(1)} = 0 \rightarrow \sum_{i=1}^{6} C_i\, r_i\left(r_i^2 + \delta^2\right) = 0 \\ H\lambda'_{(2)} = 0 \rightarrow \sum_{i=1}^{6} C_i\, r_i\left[r_i^4 - \left(\delta^2 - \eta_\theta^2\right) r_i^2 - \eta_\theta^2 \delta^2\right] e^{r_i} = 0 \end{array}\right\} \quad (843)$$

Defining:

$$\left\{\begin{array}{c} Z_i = r_i^2 - \eta_\theta^2 = q_i - \eta_\theta^2 \\ O_i = r_i^2 + \delta^2 = q_i + \delta^2 \qquad ; i = 1, 2, 3 \\ E_i = r_i^4 - \left(\delta^2 - \eta_\theta^2\right) r_i^2 - \eta_\theta^2 \delta^2 = q_i^2 - \left(\delta^2 - \eta_\theta^2\right) q_i - \eta_\theta^2 \delta^2 \end{array}\right\} \quad (844)$$

The linear algebraic system resulting from developing the boundary conditions is written in the form of a matrix:

$$\begin{bmatrix} \sqrt{q_1} Z_1 & -\sqrt{q_1} Z_1 & \sqrt{q_2} Z_2 & -\sqrt{q_2} Z_2 & \sqrt{q_3} Z_3 & -\sqrt{q_3} Z_3 \\ O_1 & O_1 & O_2 & O_2 & O_3 & O_3 \\ E_1 & E_1 & E_2 & E_2 & E_3 & E_3 \\ Z_1 e^{\sqrt{q_1}} & Z_1 e^{-\sqrt{q_1}} & Z_2 e^{\sqrt{q_2}} & Z_2 e^{-\sqrt{q_2}} & Z_3 e^{\sqrt{q_3}} & Z_3 e^{-\sqrt{q_3}} \\ \sqrt{q_1} O_1 & -\sqrt{q_1} O_1 & \sqrt{q_2} O_2 & -\sqrt{q_2} O_2 & \sqrt{q_3} O_3 & -\sqrt{q_3} O_3 \\ \sqrt{q_1} E_1 & -\sqrt{q_1} E_1 & \sqrt{q_2} E_2 & -\sqrt{q_2} E_2 & \sqrt{q_3} E_3 & -\sqrt{q_3} E_3 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (845)$$

Which has a different solution than the trivial one if the determinant is equal to zero (the matrix of coefficients is singular).

By some simple manipulations of the determinant in the equation, it can be written as:

$$\begin{vmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & O_1/Z_1 & 0 & O_2/Z_2 & 0 & O_3/Z_3 \\ 0 & E_1/Z_1 & 0 & E_2/Z_2 & 0 & E_3/Z_3 \\ S_1 & C_1 & S_2 & C_2 & S_3 & C_3 \\ O_1/Z_1 & 0 & O_2/Z_2 & 0 & O_3/Z_3 & 0 \\ E_1/Z_1 & 0 & E_2/Z_2 & 0 & E_3/Z_3 & 0 \end{vmatrix} = 0 \quad (846)$$

Where:

$$\begin{Bmatrix} s_i(z) = \dfrac{1}{2\sqrt{q_i}}\left[e^{\sqrt{q_1}z} - e^{-\sqrt{q_1}z}\right] = \begin{Bmatrix} s_i(z) = \begin{Bmatrix} \dfrac{1}{\sqrt{|q_i|}}\sinh\left(\sqrt{|q_i|}z\right); q_i > 0 \\ \dfrac{1}{\sqrt{|q_i|}}\sin\left(\sqrt{|q_i|}z\right); q_i < 0 \end{Bmatrix} \end{Bmatrix} \\ c_i(z) = \dfrac{1}{2}\left[e^{\sqrt{q_1}z} + e^{-\sqrt{q_1}z}\right] = \begin{Bmatrix} c_i(z) = \begin{Bmatrix} \cosh\left(\sqrt{|q_i|}z\right); q_i > 0 \\ \cos\left(\sqrt{|q_i|}z\right); ; q_i < 0 \end{Bmatrix} \end{Bmatrix} \\ s_i = s_i(1)\ ;\ \ c_i = c_i(1) \end{Bmatrix} \tag{847}$$

A further reduction in the equation:

$$\begin{vmatrix} \dfrac{O_2}{Z_2} - \dfrac{O_1}{Z_1} & \dfrac{O_3}{Z_3} - \dfrac{O_1}{Z_1} \\ \dfrac{E_2}{Z_2} - \dfrac{E_1}{Z_1} & \dfrac{E_3}{Z_3} - \dfrac{E_1}{Z_1} \end{vmatrix} = 0 \tag{848}$$

The determinant can be written in its simplest form:

$$\left(\frac{O_2}{Z_2} - \frac{O_1}{Z_1}\right)\left(\frac{E_3}{Z_3} - \frac{E_1}{Z_1}\right) - \left(\frac{O_3}{Z_3} - \frac{O_1}{Z_1}\right)\left(\frac{E_2}{Z_2} - \frac{E_1}{Z_1}\right) = 0 \tag{849}$$

Rewriting the determinant as $F(\delta^2)$:

$$F(\delta^2) = \left(\frac{O_2}{Z_2} - \frac{O_1}{Z_1}\right)\left(\frac{E_3}{Z_3} - \frac{E_1}{Z_1}\right) - \left(\frac{O_3}{Z_3} - \frac{O_1}{Z_1}\right)\left(\frac{E_2}{Z_2} - \frac{E_1}{Z_1}\right) = 0 \tag{850}$$

Solving this characteristic equation, solutions for δ are obtained by numerical methods and consequently the vibration periods are obtained. It is important to mention that in general q_3 tends to a large numerical value, so care must be taken to avoid numerical problems.

$$w = \frac{\delta}{H}\sqrt{\frac{K_{s2}}{\gamma_u}} \rightarrow T = \frac{2\pi}{w} = \frac{2\pi H}{\delta}\sqrt{\frac{\gamma_u}{K_{s2}}} \tag{851}$$

4.2.8.2 Case 2

The relationship between forces and displacements between two consecutive floors is obtained by taking into account the transfer matrix and the vector of external point forces. For the j-th floor:

$$\left\{\begin{array}{l}\begin{Bmatrix} u_{i+1}(0) \\ \theta_{i+1}(0) \\ \psi_{i+1}(0) \\ M_{\mathrm{l}i+1}(0) \\ M_{\mathrm{r}i+1}(0) \\ V_{i+1}(0) \end{Bmatrix} = \left\{ T_i(0) + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ m_i w^2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\} \begin{Bmatrix} u_i(0) \\ \theta_i(0) \\ \psi_i(0) \\ M_{\mathrm{l}i}(0) \\ M_{\mathrm{r}i}(0) \\ V_i(0) \end{Bmatrix} \\ \begin{Bmatrix} u_{i+1}(0) \\ \theta_{i+1}(0) \\ \psi_{i+1}(0) \\ M_{\mathrm{l}i+1}(0) \\ M_{\mathrm{r}i+1}(0) \\ V_{i+1}(0) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ m_i w^2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} T_i(0) \begin{Bmatrix} u_i(0) \\ \theta_i(0) \\ \psi_i(0) \\ M_{\mathrm{l}i}(0) \\ M_{\mathrm{r}i}(0) \\ V_i(0) \end{Bmatrix} \end{array}\right. \tag{852}$$

Rewriting:

$$\begin{Bmatrix} u_{i+1}(0) \\ \theta_{i+1}(0) \\ \psi_{i+1}(0) \\ M_{\mathrm{l}i+1}(0) \\ M_{\mathrm{r}i+1}(0) \\ V_{i+1}(0) \end{Bmatrix} = T_{wi}(0) \begin{Bmatrix} u_i(0) \\ \theta_i(0) \\ \psi_i(0) \\ M_{\mathrm{l}i}(0) \\ M_{\mathrm{r}i}(0) \\ V_i(0) \end{Bmatrix} \tag{853}$$

Where:

$$T_{wi}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ m_i w^2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} T_i(0) \tag{854}$$

Expressing the equation for the nth floor between product symbols:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ \psi_n(0) \\ M_{\mathrm{l}n}(0) \\ M_{\mathrm{r}n}(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_{wk}(0) \begin{Bmatrix} u_1(h_1) \\ \theta_1(h_1) \\ \psi_1(h_1) \\ M_{\mathrm{l}1}(h_1) \\ M_{\mathrm{r}1}(h_1) \\ V_1(h_1) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ \theta_1(h_1) \\ \psi_1(h_1) \\ M_{\mathrm{l}1}(h_1) \\ M_{\mathrm{r}1}(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{855}$$

Where:

$$t = \prod_{k=1}^{n} T_{wk}(0) \tag{856}$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 6x6 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ \theta_{(1)} = 0 \\ \psi_{(1)} = 0 \\ \theta'_{(0)} = 0 \\ \psi'_{(0)} = 0 \\ \psi_{(0)} - u'_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1(h_1) = 0 \\ \theta_1(h_1) = 0 \\ \psi_1(h_1) = 0 \\ M_{ln\,(0)} = 0 \\ M_{rn\,(0)} = 0 \\ V_{n\,(0)} = 0 \end{Bmatrix} \tag{857}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ \psi_n(0) \\ \theta_n(0) \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & t_{1,5} & t_{1,6} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} & t_{2,5} & t_{2,6} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} & t_{3,5} & t_{3,6} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,1} & t_{5,2} & t_{5,3} & t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,1} & t_{6,2} & t_{6,3} & t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ M_{\mathrm{l}1}(h_1) \\ M_{\mathrm{r}1}(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{858}$$

Solving for the bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} M_{\mathrm{l}1}(h_1) \\ M_{\mathrm{r}1}(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{859}$$

Which has a different solution than the trivial one if the determinant is equal to zero (the matrix of coefficients is singular). Solving the determinant finds the angular frequencies of the beam.

4.2.9 Modified Generalized Sandwich Beam of Two Field (MGSB)

4.2.9.1 Case 1

The differential equation according to the beam TB is:

$$\emptyset''''_{(z)} + \left[\delta^2\left(\frac{1}{\alpha^2}+\mu^2\right)\right]\emptyset''_{(z)} + \left[\delta^2\left(\frac{\mu^2}{\alpha^2}\delta^2-1\right)\right]\emptyset_{(z)} = 0 \tag{860}$$

Where:

$$\left\{\alpha = H\sqrt{\frac{K_s}{K_b}}, \mu = \frac{1}{H}\sqrt{\frac{\rho I}{\rho A}}, \delta = \sqrt{\frac{\rho A H^4}{K_b}w^2}\right\} \tag{861}$$

The roots of the equation are calculated as:

$$q_{1,2} = \frac{-\left[\delta^2\left(\frac{1}{\alpha^2}+\mu^2\right)\right] \pm \sqrt{\left[\delta^2\left(\frac{1}{\alpha^2}+\mu^2\right)\right]^2 - 4\left[\delta^2\left(\frac{\mu^2}{\alpha^2}\delta^2-1\right)\right]}}{2} \tag{862}$$

- Case 1: When the polynomial has a positive real root and a negative real root.

$$\left\{\begin{matrix} q_2^* - q_1^* = \delta^2\left(\frac{1}{\alpha^2}+\mu^2\right) \\ q_2^* q_1^* = \delta^2 - \frac{\mu^2}{\alpha^2}\delta^4 \end{matrix}\right\} \rightarrow (q_2^* - q_1^*)\left\{\left[-\frac{\alpha^2\mu^2}{(1+\alpha^2\mu^2)^2}\right](q_2^* - q_1^*) + \frac{\alpha^2}{1+\alpha^2\mu^2}\right\} - q_2^* q_1^* = 0 \tag{863}$$

Where:

$$\left\{\begin{matrix} q_1 = q_1^* \\ q_2 = -q_2^* \end{matrix}\right\} \tag{864}$$

- Case 2: When the polynomial has two negative real roots.

$$\left\{\begin{matrix} q_2^* + q_1^* = \delta^2\left(\frac{1}{\alpha^2}+\mu^2\right) \\ q_2^* q_1^* = -\delta^2 + \frac{\mu^2}{\alpha^2}\delta^4 \end{matrix}\right\} \rightarrow (q_1^* + q_2^*)\left\{\left[\frac{\alpha^2\mu^2}{(1+\alpha^2\mu^2)^2}\right](q_1^* + q_2^*) - \frac{\alpha^2}{1+\alpha^2\mu^2}\right\} - q_1^* q_2^* = 0 \tag{865}$$

Where:

$$\begin{Bmatrix} q_1 = -q_1^* \\ q_2 = -q_2^* \end{Bmatrix} \tag{866}$$

4.2.9.2 Case 2

The relationship between the forces and displacements of the upper part and the base of the beam, according to the beam TB, is:

$$\begin{Bmatrix} u_{i+1}(0) \\ \theta_{i+1}(0) \\ M_{i+1}(0) \\ V_{i+1}(0) \end{Bmatrix} = T_{wi}(0) \begin{Bmatrix} u_i(0) \\ \theta_i(0) \\ M_i(0) \\ V_i(0) \end{Bmatrix} \tag{867}$$

Where:

$$T_{wi}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m_i w^2 & 0 & 0 & 0 \end{bmatrix} T_i(0) \tag{868}$$

Expressing the equation for the nth floor between product symbols:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_{wk}(0) \begin{Bmatrix} u_1(h_1) \\ \theta_1(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ \theta_1(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{869}$$

Where:

$$\mathrm{t} = \prod_{k=1}^{n} T_{wk}(0) \tag{870}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{871}$$

Solving for the bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{872}$$

Which has a different solution than the trivial one if the determinant is equal to zero (the matrix of coefficients is singular). Solving the determinant finds the angular frequencies of the beam.

4.2.10 Parallel Coupling of Shear Beam and Timoshenko Beam of Two Field (MCTB)

4.2.10.1 Case 1

The potential energy and kinetic energy of the two-field MCTB model:

$$V = \frac{1}{2}\int_0^H \left\{K_{s1}{u'_{(x)}}^2 + K_{b2}{\theta'_{(x)}}^2 + K_{s2}\left[\theta_{(x)} - u'_{(x)}\right]^2\right\} dx,$$
$$T = \frac{1}{2}\int_0^H \left[\gamma_u {\dot{u}_{(x,t)}}^2 + \gamma_\theta {\dot{\theta}_{(x,t)}}^2\right] dx \tag{873}$$

Where:

$$\{\gamma_u = \rho(A_1 + A_2);\ \gamma_\theta = \rho(I_1 + I_2)\} \tag{874}$$

Consequently, the total potential energy of the two-field beam MCTB is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H \left[\gamma_u {\dot{u}_{(x,t)}}^2 + \gamma_\theta {\dot{\theta}_{(x,t)}}^2\right] dx - \frac{1}{2}\int_0^H \left\{K_{s1}{u'_{(x)}}^2 + K_{b2}{\theta'_{(x)}}^2 + K_{s2}\left[u'_{(x)} - \theta_{(x)}\right]^2\right\} dx \tag{875}$$

Closed-form solutions of the model are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\delta\mathcal{U} = \int_0^H \{\gamma_u \dot{u}_{(x,t)}\delta\dot{u}_{(x,t)} + \gamma_\theta\dot{\theta}_{(x,t)}\delta\dot{\theta}_{(x,t)} - K_{s1}u'_{(x,t)}\delta u'_{(x,t)} - K_{b2}\theta'_{(x,t)}\delta\theta'_{(x,t)}$$
$$- K_{s2}\left[u'_{(x,t)} - \theta_{(x,t)}\right]\delta u'_{(x,t)} + K_{s2}\left[u'_{(x,t)} - \theta_{(x,t)}\right]\delta\theta_{(x,t)}\}dx \tag{876}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\delta\mathcal{U} = \left\{\left[\gamma_u\dot{u}_{(x,t)} - (K_{s1} + K_{s2})u'_{(x,t)} + K_{s2}\theta_{(x,t)}\right]\delta u_{(x,t)}\right\}_0^H + \left[\gamma_\theta\theta_{(x,t)} - K_{b2}\theta'_{(x,t)}\right]\delta\theta_{(x,t)}\Big|_0^H$$
$$- \int_0^H \left\{\gamma_u\ddot{u}_{(x,t)} - (K_{s1} + K_{s2})u''_{(x,t)} + K_{s2}\theta'_{(x,t)}\right\}\delta u_{(x,t)}$$
$$- \int_0^H \left\{\gamma_\theta\ddot{\theta}_{(x,t)} - K_{b2}\theta''_{(x,t)} - K_{s2}\left[u'_{(x,t)} - \theta_{(x,t)}\right]\right\}\delta\theta_{(x,t)} \tag{877}$$

Setting the terms equal to zero, the following equations result:

$$\begin{Bmatrix} \gamma_u \ddot{u}_{(x,t)} - (K_{s1} + K_{s2}) u''_{(x,t)} + K_{s2} \theta'_{(x,t)} = 0 \\ \gamma_\theta \ddot{\theta}_{(x,t)} - K_{b2} \theta''_{(x,t)} - K_{s2} \left[u'_{(x,t)} - \theta_{(x,t)} \right] = 0 \end{Bmatrix} \tag{878}$$

And boundary conditions:

$$\begin{Bmatrix} (K_{s1} + K_{s2}) u'_{(H)} - K_{s2} \theta_{(H)} = 0 \\ \delta\theta_{(x,t)}:\ \theta'_{(H)} = 0 \end{Bmatrix} \tag{879}$$

The partial differential equation can be solved by separating variables, applying a solution of the following type:

$$\begin{Bmatrix} u_{(x,t)} = \emptyset_{(x)} q_{(t)} \\ \theta_{(x,t)} = \lambda_{(x)} q_{(t)} \end{Bmatrix} \tag{880}$$

Where $\emptyset_{(x)}$ defines the variation of the displacement along the length of the beam, while $q_{(t)}$ does so with time. Replacing and collecting like terms, we get the following:

$$\begin{Bmatrix} \dfrac{\ddot{q}_{(t)}}{q_{(t)}} + \left[\dfrac{-(K_{s1} + K_{s2}) \emptyset''_{(x)} + K_{s2} \lambda'_{(x)}}{\gamma_u \emptyset_{(x)}} \right] = 0 \\ \dfrac{\ddot{q}_{(t)}}{q_{(t)}} + \left[\dfrac{-K_{b2} \lambda''_{(x)} - K_{s2} \emptyset'_{(x)} + K_{s2} \lambda_{(x)}}{\gamma_\theta \lambda_{(x)}} \right] = 0 \end{Bmatrix} \tag{881}$$

Because the time and height coordinates are independent variables, each of the terms must equal a constant with opposite signs, to ensure that the net result is zero. Consequently, it can be divided into two ordinary differential equations:

$$\begin{Bmatrix} \ddot{q}_{(t)} + w^2 q_{(t)} = 0 \\ (K_{s1} + K_{s2}) \emptyset''_{(x)} - K_{s2} \lambda'_{(x)} + w^2 \gamma_u \emptyset_{(x)} = 0 \\ K_{b2} \lambda''_{(x)} + K_{s2} \emptyset'_{(x)} - K_{s2} \lambda_{(x)} + w^2 \gamma_\theta \lambda_{(x)} = 0 \end{Bmatrix} \tag{882}$$

The first equation is the same one that governs the behavior of an SDOF system with vibration frequency w.

Using the parameter variation method for the solution of the system of equations:

$$\begin{bmatrix} (K_{s1}+K_{s2})D^2+w^2\gamma_u & -K_{s2}D \\ K_{s2}D & K_{b2}D^2+(w^2\gamma_\theta-K_{s2}) \end{bmatrix}\begin{Bmatrix} \emptyset_{(x)} \\ \lambda_{(x)} \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{883}$$

i.e.,

$$\emptyset''''_{(x)}-\left[\frac{K_{s1}K_{s2}}{K_{b2}(K_{s1}+K_{s2})}-\frac{K_{b2}\gamma_u+\gamma_\theta(K_{s1}+K_{s2})}{K_{b2}(K_{s1}+K_{s2})}w^2\right]\emptyset''_{(x)}+\frac{\gamma_u w^2(w^2\gamma_\theta-K_{s2})}{K_{b2}(K_{s1}+K_{s2})}\emptyset_{(z)}=0 \tag{884}$$

A sixth order differential equation is obtained. Normalizing the differential equation by the variable $z=x/H$:

$$\emptyset''''_{(z)}-\left[\frac{K_{s1}K_{s2}}{K_{b2}(K_{s1}+K_{s2})}-\frac{K_{b2}\gamma_u+(K_{s1}+K_{s2})\gamma_\theta}{K_{b2}(K_{s1}+K_{s2})}w^2\right]H^2\emptyset''_{(z)}+\frac{\gamma_u w^2(w^2\gamma_\theta-K_{s2})H^4}{K_{b2}(K_{s1}+K_{s2})}\emptyset_{(z)}=0 \tag{885}$$

The equation can be rewritten as:

$$\emptyset''''_{(z)}-[\alpha^2\kappa^2-(1+\mu)\delta^2]\emptyset''_{(z)}-\delta^2[\alpha^2(\kappa^2+1)-\mu^2\delta^2]\emptyset_{(z)}=0 \tag{886}$$

Where:

$$\left\{\alpha=H\sqrt{\frac{K_{s2}^2}{K_{b2}(K_{s1}+K_{s2})}}\,,\kappa=\sqrt{\frac{K_{s1}}{K_{s2}}},\mu=\sqrt{\frac{K_{s1}+K_{s2}}{K_{b2}}.\frac{\gamma_\theta}{\gamma_u}},\delta=\sqrt{\frac{\gamma_u H^2}{K_{s1}+K_{s2}}w^2}\right\} \tag{887}$$

The differential equation obtained can be easily solved with the procedures presented in the previous subchapters.

4.2.10.2 Case 2

The relationship between forces and displacements between two consecutive floors is obtained by taking into account the transfer matrix and the vector of external point forces. For the j-th floor:

$$\begin{Bmatrix} u_{i+1}(0) \\ \theta_{i+1}(0) \\ M_{\mathrm{i}+1}(0) \\ V_{i+1}(0) \end{Bmatrix}=\left\{T_i(0)+\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ m_i w^2 & 0 & 0 & 0 \end{bmatrix}\right\}\begin{Bmatrix} u_i(0) \\ \theta_i(0) \\ M_{\mathrm{i}}(0) \\ V_i(0) \end{Bmatrix}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m_i w^2 & 0 & 0 & 1 \end{bmatrix}T_i(0)\begin{Bmatrix} u_i(0) \\ \theta_i(0) \\ M_{\mathrm{i}}(0) \\ V_i(0) \end{Bmatrix} \tag{888}$$

Rewriting:

$$\begin{Bmatrix} u_{i+1}(0) \\ \theta_{i+1}(0) \\ M_{\mathrm{i+1}}(0) \\ V_{i+1}(0) \end{Bmatrix} = T_{wi}(0) \begin{Bmatrix} u_i(0) \\ \theta_i(0) \\ M_{\mathrm{i}}(0) \\ V_i(0) \end{Bmatrix} \tag{889}$$

Where:

$$T_{wi}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m_i w^2 & 0 & 0 & 1 \end{bmatrix} T_i(0) \tag{890}$$

Expressing the equation for the nth floor between product symbols:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ M_{\mathrm{n}}(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_{wk}(0) \begin{Bmatrix} u_1(h_1) \\ \theta_1(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ \theta_1(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{891}$$

Where:

$$\mathrm{t} = \prod_{k=1}^{n} T_{wk}(0) \tag{892}$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 4x4 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ \theta_{(1)} = 0 \\ K_{b2}\theta'_{(0)} = 0 \\ (K_{s1} + K_{s2})u'_{(0)} - K_{s2}\theta_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1(h_1) = 0 \\ \theta_1(h_1) = 0 \\ M_{rn\,(0)} = 0 \\ V_{n\,(0)} = 0 \end{Bmatrix} \tag{893}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{894}$$

Solving for the bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{3,3} & t_{3,5} \\ t_{4,3} & t_{4,5} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{895}$$

Which has a different solution than the trivial one if the determinant is equal to zero (the matrix of coefficients is singular). Solving the determinant finds the angular frequencies of the beam.

4.2.11 Generalized Parallel Coupling of Two Beams of Three Field (GCTB)

4.2.11.1 Case 1

The potential energy and kinetic energy of the three-field GCTB model are:

$$V = \frac{1}{2}\int_0^H \left\{K_{b1}{w_{(x)}^{'}}^2 + K_{b2}{\theta_{(x)}^{'}}^2 + K_{s2}\left[u_{(x)}^{'} - \theta_{(x)}\right]^2 + K_{s1}\left[u_{(x)}^{'} + m\theta_{(x)} - nw_{(x)}\right]^2\right\} dx$$

$$T = \frac{1}{2}\int_0^H \left[\gamma_u {\dot{u}_{(x,t)}}^2 + \gamma_\theta {\dot{\theta}_{(x,t)}}^2 + \gamma_w {\dot{w}_{(x,t)}}^2\right] dx \tag{896}$$

Where:

$$\left\{\gamma_u = \rho(A_1 + A_2), \gamma_\psi = \rho(I_1 + I_2), \gamma_w = \rho\frac{A_2}{A_1}(A_1 + A_2)\right\} \tag{897}$$

Consequently, the total potential energy of the three-field beam GCTB is expressed as:

$$\begin{aligned}\mathcal{U} = &\frac{1}{2}\int_0^H \left[\gamma_u {\dot{u}_{(x,t)}}^2 + \gamma_\theta {\dot{\theta}_{(x,t)}}^2 + \gamma_w {\dot{w}_{(x,t)}}^2\right] dx \\ &- \frac{1}{2}\int_0^H \left\{K_{b1}{w_{(x)}^{'}}^2 + K_{b2}{\theta_{(x)}^{'}}^2 + K_{s2}\left[u_{(x)}^{'} - \theta_{(x)}\right]^2 + K_{s1}\left[u_{(x)}^{'} + m\theta_{(x)} - nw_{(x)}\right]^2\right\} dx\end{aligned} \tag{898}$$

Closed-form solutions of the model are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\begin{aligned}\delta\mathcal{U} = \int_0^H \{&\gamma_u \dot{u}_{(x,t)}\delta\dot{u}_{(x,t)} + \gamma_\theta\dot{\theta}_{(x,t)}\delta\dot{\theta}_{(x,t)} + \gamma_w\dot{w}_{(x,t)}\delta\dot{w}_{(x,t)} - K_{b1}w'_{(x)}\delta w'_{(x)} \\ &- K_{b2}\theta'_{(x)}\delta\theta'_{(x)} - K_{s2}\left[u'_{(x)} - \theta_{(x)}\right]\left[\delta u'_{(x)} - \delta\theta_{(x)}\right] \\ &- K_{s1}\left[u'_{(x)} + m\theta_{(x)} - nw_{(x)}\right]\left[\delta u'_{(x)} + m\delta\theta_{(x)} - n\delta w_{(x)}\right]\}dx\end{aligned} \tag{899}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\delta\mathcal{U} = \left[\gamma_u \dot{u}_{(x,t)} - (K_{s1}+K_{s2})u'_{(x)} + (K_{s2} - mK_{s1})\theta_{(x)} + nK_{s1}w_{(x)}\right]\delta u_{(x)}\Big|_0^H$$
$$+ \left[\gamma_\theta \dot{\theta}_{(x,t)} - K_{b2}\theta'_{(x)}\right]\delta\theta_{(x)}\Big|_0^H + \left[\gamma_w \dot{w}_{(x,t)} - K_{b1}w'_{(x)}\right]\delta\theta_{(x)}\Big|_0^H$$
$$- \int_0^H \left\{\gamma_u \ddot{u}_{(x,t)} - (K_{s1}+K_{s2})u''_{(x)} + (K_{s2} - mK_{s1})\theta'_{(x)} + nK_{s1}w'_{(x)}\right\}\delta u_{(x)}$$
$$- \int_0^H \left\{\gamma_\theta \ddot{\theta}_{(x,t)} - K_{b2}\theta''_{(x)} - (K_{s2} - mK_{s1})u'_{(x)} + (K_{s2} + m^2K_{s1})\theta_{(x)}\right.$$
$$\left. - mnK_{s1}w_{(x)}\right\}\delta\theta_{(x)}$$
$$- \int_0^H \left\{\gamma_w \ddot{w}_{(x,t)} - K_{b1}w''_{(x)} - nK_{s1}u'_{(x)} - mnK_{s1}\theta_{(x)} + n^2K_{s1}w_{(x)}\right\}\delta w_{(x)} \tag{900}$$

Setting the terms equal to zero, the following equations result:

$$\begin{Bmatrix} \gamma_u \ddot{u}_{(x,t)} - (K_{s1}+K_{s2})u''_{(x)} + (K_{s2} - mK_{s1})\theta'_{(x)} + nK_{s1}w'_{(x)} = 0 \\ \gamma_\theta \ddot{\theta}_{(x,t)} - K_{b2}\theta''_{(x)} - (K_{s2} - mK_{s1})u'_{(x)} + (K_{s2} + m^2K_{s1})\theta_{(x)} - mnK_{s1}w_{(x)} = 0 \\ \gamma_w \ddot{w}_{(x,t)} - K_{b1}w''_{(x)} - nK_{s1}u'_{(x)} - mnK_{s1}\theta_{(x)} + n^2K_{s1}w_{(x)} = 0 \end{Bmatrix} \tag{901}$$

And boundary conditions:

$$\begin{Bmatrix} (K_{s1}+K_{s2})u'_{(H)} - (K_{s2} - mK_{s1})\theta_{(H)} - nK_{s1}w_{(H)} = 0 \\ \theta'_{(H)} = 0 \\ w'_{(H)} = 0 \end{Bmatrix} \tag{902}$$

The partial differential equation can be solved by separating variables, applying a solution of the following type:

$$\begin{Bmatrix} u_{(x,t)} = \emptyset_{(x)}q_{(t)} \\ \theta_{(x,t)} = \lambda_{1(x)}q_{(t)} \\ w_{(x,t)} = \lambda_{2(x)}q_{(t)} \end{Bmatrix} \tag{903}$$

Where $\emptyset_{(x)}$ defines the variation of the displacement along the length of the beam, while $q_{(t)}$ does so with time. Replacing and collecting like terms, we get the following:

$$\left\{\begin{array}{c} \frac{\ddot{q}_{(t)}}{q_{(t)}} + \left[\frac{-(K_{s1}+K_{s2})\emptyset''_{(x)} + (K_{s2}-mK_{s1})\lambda'_{1(x)} + nK_{s1}\lambda'_{2(x)} =}{\gamma_u \emptyset_{(x)}}\right] = 0 \\ \frac{\ddot{q}_{(t)}}{q_{(t)}} + \left[\frac{-K_{b2}\lambda''_{1(x)} - (K_{s2}-mK_{s1})\emptyset'_{(x)} + (K_{s2}+m^2K_{s1})\lambda_{1(x)} - mnK_{s1}\lambda_{2(x)}}{\gamma_\theta \lambda_{1(x)}}\right] = 0 \\ \frac{\ddot{q}_{(t)}}{q_{(t)}} + \left[\frac{-K_{b1}\lambda''_{2(x)} - nK_{s1}\emptyset'_{(x)} - mnK_{s1}\lambda_{1(x)} + n^2K_{s1}\lambda_{2(x)}}{\gamma_w \lambda_{2(x)}}\right] = 0 \end{array}\right\} \tag{904}$$

Because the time and height coordinates are independent variables, each of the terms must equal a constant with opposite signs, to ensure that the net result is zero. Consequently, it can be divided into two ordinary differential equations:

$$\left\{\begin{array}{c} \ddot{q}_{(t)} + w^2 q_{(t)} = 0 \\ (K_{s1}+K_{s2})\emptyset''_{(x)} - (K_{s2}-mK_{s1})\lambda'_{1(x)} - nK_{s1}\lambda'_{2(x)} + w^2\gamma_u\emptyset_{(x)} = 0 \\ K_{b2}\lambda''_{1(x)} + (K_{s2}-mK_{s1})\emptyset'_{(x)} - (K_{s2}+m^2K_{s1})\lambda_{1(x)} + mnK_{s1}\lambda_{2(x)} + w^2\gamma_\theta\lambda_{1(x)} = 0 \\ K_{b1}\lambda''_{2(x)} + nK_{s1}\emptyset'_{(x)} + mnK_{s1}\lambda_{1(x)} - n^2K_{s1}\lambda_{2(x)} + w^2\gamma_w\lambda_{2(x)} = 0 \end{array}\right\} \tag{905}$$

The first equation is the same one that governs the behavior of an SDOF system with vibration frequency w.

Using the method of differential operators for the solution of the system of equations:

$$\begin{bmatrix} (K_{s1}+K_{s2})D^2 + w^2\gamma_u & -(K_{s2}-mK_{s1})D & -nK_{s1}D \\ (K_{s2}-mK_{s1})D & K_{b2}D^2 + [w^2\gamma_\theta - (K_{s2}+m^2K_{s1})] & mnK_{s1} \\ nK_{s1}D & mnK_{s1} & K_{b1}D^2 + [w^2\gamma_w - n^2K_{s1}] \end{bmatrix} \begin{Bmatrix} \emptyset_{(x)} \\ \lambda_{1(x)} \\ \lambda_{2(x)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{906}$$

i.e.,

$$\begin{aligned} \emptyset''''''_{(x)} &- \left[\frac{K_{s1}K_{s2}[K_{b2}n^2 + K_{b1}(m+1)^2]}{K_{b1}K_{b2}(K_{s1}+K_{s2})} - \frac{K_{b1}K_{b2}\gamma_u + (K_{b1}\gamma_\theta + K_{b2}\gamma_w)(K_{s1}+K_{s2})}{K_{b1}K_{b2}(K_{s1}+K_{s2})}w^2\right]\emptyset''''_{(x)} \\ &- w^2\left\{\frac{K_{s1}K_{s2}[\gamma_\theta n^2 + \gamma_w(m+1)^2] + [K_{s1}K_{b2}n^2 + (K_{s2}+m^2K_{s1})K_{b1}]\gamma_u}{K_{b1}K_{b2}(K_{s1}+K_{s2})}\right. \\ &\left. - \frac{(K_{s1}+K_{s2})\gamma_\theta\gamma_w + (K_{b1}\gamma_\theta + K_{b2}\gamma_w)\gamma_u}{K_{b1}K_{b2}(K_{s1}+K_{s2})}w^2\right\}\emptyset''_{(x)} \\ &+ \frac{\gamma_u w^2\{(w^2\gamma_w - n^2K_{s1})[w^2\gamma_\theta - (K_{s2}+m^2K_{s1})] - m^2n^2K_{s1}^2\}}{K_{b1}K_{b2}(K_{s1}+K_{s2})}\emptyset_{(z)} = 0 \end{aligned} \tag{907}$$

A sixth order differential equation is obtained. Normalizing the differential equation by the variable $z = x/H$:

$$\begin{aligned}
&\emptyset_{(z)}^{''''''} - \left[\frac{K_{s1}K_{s2}[K_{b2}n^2 + K_{b1}(m+1)^2]}{K_{b1}K_{b2}(K_{s1}+K_{s2})} - \frac{K_{b1}K_{b2}\gamma_u + (K_{b1}\gamma_\theta + K_{b2}\gamma_w)(K_{s1}+K_{s2})}{K_{b1}K_{b2}(K_{s1}+K_{s2})}w^2\right]H^2\emptyset_{(z)}^{''''} \\
&\quad - w^2\left\{\frac{K_{s1}K_{s2}[\gamma_\theta n^2 + \gamma_w(m+1)^2] + [K_{s1}K_{b2}n^2 + (K_{s2}+m^2K_{s1})K_{b1}]\gamma_u}{K_{b1}K_{b2}(K_{s1}+K_{s2})}\right. \\
&\quad \left. - \frac{(K_{s1}+K_{s2})\gamma_\theta\gamma_w + (K_{b1}\gamma_\theta + K_{b2}\gamma_w)\gamma_u}{K_{b1}K_{b2}(K_{s1}+K_{s2})}w^2\right\}H^4\emptyset_{(z)}^{''} \\
&\quad + \frac{\gamma_u w^2\{(w^2\gamma_w - n^2K_{s1})[w^2\gamma_\theta - (K_{s2}+m^2K_{s1})] - m^2n^2K_{s1}^2\}}{K_{b1}K_{b2}(K_{s1}+K_{s2})}H^6\emptyset_{(z)} = 0
\end{aligned} \tag{908}$$

The equation can be rewritten as:

$$\begin{aligned}
&\emptyset_{(z)}^{''''''} - \left[\alpha^2\kappa^2 - \left(1 + \mu_\psi^2 + \mu_\theta^2\right)\delta^2\right]\emptyset_{(z)}^{''''} \\
&\quad - \delta^2\left\{\alpha^2\left[\frac{\kappa^2 - (m+1)^2}{n^2}\mu_\theta^2 + \mu_\psi^2\right] + \left(\eta_\theta^2 + n^2\eta_w^2\right)\right. \\
&\quad \left. - \left[\left(\frac{\mu_w}{m+1}\right)^2\left(\frac{\mu_\theta}{n}\right)^2 + \left(\frac{\mu_w}{m+1}\right)^2 + \left(\frac{\mu_\theta}{n}\right)^2\right]\delta^2\right\}\emptyset_{(z)}^{''} \\
&\quad + \delta^2\left\{\left[\delta^2\left(\frac{\mu_w}{m+1}\right)^2 - n^2\eta_w^2\right]\left[\delta^2\left(\frac{\mu_\theta}{n}\right)^2 - \eta_\theta^2\right] - n^2\eta_w^2\left(\eta_\theta^2 - H^2\right)\right\}\emptyset_{(z)} = 0
\end{aligned} \tag{909}$$

Where:

$$\left\{\begin{array}{c}
\alpha = H\sqrt{\dfrac{K_{s1}K_{s2}}{K_{b2}(K_{s1}+K_{s2})}}, \kappa = \sqrt{(m+1)^2 + n^2\dfrac{K_{b2}}{K_{b1}}} \\
\mu_w = (m+1)\sqrt{\dfrac{K_{s1}+K_{s2}}{K_{b1}}\dfrac{\gamma_w}{\gamma_u}}, \mu_\theta = n\sqrt{\dfrac{K_{s1}+K_{s2}}{K_{b2}}\dfrac{\gamma_\theta}{\gamma_u}} =, \delta = \sqrt{\dfrac{\gamma_u H^2}{K_{s1}+K_{s2}}w^2} \\
\eta_w = H\sqrt{\dfrac{K_{s1}}{K_{b1}}}, \eta_\theta = H\sqrt{\dfrac{K_{s2}+m^2K_{s1}}{K_{b2}}}
\end{array}\right\} \tag{910}$$

The differential equation obtained can be easily solved with the procedures presented in the previous subchapters.

4.2.11.2 Case 2

The relationship between forces and displacements between two consecutive floors is obtained by taking into account the transfer matrix and the vector of external point forces. For the j-th floor:

$$\left\{\begin{array}{l} \begin{Bmatrix} u_{i+1}(0) \\ w_{i+1}(0) \\ \theta_{i+1}(0) \\ M_{1i+1}(0) \\ M_{2i+1}(0) \\ V_{i+1}(0) \end{Bmatrix} = \left\{ T_i(0) + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ m_i w^2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\} \begin{Bmatrix} u_i(0) \\ w_i(0) \\ \theta_i(0) \\ M_{1i}(0) \\ M_{2i}(0) \\ V_i(0) \end{Bmatrix} \\ \begin{Bmatrix} u_{i+1}(0) \\ w_{i+1}(0) \\ \theta_{i+1}(0) \\ M_{1i+1}(0) \\ M_{2i+1}(0) \\ V_{i+1}(0) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ m_i w^2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} T_i(0) \begin{Bmatrix} u_i(0) \\ w_i(0) \\ \theta_i(0) \\ M_{1i}(0) \\ M_{2i}(0) \\ V_i(0) \end{Bmatrix} \end{array}\right. \tag{911}$$

Rewriting:

$$\begin{Bmatrix} u_{i+1}(0) \\ w_{i+1}(0) \\ \theta_{i+1}(0) \\ M_{1i+1}(0) \\ M_{2i+1}(0) \\ V_{i+1}(0) \end{Bmatrix} = T_{wi}(0) \begin{Bmatrix} u_i(0) \\ w_i(0) \\ \theta_i(0) \\ M_{1i}(0) \\ M_{2i}(0) \\ V_i(0) \end{Bmatrix} \tag{912}$$

Where:

$$T_{wi}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ m_i w^2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} T_i(0) \tag{913}$$

Expressing the equation for the nth floor between product symbols:

$$\begin{Bmatrix} u_n(0) \\ w_n(0) \\ \theta_n(0) \\ M_{1n}(0) \\ M_{2n}(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_{wk}(0) \begin{Bmatrix} u_1(h_i) \\ w_1(h_i) \\ \theta_1(h_i) \\ M_{11}(h_i) \\ M_{21}(h_i) \\ V_1(h_i) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_i) \\ w_1(h_i) \\ \theta_1(h_i) \\ M_{11}(h_i) \\ M_{21}(h_i) \\ V_1(h_i) \end{Bmatrix} \qquad (914)$$

Where:

$$\mathrm{t} = \prod_{k=1}^{n} T_{wk}(0) \qquad (915)$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 6x6 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ w_{(1)} = 0 \\ \theta_{(1)} = 0 \\ w'_{(0)} = 0 \\ \theta'_{(0)} = 0 \\ (K_{s1} + K_{s2})u'_{(0)} - (K_{s2} - mK_{s1})\theta_{(0)} - nK_{s1}w_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1(h_1) = 0 \\ w_1(h_1) = 0 \\ \theta_1(h_1) = 0 \\ M_{1n\,(0)} = 0 \\ M_{2n\,(0)} = 0 \\ V_{n\,(0)} = 0 \end{Bmatrix} \qquad (916)$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ w_n(0) \\ \theta_n(0) \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & t_{1,5} & t_{1,6} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} & t_{2,5} & t_{2,6} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} & t_{3,5} & t_{3,6} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,1} & t_{5,2} & t_{5,3} & t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,1} & t_{6,2} & t_{6,3} & t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ M_{11}(h_1) \\ M_{21}(h_1) \\ V_1(h_1) \end{Bmatrix} \qquad (917)$$

Solving for the bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} M_{11}(h_1) \\ M_{21}(h_1) \\ V_1(h_1) \end{Bmatrix} \qquad (918)$$

Which has a different solution than the trivial one if the determinant is equal to zero (the matrix of coefficients is singular). Solving the determinant finds the angular frequencies of the beam.

4.2.12 Modified Generalized Parallel Coupling of Two Beams (GCTB) of Two Field

4.2.12.1 Case 1

The potential energy and kinetic energy of the three-field GSB model are:

$$V = \frac{1}{2}\int_0^H \left\{K_{b1}{w'_{(x)}}^2 + K_{b2}{u''_{(x)}}^2 + K_{s1}\left[(m+1)u'_{(x)} - nw_{(x)}\right]^2\right\} dx$$
$$T = \frac{1}{2}\int_0^H \left[\gamma_u {\dot{u}_{(x,t)}}^2 + \gamma_w {\dot{w}_{(x,t)}}^2\right] dx \tag{919}$$

Where:

$$\left\{\gamma_u = \rho(A_1 + A_2), \gamma_w = \rho\frac{A_2}{A_1}(A_1 + A_2)\right\} \tag{920}$$

Consequently, the total potential energy of the three-field beam GSB is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H \left[\gamma_u {\dot{u}_{(x,t)}}^2 + \gamma_w {\dot{w}_{(x,t)}}^2\right] dx - \frac{1}{2}\int_0^H \left\{K_{b1}{w'_{(x)}}^2 + K_{b2}{u''_{(x)}}^2 + K_{s1}\left[(m+1)u'_{(x)} - nw_{(x)}\right]^2\right\} dx \tag{921}$$

Closed-form solutions of the model are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\delta\mathcal{U} = \int_0^H \{\gamma_u \dot{u}_{(x,t)}\delta\dot{u}_{(x,t)} + \gamma_w \dot{w}_{(x,t)}\delta\dot{w}_{(x,t)} - K_{b1}w'_{(x)}\delta w'_{(x)} - K_{b2}\theta'_{(x)}\delta\theta'_{(x)}$$
$$- K_{s1}\left[(m+1)u'_{(x)} - nw_{(x)}\right]\left[(m+1)\delta u'_{(x)} - n\delta w_{(x)}\right]\}dx \tag{922}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\delta\mathcal{U} = \{\left[\gamma_u \dot{u}_{(x,t)} + K_{b2}u'''_{(x)} - (m+1)^2 K_{s1}u'_{(x)} + n(m+1)K_{s1}w_{(x)}\right]\delta u_{(x)}\}_0^H - K_{b2}u''_{(x)}\delta {u'_{(x)}}_0^H$$
$$+ \left[\gamma_w \dot{w}_{(x,t)} - K_{b1}w'_{(x)}\right]\delta {w_{(x)}}_0^H$$
$$- \int_0^H \{\gamma_u \ddot{u}_{(x,t)} + K_{b2}u''''_{(x)} - (m+1)^2 K_{s1}u''_{(x)} + n(m+1)K_{s1}w'_{(x)}\}\delta u_{(x)}$$
$$- \int_0^H \{\gamma_w \ddot{w}_{(x,t)} - K_{b1}w''_{(x)} - n(m+1)K_{s1}u'_{(x)} + n^2 K_{s1}w_{(x)}\}\delta w_{(x)} \tag{923}$$

Setting the terms equal to zero, the following equations result:

$$\begin{Bmatrix} \delta u_{(x,t)}: \gamma_u \ddot{u}_{(x,t)} + K_{b2} u''''_{(x)} - (m+1)^2 K_{s1} u''_{(x)} + n(m+1) K_{s1} w'_{(x)} = 0 \\ \delta w_{(x,t)}: \gamma_w \ddot{w}_{(x,t)} - K_{b1} w''_{(x)} - n(m+1) K_{s1} u'_{(x)} + n^2 K_{s1} w_{(x)} = 0 \end{Bmatrix} \qquad (924)$$

And boundary conditions:

$$\begin{Bmatrix} \delta u_{(x,t)}:\ K_{b2} u'''_{(H)} - (m+1)^2 K_{s1} u'_{(H)} + n(m+1) K_{s1} w_{(H)} = 0 \\ \delta u'_{(x,t)}:\ u''_{(H)} = 0 \\ \delta w_{(x,t)}:\ w'_{(H)} = 0 \end{Bmatrix} \qquad (925)$$

The partial differential equation can be solved by separating variables, applying a solution of the following type:

$$\begin{Bmatrix} u_{(x,t)} = \emptyset_{(x)} q_{(t)} \\ w_{(x,t)} = \lambda_{(x)} q_{(t)} \end{Bmatrix} \qquad (926)$$

Where $\emptyset_{(x)}$ defines the variation of the displacement along the length of the beam, while $q_{(t)}$ does so with time. Replacing and collecting like terms, we get the following:

$$\begin{Bmatrix} \dfrac{\ddot{q}_{(t)}}{q_{(t)}} + \left[\dfrac{K_{b2}\emptyset'''_{(x)} - (m+1)^2 K_{s1}\emptyset''_{(x)} + n(m+1)K_{s1}\lambda'_{(x)} =}{\gamma_u \emptyset_{(x)}}\right] = 0 \\ \dfrac{\ddot{q}_{(t)}}{q_{(t)}} + \left[\dfrac{-K_{b1}\lambda''_{(x)} - n(m+1)K_{s1}\emptyset'_{(x)} + n^2 K_{s1}\lambda_{(x)}}{\gamma_w \lambda_{(x)}}\right] = 0 \end{Bmatrix} \qquad (927)$$

Because the time and height coordinates are independent variables, each of the terms must equal a constant with opposite signs, to ensure that the net result is zero. Consequently, it can be divided into two ordinary differential equations:

$$\begin{Bmatrix} \ddot{q}_{(t)} + w^2 q_{(t)} = 0 \\ -K_{b2}\emptyset'''_{(x)} + (m+1)^2 K_{s1}\emptyset''_{(x)} - n(m+1)K_{s1}\lambda'_{(x)} + w^2\gamma_u\emptyset_{(x)} = 0 \\ K_{b1}\lambda''_{(x)} + n(m+1)K_{s1}\emptyset'_{(x)} - n^2 K_{s1}\lambda_{(x)} + w^2\gamma_w\lambda_{(x)} = 0 \end{Bmatrix} \qquad (928)$$

The first equation is the same one that governs the behavior of an SDOF system with vibration frequency w.

Using the method of differential operators for the solution of the system of equations:

$$\begin{bmatrix} -K_{b2}D^4 + (m+1)^2 K_{s1} D^2 + w^2 \gamma_u & -n(m+1)K_{s1}D \\ n(m+1)K_{s1}D & K_{b1}D^2 + [w^2\gamma_w - n^2 K_{s1}] \end{bmatrix} \begin{Bmatrix} \emptyset_{(x)} \\ \lambda_{(x)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{929}$$

i.e.,

$$\emptyset_{(x)}'''''' - \left[\frac{K_{s1}[K_{b2}n^2 + K_{b1}(m+1)^2]}{K_{b1}K_{b2}} - \frac{\gamma_w}{K_{b1}} w^2\right] \emptyset_{(x)}'''' - w^2 \left[\frac{\gamma_u}{K_{b2}} + \frac{(m+1)^2 K_{s1}\gamma_w}{K_{b1}K_{b2}}\right] \emptyset_{(x)}'' + \frac{\gamma_u w^2 (w^2\gamma_w - n^2 K_{s1})}{K_{b1}K_{b2}} \emptyset_{(z)} = 0 \tag{930}$$

A sixth order differential equation is obtained. Normalizing the differential equation by the variable $z = x/H$:

$$\emptyset_{(z)}'''''' - \left[\frac{K_{s1}[K_{b2}n^2 + K_{b1}(m+1)^2]}{K_{b1}K_{b2}} - \frac{\gamma_w}{K_{b1}} w^2\right] H^2 \emptyset_{(z)}'''' - w^2 \left[\frac{\gamma_u}{K_{b2}} + \frac{(m+1)^2 K_{s1}\gamma_w}{K_{b1}K_{b2}}\right] H^4 \emptyset_{(z)}'' + \frac{\gamma_u w^2 (w^2\gamma_w - n^2 K_{s1}) H^6}{K_{b1}K_{b2}} \emptyset_{(z)} = 0 \tag{931}$$

The equation can be rewritten as:

$$\emptyset_{(z)}'''''' - [\alpha^2\kappa^2 - \mu^2\delta^2]\emptyset_{(z)}'''' - \delta^2[1 + (m+1)^2\mu^2\alpha^2]\emptyset_{(z)}'' + \delta^2\{\alpha^2[\kappa^2 - (m+1)^2] - \mu^2\delta^2\}\emptyset_{(z)} = 0 \tag{932}$$

Where:

$$\left\{\alpha = H\sqrt{\frac{K_{s1}}{K_{b2}}}, \kappa = \sqrt{(m+1)^2 + n^2\frac{K_{b2}}{K_{b1}}}, \mu = \frac{1}{H}\sqrt{\frac{K_{b2}}{K_{b1}}\frac{\gamma_w}{\gamma_u}} =, \delta = \sqrt{\frac{\gamma_u H^4}{K_{b2}} w^4}\right\} \tag{933}$$

The differential equation obtained can be easily solved with the procedures presented in the previous subchapters.

4.2.12.2 Case 2

The relationship between forces and displacements between two consecutive floors is obtained by taking into account the transfer matrix and the vector of external point forces. For the j-th floor:

$$\left\{\begin{array}{l}\begin{Bmatrix} u_{i+1}(0) \\ u_{i+1}^{'}(0) \\ w_{i+1}(0) \\ M_{\mathrm{li+1}}(0) \\ M_{\mathrm{ri+1}}(0) \\ V_{i+1}(0) \end{Bmatrix} = \left\{ T_i(0) + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ m_i w^2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\} \begin{Bmatrix} u_i(0) \\ u_i^{'}(0) \\ w_i(0) \\ M_{\mathrm{li}}(0) \\ M_{\mathrm{ri}}(0) \\ V_i(0) \end{Bmatrix} \\ \begin{Bmatrix} u_{i+1}(0) \\ u_{i+1}^{'}(0) \\ w_{i+1}(0) \\ M_{\mathrm{li+1}}(0) \\ M_{\mathrm{ri+1}}(0) \\ V_{i+1}(0) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ m_i w^2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} T_i(0) \begin{Bmatrix} u_i(0) \\ u_i^{'}(0) \\ w_i(0) \\ M_{\mathrm{li}}(0) \\ M_{\mathrm{ri}}(0) \\ V_i(0) \end{Bmatrix} \end{array}\right\} \tag{934}$$

Rewriting:

$$\begin{Bmatrix} u_{i+1}(0) \\ u'_{i+1}(0) \\ w_{i+1}(0) \\ M_{\mathrm{li+1}}(0) \\ M_{\mathrm{ri+1}}(0) \\ V_{i+1}(0) \end{Bmatrix} = T_{wi}(0) \begin{Bmatrix} u_i(0) \\ u'_i(0) \\ w_i(0) \\ M_{\mathrm{li}}(0) \\ M_{\mathrm{ri}}(0) \\ V_i(0) \end{Bmatrix} \tag{935}$$

Where:

$$T_{wi}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ m_i w^2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} T_i(0) \tag{936}$$

Expressing the equation for the nth floor between product symbols:

$$\begin{Bmatrix} u_n(0) \\ u'_n(0) \\ \theta_n(0) \\ M_{\mathrm{ln}}(0) \\ M_{\mathrm{rn}}(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_{wk}(0) \begin{Bmatrix} u_1(h_1) \\ u'_1(h_1) \\ \theta_1(h_1) \\ M_{\mathrm{l1}}(h_1) \\ M_{\mathrm{r1}}(h_1) \\ V_1(h_1) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ u'_1(h_1) \\ \theta_1(h_1) \\ M_{\mathrm{l1}}(h_1) \\ M_{\mathrm{r1}}(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{937}$$

Where:

$$t = \prod_{k=1}^{n} T_{wk}(0) \tag{938}$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 6x6 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\left\{\begin{matrix} u_{(1)} = 0 \\ u'_{(1)} = 0 \\ \theta_{(1)} = 0 \\ \theta'_{(0)} = 0 \\ u''_{(0)} = 0 \\ -K_{b2}u'''_{(0)} + (m+1)^2 K_{s1} u'_{(0)} - n(m+1)K_{s1} w_{(0)} = 0 \end{matrix}\right\} \rightarrow \left\{\begin{matrix} u_1(h_1) = 0 \\ u'_1(h_1) = 0 \\ \theta_1(h_1) = 0 \\ M_{ln}(0) = 0 \\ M_{rn}(0) = 0 \\ V_n(0) = 0 \end{matrix}\right\} \tag{939}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ u'_n(0) \\ \theta_n(0) \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & t_{1,5} & t_{1,6} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} & t_{2,5} & t_{2,6} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} & t_{3,5} & t_{3,6} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,1} & t_{5,2} & t_{5,3} & t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,1} & t_{6,2} & t_{6,3} & t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ M_{l1}(h_1) \\ M_{r1}(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{940}$$

Solving for the bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} M_{l1}(h_1) \\ M_{r1}(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{941}$$

Which has a different solution than the trivial one if the determinant is equal to zero (the matrix of coefficients is singular). Solving the determinant finds the angular frequencies of the beam.

4.2.13 Generalized Parallel Coupling of Two Beams of a Field (GCTB)

4.2.13.1 Case 1

The potential energy and kinetic energy of the three-field GSB model are:

$$V = \frac{1}{2}\int_0^H \left\{K_b {u''_{(x)}}^2 + (m+1)^2 K_s {u'_{(x)}}^2\right\} dx, T = \frac{1}{2}\int_0^H \left[\gamma_u {\dot{u}_{(x,t)}}^2\right] dx \tag{942}$$

Where:

$$\{\gamma_u = \rho(A_1 + A_2)\} \tag{943}$$

Consequently, the total potential energy of the three-field beam GSB is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H \left[\gamma_u {\dot{u}_{(x,t)}}^2\right] dx - \frac{1}{2}\int_0^H \left\{K_b {u''_{(x)}}^2 + (m+1)^2 K_s {u'_{(x)}}^2\right\} dx \tag{944}$$

Closed-form solutions of the model are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\delta\mathcal{U} = \int_0^H \{\gamma_u \dot{u}_{(x,t)} \delta\dot{u}_{(x,t)} - K_b u''_{(x)} \delta u''_{(x)} - (m+1)^2 K_s u'_{(x)} \delta u'_{(x)}\} dx \tag{945}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\begin{aligned}\delta\mathcal{U} = &\left[\gamma_u \dot{u}_{(x,t)} + K_b u'''_{(x)} - (m+1)^2 K_s u'_{(x)}\right] {\delta u_{(x)}}_0^H - K_b u''_{(x)} {\delta u'_{(x)}}_0^H \\ &- \int_0^H \left[\gamma_u \ddot{u}_{(x,t)} + K_b u''''_{(x)} - (m+1)^2 K_s u''_{(x)}\right] \delta u_{(x)} dx\end{aligned} \tag{946}$$

Setting the terms equal to zero, the following equations result:

$$\delta u_{(x,t)}\colon \gamma_u \ddot{u}_{(x,t)} + K_b u''''_{(x)} - (m+1)^2 K_s u''_{(x)} = 0 \tag{947}$$

And boundary conditions:

$$\begin{Bmatrix} \delta u_{(x,t)}\colon\ K_b u'''_{(H)} - (m+1)^2 K_s u'_{(H)} = 0 \\ \delta u'_{(x,t)}\colon\ u''_{(H)} = 0 \end{Bmatrix} \tag{948}$$

The partial differential equation can be solved by separating variables, applying a solution of the following type:

$$u_{(x,t)} = \emptyset_{(x)} q_{(t)} \tag{949}$$

Where $\emptyset_{(x)}$ defines the variation of the displacement along the length of the beam, while $q_{(t)}$ does so with time. Replacing and collecting like terms, we get the following:

$$\frac{\ddot{q}_{(t)}}{q_{(t)}} + \left[\frac{K_b \emptyset''''_{(x)} - (m+1)^2 K_s \emptyset''_{(x)}}{\gamma_u \emptyset_{(x)}}\right] = 0 \tag{950}$$

Because the time and height coordinates are independent variables, each of the terms must equal a constant with opposite signs, to ensure that the net result is zero. Consequently, it can be divided into two ordinary differential equations:

$$\begin{Bmatrix} \ddot{q}_{(t)} + w^2 q_{(t)} = 0 \\ K_b \emptyset''''_{(x)} - (m+1)^2 K_s \emptyset''_{(x)} - w^2 \gamma_u \emptyset_{(x)} = 0 \end{Bmatrix} \tag{951}$$

The first equation is the same one that governs the behavior of an SDOF system with vibration frequency w.

Using the method of differential operators for the solution of the system of equations:

$$\begin{bmatrix} -K_{b2} D^4 + (m+1)^2 K_{s1} D^2 + w^2 \gamma_u & -n(m+1) K_{s1} D \\ n(m+1) K_{s1} D & K_{b1} D^2 + [w^2 \gamma_w - n^2 K_{s1}] \end{bmatrix} \begin{Bmatrix} \emptyset_{(x)} \\ \lambda_{(x)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{952}$$

i.e.,

$$\emptyset''''_{(x)} - \left[\frac{(m+1)^2 K_s}{K_b}\right] \emptyset''_{(x)} - \frac{\gamma_u w^2}{K_b} \emptyset_{(x)} = 0 \tag{953}$$

A sixth order differential equation is obtained. Normalizing the differential equation by the variable z=x/H:

$$\emptyset''''_{(z)} - \frac{(m+1)^2 K_s H^2}{K_b} \emptyset''_{(z)} - \frac{\gamma_u w^2 H^4}{K_b} \emptyset_{(z)} = 0 \tag{954}$$

The equation can be rewritten as:

$$\emptyset''''_{(z)} - \alpha^2\emptyset''_{(z)} - \delta^2\emptyset_{(z)} = 0 \tag{955}$$

Where:

$$\left\{\alpha = H\sqrt{\frac{K_s}{K_b}}, \delta = \sqrt{\frac{\gamma_u H^4}{K_{b2}} w^2}\right\} \tag{956}$$

The differential equation obtained can be easily solved with the procedures presented in the previous subchapters.

4.2.13.2 Case 2

The relationship between forces and displacements between two consecutive floors is obtained by taking into account the transfer matrix and the vector of external point forces. For the j-th floor:

$$\begin{Bmatrix} u_{i+1}(0) \\ u'_{i+1}(0) \\ M_{i+1}(0) \\ V_{i+1}(0) \end{Bmatrix} = \left\{ T_i(0) + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ m_i w^2 & 0 & 0 & 0 \end{bmatrix} \right\} \begin{Bmatrix} u_i(0) \\ u'_i(0) \\ M_i(0) \\ V_i(0) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m_i w^2 & 0 & 0 & 1 \end{bmatrix} T_i(0) \begin{Bmatrix} u_i(0) \\ u'_i(0) \\ M_i(0) \\ V_i(0) \end{Bmatrix} \tag{957}$$

Rewriting:

$$\begin{Bmatrix} u_{i+1}(0) \\ u'_{i+1}(0) \\ M_{i+1}(0) \\ V_{i+1}(0) \end{Bmatrix} = T_{wi}(0) \begin{Bmatrix} u_i(0) \\ u'_i(0) \\ M_i(0) \\ V_i(0) \end{Bmatrix} \tag{958}$$

Where:

$$T_{wi}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m_i w^2 & 0 & 0 & 1 \end{bmatrix} T_i(0) \tag{959}$$

Expressing the equation for the nth floor between product symbols:

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_{wk}(0) \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{960}$$

Where:

$$\mathrm{t} = \prod_{k=1}^{n} T_{wk}(0) \tag{961}$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 4x4 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ u_{(1)}' = 0 \\ u_{(0)}'' = 0 \\ K_b u_{(0)}''' - K_s u_{(0)}' = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1(h_1) = 0 \\ u_1'(h_1) = 0 \\ M_n(0) = 0 \\ V_n(0) = 0 \end{Bmatrix} \tag{962}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \rightarrow \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{963}$$

Which has a different solution than the trivial one if the determinant is equal to zero (the matrix of coefficients is singular). Solving the determinant finds the angular frequencies of the beam.

4.3 STABILITY ANALYSIS OF INDIVIDUAL STRUCTURAL SYSTEMS

The objective of this section is to develop a methodology for calculating the critical load of replacement beams.

Stability analysis of tall buildings by conventional methods results in a very cumbersome and complicated procedure. As mentioned by Zalka (2020), stability analysis presents a greater mathematical challenge than dynamic analysis and even greater than static analysis. It is due to these complications that investigations in the field of stability of tall buildings are very limited and special cases are usually analyzed ignoring certain characteristics that do not have an important influence on the analysis. To overcome this problem, two cases of analysis are considered with the aim of covering as many buildings as possible.

- **Case 1**: A continuous analysis is considered because the method used is based solely on the continuous method and a uniformly distributed vertical load is assumed over the height of the element.

  To take into account that the vertical load is applied at the level of the floors and that it is not distributed over the height of the building, Zalka (2020), using the Dunkerley sum theorem, proposes considering a correction factor in the stability analysis.

$$r_s = \frac{n}{n + 1.588} \qquad (964)$$

  Where n is the number of floors of the building. It is true that this effect is negligible in tall buildings, but for medium and low buildings, not considering this correction coefficient is not conservative because the centroid of the total vertical load shifts downwards, resulting in critical load values greater than the actual load.

  The main disadvantage is that it is only applicable to structures where the cross section is uniform in height and the lateral load is continuous. The main advantage is that continuous closed-form solutions are obtained that allow parametric analysis.

- **Case 2**: A discrete analysis is considered because the methods used are the continuous method and the transfer matrix method, and an arbitrary point vertical load applied at floor level is assumed.

The main disadvantage is that closed continuous solutions that allow parametric analysis are not obtained. The main advantage is that it allows the analysis of structures whose cross section is not continuous in height and/or for structures where the loads are applied at the level of the floors, whether their cross section is uniform or not; that is, it is considered a case of general analysis because it even serves as a verification of case 1.

The following assumptions will be used:

a) The material is elastic and linear.

b) The floors are considered as rigid diaphragms and their rigidity perpendicular to their plane is neglected.

c) Vertical loads are applied statically and do not change their direction during buckling.

d) The deformations are considered small.

e) Structures have a minimum of four stories.

Mikhlin (1964) proposes that the approximate solution of the eigenvalue problem generally reduces to the integration of a differential equation of the form:

$$Lw - \lambda Mw = 0 \tag{965}$$

Where w is the displacement that satisfies the differential equation and the boundary conditions, L and M are differential operators, and λ is the unknown numerical parameter to calculate.

In most practical cases, it is not possible to assume a function that gives a close enough approximation to the exact deflection. So the approximation of the upper limit obtained from the Rayleigh ratio is not very close. Substantial improvement in accuracy is possible if we consider a linear combination of several superimposed functions and then apply the Rayleigh-Ritz method by minimizing the Rayleigh quotient with respect to the unknown coefficients of this linear combination.

By Ritz's theorem, the limit of the approximation $u_{N(z)} = \sum_{k=1}^{N} q_k \phi_{k(z)}$ for N→∞ is the exact solution $u_{(z)}$ if the system of the chosen functions $\phi_{k(z)}$ satisfies the following conditions:

a) The functions $\phi_{k(z)}$ are linearly independent.

b) The functions $\phi_{k(z)}$ form a complete system of functions.

c) The functions $\phi_{k(z)}$ satisfy the boundary conditions.

As an approximate displacement function, a power series will be used, whose center is equal to zero:

$$\phi_{(z)} = \sum_{k=0}^{N} C_k z^k \tag{966}$$

Where C_k are constant coefficients, which represent the parameters of the approximate function.

$$\phi_{(z)} = \sum_{k=0}^{N} C_k z^k = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + C_4 z^4 + C_5 z^5 + \cdots \tag{967}$$

4.3.1 Bending Beam of a Field (EBB)

4.3.1.1 Case 1

The potential energy of the EBB model of a field is:

$$V = \frac{1}{2}\int_0^H K_b u''^{\,2}_{(x)}\, dx \tag{968}$$

The work done by the external force is:

$$W = -\int_0^H f_{(x)} dl \tag{969}$$

Where $f_{(x)}$ is the generalized axial load distributed along the height and dl is the axial shortening of the beam.

Taking into account that the displacements are small, the axial shortening is:

$$dl = ds - dx = \left(\sqrt{dx^2 + du^2}\right) - dx = \left(\sqrt{1 + \left(\frac{du}{dx}\right)^2} - 1\right) dx \tag{970}$$

Replacing:

$$W = -\int_0^H f_{(x)} dl = -\int_0^H f_{(x)} \left(\sqrt{1 + \left(\frac{du}{dx}\right)^2} - 1\right) dx \tag{971}$$

Expanding this function by the Maclaurin series:

$$\sqrt{1 + \left(\frac{du}{dx}\right)^2} = 1 + \frac{1}{2}\left(\frac{du}{dx}\right)^2 + \cdots \approx 1 + \frac{1}{2}\left(\frac{du}{dx}\right)^2 \rightarrow \sqrt{1 + \left(\frac{du}{dx}\right)^2} - 1 = \frac{1}{2}\left(\frac{du}{dx}\right)^2 \tag{972}$$

Finally the work results:

$$W = -\frac{1}{2}\int_0^H f_{(x)} \left(\frac{du}{dx}\right)^2 dx = -\frac{1}{2}\int_0^H f_{(x)} u'^{\,2}_{(x)}\, dx \tag{973}$$

Consequently, the total potential energy of the model is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H K_b {u''_{(x)}}^2\, dx - \frac{1}{2}\int_0^H f(x) {u'_{(x)}}^2\, dx \tag{974}$$

Closed-form solutions of the model are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\delta\mathcal{U} = \int_0^H \left[K_b u''_{(x)}\delta u''_{(x)} - f_{(x)} u'_{(x)}\delta u'_{(x)}\right]dx - \frac{1}{2}\int_0^H f(x) u'^2_{(x)}\delta f_{(x)} dx \tag{975}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\begin{aligned}\delta\mathcal{U} = \left[K_b u''_{(x)}\delta u'_{(x)}\right]_0^H - \left\{\left[K_{b2} u'''_{(x)} + f_{(x)} u'_{(x)}\right]\delta u_{(x)}\right\}_0^H \\ + \int_0^H \left[K_b u''''_{(x)} + f_{(x)} u''_{(x)} + f'_{(x)} u'_{(x)}\right]\delta u_{(x)} dx - \frac{1}{2}\int_0^H f(x) u'^2_{(x)}\delta f_{(x)} dx\end{aligned} \tag{976}$$

Setting the terms equal to zero, the following equations result:

$$K_b u''''_{(x)} + f_{(x)} u''_{(x)} + f'_{(x)} u'_{(x)} = 0 \tag{977}$$

And boundary conditions:

$$\left\{\begin{matrix} u''_{(0)} = 0 \\ K_b u'''_{(0)} + f_{(0)} u'_{(0)} = 0 \end{matrix}\right\} \tag{978}$$

Integrating the equation once and evaluating at x=0:

$$K_b u'''_{(x)} + f_{(x)} u'_{(x)} = 0 \tag{979}$$

A third order differential equation is obtained, where the critical load results from the smallest eigenvalue. Normalizing the differential equation by the variable $z = x/H$:

$$u'''_{(z)} + f_{(z)}\left[\frac{H^3}{K_b} u'_{(z)}\right] = 0 \tag{980}$$

The equation can be rewritten as:

$$u'''_{(z)} + \lambda\alpha_{(z)}u'_{(z)} = 0 \tag{981}$$

Where:

$$\lambda = \frac{qH^3}{K_b}, f_{(z)} = q\alpha_{(z)} \tag{982}$$

Expressing the boundary conditions:

$$\begin{Bmatrix} u_{(1)} = 0 \\ u'_{(1)} = 0 \\ u''_{(0)} = 0 \end{Bmatrix} \tag{983}$$

- **Uniformly Distributed Load**

For beam stability, the governing differential equation is of the form:

$$\left[\frac{d^4}{dz^4} - (\alpha\kappa)^2\frac{d^2}{dz^2}\right]\theta_{(z)} - \lambda\left\{-\alpha_{(z)}\left[\frac{d^2}{dz^2} - \alpha^2(\kappa^2 - 1)\right]\right\}\theta_{(z)} = 0 \tag{984}$$

Where:

$$L = \frac{d^3}{dz^3}, M = -\alpha_{(z)}\left(\frac{d}{dz}\right) \tag{985}$$

Multiplying the equation by $u'_{(z)}$ and integrating from 0 to 1:

$$\int_0^1 u'_{(z)}u'''_{(z)}dz + \lambda\int_0^1 \alpha_{(z)}{u'}^2_{(z)}dz = 0 \tag{986}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\int_0^1 {u''}^2_{(z)}dz - \lambda\int_0^1 \alpha_{(z)}{u'}^2_{(z)}dz = 0 \tag{987}$$

Solving the parameter γ:

$$\lambda = \frac{\int_0^1 u''^2_{(z)} dz}{\int_0^1 \alpha_{(z)} u'^2_{(z)} dz} \tag{988}$$

This Rayleigh ratio represents an approximation of the upper limit of the critical load, and it is exact if and only if the exact equilibrium curve $u_{(z)}$ is used to calculate λ.

For the case of a uniformly distributed load, the function $\alpha_{(z)}$:

$$\alpha_{(z)} = z \rightarrow f_{(z)} = qz \tag{989}$$

The Rayleigh quotient becomes:

$$\lambda = \frac{\int_0^1 u''^2_{(z)} dz}{\int_0^1 z u'^2_{(z)} dz} \tag{990}$$

- 1st Iteration:

Taking into account the boundary conditions. We consider two simple polynomials of different degrees that satisfy the boundary condition:

$$\phi_1^1 = 1 - \frac{4}{3}z + \frac{1}{3}z^4 \quad y \quad \phi_2^1 = 1 - \frac{5}{4}z + \frac{1}{4}z^5 \tag{991}$$

Taking a linear combination of both terms:

$$u_{(z)} = A\phi_1^1 + B\phi_2^1 = A\left(1 - \frac{4}{3}z + \frac{1}{3}z^4\right) + B\left(1 - \frac{5}{4}z + \frac{1}{4}z^5\right) \tag{992}$$

We expand the integrals and substitute into the Rayleigh quotient:

$$\mathcal{U} = \int_0^1 u''^2_{(z)}\, dz - \lambda \int_0^1 z u'^2_{(z)} dz \tag{993}$$

Expanding the integrals and grouping common terms:

$$\begin{gathered} \mathcal{U} = (3.2A^2 + 3.5714B^2 + 6.6667AB) - \lambda(0.4A^2 + 0.4167B^2 + 0.8148AB) \\ \mathcal{U} = (3.2 - 0.4\lambda)A^2 + (3.5714 - 0.4167\lambda)B^2 + (3.3333 - 0.4074\lambda)AB \end{gathered} \tag{994}$$

The condition for the critical load to be the minimum is expressed as:

$$\left\{\begin{aligned}\frac{\partial\mathcal{U}}{\partial A}=0\rightarrow(6.4-0.8\lambda)A+(6.6667-0.8148\lambda)B=0\\ \frac{\partial\mathcal{U}}{\partial B}=0\rightarrow(6.6667-0.8148\lambda)A+(7.1429-0.8333\lambda)B=0\end{aligned}\right\} \tag{995}$$

Expressing in matrix form:

$$\begin{bmatrix}6.4-0.8\lambda & 6.6667-0.8148\lambda\\ 6.6667-0.8148\lambda & 7.1429-0.8333\lambda\end{bmatrix}\begin{Bmatrix}A\\ B\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix} \tag{996}$$

For a nontrivial solution (a and b cannot be equal to zero simultaneously), the determinant of the coefficient matrix for a and b must be equal to zero. Operating the determinant:

$$0.00274348\lambda^2+-0.183421552\lambda+1.26984127=0 \tag{997}$$

The minimum eigenvalue is obtained from the minimum root of the quadratic equation.

$$\left\{\begin{matrix}\lambda=7.843180189\\ \lambda=59.013962668\end{matrix}\rightarrow\lambda_1=7.843180189\rightarrow q_{cr}H=\lambda_1\frac{K_b}{H^2}\rightarrow q_{cr}H=7.843180189\frac{K_b}{H^2}\right\} \tag{998}$$

Which is the first approximation to the value of the critical load of the beam. For most practical cases the resulting critical load is accurate enough; In order to obtain a better approximation to the exact critical load, it is necessary to repeat the previous procedure with two new higher degree polynomials.

- 2nd Iteration:

The first polynomial to be considered will be the one with the highest degree of the previous iteration:

$$\phi_1^2=1-\frac{5}{4}z+\frac{1}{4}z^5 \tag{999}$$

To obtain a new polynomial of higher degree and that takes into account the eigenvalue calculated in the previous iteration, we will integrate the differential equation resulting from the beam model four times:

$$u_{(z)}=-\lambda\iiint_0^z\alpha_{(z)}u'_{(z)}dz+C_2z^2+C_1z+C_0 \tag{1000}$$

For the case of a uniform load:

$$u_{(z)} = -\lambda\left(\frac{1}{24}z^4 - \frac{1}{48}z^5 + \frac{1}{2016}z^9\right) + C_2 z^2 + C_1 z + C_0 \tag{1001}$$

When evaluating the boundary conditions, the constants C_0, C_1, C_2 and C_3 are determined and the new polynomial to be used in the second iteration is determined.

$$\phi_2^2 = 1.0212 - 1.4006z + 0.4085z^4 - 0.0292z^8 \tag{1002}$$

Taking a linear combination of both terms:

$$u_{(z)} = A\phi_1^2 + B\phi_2^2 \tag{1003}$$

Solving similarly to iteration 1:

$$\lambda_2 == 7.838442004 \rightarrow q_{cr}H = \lambda_2 \frac{K_b}{H^2} \rightarrow q_{cr}H = 7.838442004 \frac{K_b}{H^2} \tag{1004}$$

- 3rd Iteration:

$$\begin{gathered} \phi_1^3 = 1.0212 - 1.4006z + 0.4085z^4 - 0.0292z^8 \\ \phi_2^3 = 1.0248 - 1.4231z + 0.4574z^4 - 0.0610z^7 + 0.0018z^{11} \\ u_{(z)} = A\phi_1^3 + B\phi_2^3 \\ \lambda_3 = 7.837349280 \rightarrow q_{cr}\mathrm{H} = \lambda_3 \frac{K_b}{H^2} \rightarrow q_{cr}\mathrm{H} = 7.837349280 \frac{K_b}{H^2} \end{gathered} \tag{1005}$$

- 4th Iteration:

$$\begin{gathered} \phi_1^4 = 1.0248 - 1.4231z + 0.4574z^4 - 0.0610z^7 + 0.0018z^{11} \\ \phi_2^4 = 1.0253 - 1.4263z + 0.4647z^4 - 0.0683z^7 + 0.0046z^{10} - 0.0001z^{14} \\ u_{(z)} = A\phi_1^4 + B\phi_2^4 \\ \lambda_4 = 7.837347443 \rightarrow q_{cr}\mathrm{H} = \lambda_4 \frac{K_b}{H^2} \rightarrow q_{cr}\mathrm{H} = 7.837347443 \frac{K_b}{H^2} \end{gathered} \tag{1006}$$

- 5th Iteration:

$$\phi_1^5 = 1.0253 - 1.4263z + 0.4647z^4 - 0.0683z^7 + 0.0046z^{10} - 0.0001z^{14}$$
$$\phi_2^5 = 1.0254 - 1.4268z + 0.4658z^4 - 0.0694z^7 + 0.0052z^{10} - 0.0002z^{13} + 0.000002z^{17}$$
$$u_{(z)} = A\phi_1^5 + B\phi_2^5$$
$$\lambda_5 = 7.837347435 \rightarrow q_{cr}\mathrm{H} = \lambda_5 \frac{K_b}{H^2} \rightarrow q_{cr}\mathrm{H} = 7.837347435 \frac{K_b}{H^2} \tag{1007}$$

Numerically it is observed that with a third iteration the approximation can be considered exact.

$$\lambda = 7.837 \rightarrow q_{cr}H = \lambda \frac{K_b}{H^2} \rightarrow q_{cr}H = 7.837 \frac{K_b}{H^2} \tag{1008}$$

This found value is identical to the exact value given by Timoshenko and Gere using Bessel functions; therefore, it can be considered practical for engineering purposes.

- **Point Load at x=0 (z=0)**

For the case of a point load applied at x=0 (z=0), the function $\alpha_{(z)}$:

$$\alpha_{(z)} = 1 \rightarrow f_{(z)} = q \tag{1009}$$

Substituting into the differential equation:

$$u'''_{(z)} + \frac{\lambda}{H} u'_{(z)} = 0 \tag{1010}$$

The expression for $u_{(z)}$ can be derived as:

$$u_{(z)} = C_0 + C_1 \cos\left(\sqrt{\lambda/H}\, z\right) + C_2 \sin\left(\sqrt{\lambda/H}\, z\right) \tag{1011}$$

The linear algebraic system resulting from the boundary conditions, written in matrix form, is:

$$\begin{bmatrix} 1 & \cos\left(\sqrt{\lambda/H}\right) & \sin\left(\sqrt{\lambda/H}\right) \\ 0 & -\sin\left(\sqrt{\lambda/H}\right) & \cos\left(\sqrt{\lambda/H}\right) \\ 0 & \cos\left(\sqrt{\lambda/H}\right) & 0 \end{bmatrix} \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \end{Bmatrix} = 0 \tag{1012}$$

Which has a different solution than the trivial one $(C_0 = C_1 = C_2 = 0)$ if the determinant is equal to zero (the matrix of coefficients is singular), that is:

$$Cos\left(\sqrt{\lambda/H}\right) = 0 \rightarrow \sqrt{\lambda/H} = (2n-1)\frac{\pi}{2}\ /\ n = 1,2,3\ldots \qquad (1013)$$

Solving it is found that the critical load:

$$q_{cr} = (2n-1)^2\frac{\pi^2 K_b}{4H^2} \qquad (1014)$$

For the case when n=1, we have:

$$q_{cr} = \frac{\pi^2}{4}\frac{K_b}{H^2} \qquad (1015)$$

4.3.1.2 Case 2

- **Calculation of the Transfer Matrix**

According to the fourth order differential equation:

$$K_b u''''_{(x)} + f_{(x)} u''_{(x)} = 0 \qquad (1016)$$

Using the method of coefficients:

$$D^2(D^2 + r^2) = 0 \qquad (1017)$$

Where:

$$\xi = \frac{q}{K_b} \qquad (1018)$$

The expression for $u_{(z)}$ and $u'_{(z)}$ is proposed:

$$\begin{Bmatrix} u_{(z)} = C_0 + C_1 z + C_2\cos\left(\sqrt{\xi}z\right) + C_3\sin\left(\sqrt{\xi}z\right) \\ u'_{(z)} = C_1 - C_2\sqrt{\xi}\sin\left(\sqrt{\xi}z\right) + C_3\sqrt{\xi}\cos\left(\sqrt{\xi}z\right) \end{Bmatrix} \qquad (1019)$$

Internal forces such as bending moment and shear force result:

$$\begin{Bmatrix} M_{(z)} = K_b u''_{(z)} = -[\xi K_b \cos(\sqrt{\xi} z)] C_2 - [\xi K_b \sin(\sqrt{\xi} z)] C_3 \\ V_{(z)} = K_b u'''_{(z)} + q u'_{(z)} = (q) C_1 + [(\xi K_b - q)\sqrt{\xi} \sin(\sqrt{\xi} z)] C_2 - [(\xi K_b - q)\sqrt{\xi} \cos(\sqrt{\xi} z)] C_2 \end{Bmatrix} \tag{1020}$$

Writing the equations in matrix form:

$$\begin{Bmatrix} u_i(z_i) \\ u'_i(z_i) \\ M_i(z_i) \\ V_i(z_i) \end{Bmatrix} = K_i(z_i) \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{Bmatrix} \tag{1021}$$

Where:

$$K_i(z_i) = \begin{bmatrix} 1 & z_i & \cos(\sqrt{\xi} z) & \sin(\sqrt{\xi} z) \\ 0 & 1 & -\sqrt{\xi} \sin(\sqrt{\xi} z) & \sqrt{\xi} \cos(\sqrt{\xi} z) \\ 0 & 0 & -\xi K_b \cos(\sqrt{\xi} z) & -\xi K_b \sin(\sqrt{\xi} z) \\ 0 & q & (\xi K_b - q)\sqrt{\xi} \sin(\sqrt{\xi} z) & -[(\xi K_b - q)\sqrt{\xi} \cos(\sqrt{\xi} z)] \end{bmatrix}_i \tag{1022}$$

Evaluating at the base of the i-th floor; that is, for $z_i = h_i$:

$$\begin{Bmatrix} u_i(h_i) \\ u'_i(h_i) \\ M_i(h_i) \\ V_i(h_i) \end{Bmatrix} = K_i(h_i) \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{Bmatrix} \to \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{Bmatrix} = K_i^{-1}(h_i) \begin{Bmatrix} u_i(h_i) \\ u'_i(h_i) \\ M_i(h_i) \\ V_i(h_i) \end{Bmatrix} \tag{1023}$$

Replacing the vector of coefficients:

$$\begin{Bmatrix} u_i(z_i) \\ u'_i(z_i) \\ M_i(z_i) \\ V_i(z_i) \end{Bmatrix} = K_i(z_i) K_i^{-1}(h_i) \begin{Bmatrix} u_i(h_i) \\ u'_i(h_i) \\ M_i(h_i) \\ V_i(h_i) \end{Bmatrix} = T_i(z_i) \begin{Bmatrix} u_i(h_i) \\ u'_i(h_i) \\ M_i(h_i) \\ V_i(h_i) \end{Bmatrix} \tag{1024}$$

Where:

$$T_i(\mathrm{z}) = K_i(z_i) K_i^{-1}(h_i) \tag{1025}$$

If we evaluate the forces and displacements at the top of the i-th floor, we have:

$$\begin{Bmatrix} u_i(0) \\ u_i'(0) \\ M_i(0) \\ V_i(0) \end{Bmatrix} = T_i(0) \begin{Bmatrix} u_i(h_i) \\ u_i'(h_i) \\ M_i(h_i) \\ V_i(h_i) \end{Bmatrix} \tag{1026}$$

This equation shows the relationship of forces and displacements between the top and bottom of the ith floor.

- **Stability Analysis Under Point Loads Applied at Floor Level**

Applying sequentially from the bottom to the top of the beam:

For the first floor:

$$\begin{Bmatrix} u_1(0) \\ u_1'(0) \\ M_1(0) \\ V_1(0) \end{Bmatrix} = T_1(0) \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1027}$$

For the second floor:

$$\begin{Bmatrix} u_2(0) \\ u_2'(0) \\ M_2(0) \\ V_2(0) \end{Bmatrix} = T_2(0)T_1(0) \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1028}$$

For the third floor:

$$\begin{Bmatrix} u_3(0) \\ u_3'(0) \\ M_3(0) \\ V_3(0) \end{Bmatrix} = T_3(0)T_2(0)T_1(0) \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1029}$$

For the nth floor (top of the beam):

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = T_n(0) \ldots T_2(0)T_1(0) \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1030}$$

Expressing the equation between product symbol:

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_k(0) \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1031}$$

Where:

$$\mathrm{t} = \prod_{k=1}^{n} T_k(0) \tag{1032}$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 4x4 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ u_{(1)}' = 0 \\ K_b u_{(0)}'' = 0 \\ K_b u_{(0)}''' + f_{(0)} u_{(0)}' = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1(h_1) = 0 \\ u_1'(h_1) = 0 \\ M_n(0) = 0 \\ V_n(0) = 0 \end{Bmatrix} \tag{1033}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1034}$$

Solving for the bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1035}$$

Which has a different solution than the trivial if the determinant is equal to zero (the coefficient matrix is singular). Solving the critical loads of the beam.

4.3.2 Shear Beam of a Field (SB)

4.3.2.1 Case 1

The potential energy of the SB model of a field is:

$$V = \frac{1}{2}\int_0^H K_s {u'_{(x)}}^2 \, dx \tag{1036}$$

Where:

$$\left\{K_s = \left(K_b^{-1} + K_c^{-1}\right)^{-1}, K_b = \frac{12EI_v}{lh}, K_c = \frac{\pi^2 EI_c}{h^2}\right\} \tag{1037}$$

The work done by the external force is:

$$W = -\int_0^H f_{(x)} dl = -\frac{1}{2}\int_0^H f_{(x)} {u'_{(x)}}^2 \, dx \tag{1038}$$

Consequently, the total potential energy of the model is expressed as:

$$\mathcal{U} = \frac{1}{2} K_s {u'_{(x)}}^2 dx - \frac{1}{2}\int_0^H f(x) {u'_{(x)}}^2 \, dx \tag{1039}$$

Closed-form solutions of the model are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\delta\mathcal{U} = \int_0^H \left[K_s u'_{(x)} \delta u'_{(x)} - f_{(x)} u'_{(x)} \delta u'_{(x)}\right] dx - \frac{1}{2}\int_0^H f(x) u'^2_{(x)} \delta f_{(x)} dx \tag{1040}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\begin{aligned}\delta\mathcal{U} = &\left[K_s u'_{(x)} \delta u_{(x)}\right]_0^H - \left[f_{(x)} u'_{(x)} \delta u_{(x)}\right]_0^H - \int_0^H \left[K_s u''_{(x)} - f_{(x)} u''_{(x)} - f'_{(x)} u'_{(x)}\right] \delta u_{(x)} dx \\ &- \frac{1}{2}\int_0^H f(x) u'^2_{(x)} \delta f_{(x)} dx\end{aligned} \tag{1041}$$

Setting the terms equal to zero, the following equation results:

$$K_s u''_{(x)} - f_{(x)} u''_{(x)} - f'_{(x)} u'_{(x)} = 0 \tag{1042}$$

And boundary conditions:

$$\begin{Bmatrix} u_{(1)} = 0 \\ u'_{(0)} = 0 \end{Bmatrix} \tag{1043}$$

Integrating the equation once and evaluating at x=0:

$$\left[K_s - f_{(x)}\right] u'_{(x)} = 0 \tag{1044}$$

Equaling zero:

$$K_s - f_{(x)} = 0 \tag{1045}$$

Normalizing the differential equation by the variable $z = x/H$:

$$K_s - H f_{(z)} = 0 \tag{1046}$$

- **Uniformly Distributed Load**

For the case of a uniformly distributed load, the function $\alpha_{(z)}$:

$$\alpha_{(z)} = z \rightarrow f_{(z)} = qz \tag{1047}$$

Solving in z=1, we obtain:

$$K_s - qH = 0 \rightarrow q_{cr\grave{\imath}tico} H = K_s \tag{1048}$$

- **Point Load at x=0 (z=0)**

For the case of a point load applied at x=0 (z=0), the function $\alpha_{(z)}$:

$$\alpha_{(z)} = 1 \rightarrow f_{(z)} = q \tag{1049}$$

Solving it is found that the critical load:

$$K_s - q = 0 \rightarrow q_{critico} = K_s \tag{1050}$$

4.3.2.2 Case 2

According to the second order differential equation:

$$K_s u''_{(x)} - f_{(x)} u''_{(x)} = 0 \tag{1051}$$

Using the method of coefficients:

$$D^2\left[K_s - f_{(x)}\right] = 0 \rightarrow q_{crìtico} = K_s \tag{1052}$$

4.3.3 Timoshenko Beam of Two Field (TB)

4.3.3.1 Case 1

The potential energy of the two-field TB model is:

$$V = \frac{1}{2}\int_0^H \left\{K_b {\theta'_{(x)}}^2 + K_s\left[\theta_{(x)} - u'_{(x)}\right]^2\right\} dx \tag{1053}$$

The work done by the external force is expressed as:

$$W = -f_{(x)}dl = -\frac{1}{2}\int_0^H f_{(x)} {u'_{(x)}}^2\, dx \tag{1054}$$

Consequently, the total potential energy of the model is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H \left\{K_b {\theta'_{(x)}}^2 + K_s\left[\theta_{(x)} - u'_{(x)}\right]^2\right\} dx - \frac{1}{2}\int_0^H f(x) {u'_{(x)}}^2\, dx \tag{1055}$$

Closed-form solutions of the model are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\begin{aligned}\delta\mathcal{U} = &\int_0^H \left[K_b\theta'_{(x)}\delta\theta'_{(x)} + K_s\left(\theta_{(x)} - u'_{(x)}\right)\left(\delta\theta_{(x)} - \delta u'_{(x)}\right) - f_{(x)}u'_{(x)}\delta u'_{(x)}\right]dx \\ &- \frac{1}{2}\int_0^H f(x) u'^2_{(x)}\delta f_{(x)} dx\end{aligned} \tag{1056}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\begin{aligned}\delta\mathcal{U} = &\left[K_b\theta'_{(x)}\delta\theta_{(x)}\right]_0^H - \left\{\left\{K_s\left[\theta_{(x)} - u'_{(x)}\right] + f_{(x)}u'_{(x)}\right\}\delta u_{(x)}\right\}_0^H \\ &- \int_0^H \left\{K_b\theta''_{(x)} - K_s\left[\theta_{(x)} - u'_{(x)}\right]\right\}\delta\theta_{(x)} dx \\ &+ \int_0^H \left\{K_s\left[\theta'_{(x)} - u''_{(x)}\right] + f_{(x)}u''_{(x)} + f'_{(x)}u'_{(x)}\right\}\delta u_{(x)} dx \\ &- \frac{1}{2}\int_0^H f(x) u'^2_{(x)}\delta f_{(x)} dx\end{aligned} \tag{1057}$$

Setting the terms equal to zero, the following equations result:

$$\begin{Bmatrix} K_b\theta''_{(x)} - K_s\left[\theta_{(x)} - u'_{(x)}\right] = 0 \\ K_s\left[\theta'_{(x)} - u''_{(x)}\right] + f_{(x)}u''_{(x)} + f'_{(x)}u'_{(x)} = 0 \end{Bmatrix} \tag{1058}$$

And boundary conditions:

$$\begin{Bmatrix} \theta'_{(0)} = 0 \\ K_s\left[\theta_{(0)} - u'_{(0)}\right] + f_{(0)}u'_{(0)} = 0 \end{Bmatrix} \tag{1059}$$

Integrating the equation once and evaluating at x=0:

$$K_s\left[\theta_{(x)} - u'_{(x)}\right] + f_{(x)}u'_{(x)} = 0 \tag{1060}$$

We have a new system of coupled differential equations:

$$\begin{Bmatrix} K_b\theta''_{(x)} - K_s\left[\theta_{(x)} - u'_{(x)}\right] = 0 \\ K_s\left[\theta_{(x)} - u'_{(x)}\right] + f_{(x)}u'_{(x)} = 0 \end{Bmatrix} \tag{1061}$$

From the equation we isolate the value of $u'_{(x)}$:

$$u'_{(x)} = \frac{K_s\theta_{(x)} - K_b\theta''_{(x)}}{K_s} \tag{1062}$$

Substituting the equations into the equation:

$$K_b\theta''_{(x)} - f_{(x)}\left[\frac{K_b}{K_s}\theta''_{(x)} - \theta_{(x)}\right] = 0 \tag{1063}$$

Reordering:

$$\theta''_{(x)} - f_{(x)}\left[\frac{1}{K_s}\theta''_{(x)} - \frac{1}{K_b}\theta_{(x)}\right] = 0 \tag{1064}$$

A fourth order differential equation is obtained, where the critical load results from the smallest eigenvalue. Normalizing the differential equation by the variable $z = x/H$:

$$\theta''_{(z)} - f_{(z)}\left[\frac{1}{K_s}\theta''_{(z)} - \frac{H^2}{K_b}\theta_{(z)}\right] = 0 \tag{1065}$$

We define two additional parameters:

$$\left\{\alpha = H\sqrt{\frac{K_s}{K_b}}, \lambda = \frac{qH^3}{K_b}\right\} \tag{1066}$$

The equation can be rewritten as:

$$\theta''_{(z)} - \lambda\alpha_{(z)}\left[\frac{1}{\alpha^2}\theta''_{(z)} - \theta_{(z)}\right] = 0 \tag{1067}$$

Where:

$$f_{(z)} = q\alpha_{(z)} \tag{1068}$$

Expressing the boundary conditions as a function of $\theta_{(z)}$:

$$\left\{\begin{matrix} \theta_{(1)} = 0 \\ \theta'_{(0)} = 0 \\ \theta''_{(0)} = 0 \end{matrix}\right\} \tag{1069}$$

- **Uniformly Distributed Load**

The stability of the Tymoshenko beam (TB), the governing differential equation is of the form:

$$\left[\frac{d^2}{dz^2}\right]\theta_{(z)} - \lambda\left\{\alpha_{(z)}\left[\frac{1}{\alpha^2}\frac{d^2}{dz^2} - 1\right]\right\}\theta_{(z)} = 0 \tag{1070}$$

Multiplying the equation by $\left[\frac{1}{\alpha^2}\theta''_{(z)} - \theta_{(z)}\right]$ and integrating from 0 to 1:

$$\int_0^1 \left[\frac{1}{\alpha^2}\theta''^2_{(z)} - \theta_{(z)}\theta''_{(z)}\right]dz - \lambda\int_0^1 \alpha_{(z)}\left[\frac{1}{\alpha^2}\theta''_{(z)} - \theta_{(z)}\right]^2 dz = 0 \tag{1071}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\int_0^1 \left[\frac{1}{\alpha^2}\theta''^2_{(z)} + \theta'^2_{(z)}\right]dz - \lambda\int_0^1 \alpha_{(z)}\left[\frac{1}{\alpha^2}\theta''_{(z)} - \theta_{(z)}\right]^2 dz = 0 \tag{1072}$$

Solving the parameter γ:

$$\lambda = \frac{\int_0^1 \left[\frac{1}{\alpha^2}\theta''^2_{(z)} + \theta'^2_{(z)}\right] dz}{\int_0^1 \alpha_{(z)} \left[\frac{1}{\alpha^2}\theta''_{(z)} - \theta_{(z)}\right]^2 dz} \qquad (1073)$$

This Rayleigh ratio represents an approximation of the upper limit of the critical load, and it is exact if and only if the exact equilibrium curve $\theta_{(z)}$ is used to calculate λ.

For the case of a uniformly distributed load, the function $\alpha_{(z)}$ results in:

$$\alpha_{(z)} = z \rightarrow f_{(z)} = qz \qquad (1074)$$

The Rayleigh quotient becomes:

$$\lambda = \frac{\int_0^1 \left[\frac{1}{\alpha^2}\theta''^2_{(z)} + \theta'^2_{(z)}\right] dz}{\int_0^1 z \left[\frac{1}{\alpha^2}\theta''_{(z)} - \theta_{(z)}\right]^2 dz} \qquad (1075)$$

Taking into account the boundary conditions. We consider two simple polynomials of different degrees that satisfy the boundary condition:

$$\phi_1^1 = 1 - z^4, \phi_2^1 = 1 - z^5 \qquad (1076)$$

Taking a linear combination of both terms:

$$\theta_{(z)} = A\phi_1^1 + B\phi_2^1 = A(1 - z^4) + B(1 - z^5) \qquad (1077)$$

We expand the integrals and substitute into the Rayleigh quotient:

$$\mathcal{U} = \int_0^1 \left[\frac{1}{\alpha^2}\theta''^2_{(z)} + \theta'^2_{(z)}\right] dz - \lambda \int_0^1 z \left[\frac{1}{\alpha^2}\theta''_{(z)} - \theta_{(z)}\right]^2 dz$$

Expanding the integrals and joining common terms:

$$\mathcal{U} = A^2(a_1 - \lambda a_2) + B^2(b_1 - \lambda b_2) + AB[(ab)_1 - \lambda(ab)_2] \qquad (1078)$$

Where:

$$\left\{\begin{array}{c} a_1 = 2.2857 + 28.8\frac{1}{\alpha^2}, a_2 = 0.2667 + 3\frac{1}{\alpha^2} + 24\frac{1}{\alpha^4} \\ b_1 = 2.7778 + 57.1429\frac{1}{\alpha^2}, b_2 = 0.2976 + 4\frac{1}{\alpha^2} + 50\frac{1}{\alpha^4} \\ (ab)_1 = 5 + 80\frac{1}{\alpha^2}, (ab)_2 = 0.5628 + 6.8889\frac{1}{\alpha^2} + 68.5714\frac{1}{\alpha^4} \end{array}\right\} \quad (1079)$$

The condition for the critical load to be the minimum is expressed as:

$$\left\{\begin{array}{l} \frac{\partial \mathcal{U}}{\partial A} = 0 \to 2(a_1 - \lambda a_2)A + [(ab)_1 - \lambda(ab)_2]B = 0 \\ \frac{\partial \mathcal{U}}{\partial B} = 0 \to [(ab)_1 - \lambda(ab)_2]A + 2(b_1 - \lambda b_2)B = 0 \end{array}\right\} \quad (1080)$$

Expressing in matrix form:

$$\begin{bmatrix} 2(a_1 - \lambda a_2) & [(ab)_1 - \lambda(ab)_2] \\ [(ab)_1 - \lambda(ab)_2] & 2(b_1 - \lambda b_2) \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (1081)$$

For a nontrivial solution (a and b cannot be equal to zero simultaneously), the determinant of the coefficient matrix for a and b must be equal to zero. Operating the determinant:

$$[4a_2b_2 - (ab)_2^2]\lambda^2 + [2(ab)_1(ab)_2 - 4(a_1b_2 + a_2b_1)]\lambda + [4a_1b_1 - (ab)_1^2] = 0 \quad (1082)$$

The minimum eigenvalue is obtained from the minimum root of the quadratic equation.

$$\left\{\lambda = \frac{qH^3}{K_{b2}} \to q_{cr}H = \lambda\frac{K_{b2}}{H^2}\right\} \quad (1083)$$

Which is the first approximation to the value of the critical load of the beam TB. For most practical cases the resulting critical load is accurate enough; In order to obtain a better approximation to the exact critical load, it is necessary to repeat the previous procedure with two new higher degree polynomials.

The first polynomial to be considered will be the one with the highest degree of the previous iteration:

$$\phi_1^2 = 1 - z^5 \quad (1084)$$

To obtain a new polynomial of higher degree and that takes into account the eigenvalue calculated in the previous iteration, we will integrate the resulting differential equation of the TB beam model four times:

$$\theta_{(z)} = \frac{1}{\alpha^2}\lambda \iint_0^z \alpha_{(z)}\theta''_{(z)}dz - \lambda \iint_0^z \alpha_{(z)}\theta_{(z)}dz + C_1 z + C_0 \tag{1085}$$

For the case of a uniform load:

$$\theta_{(z)} = \frac{1}{\alpha^2}\lambda \iint_0^z z\theta''_{(z)}dz - \lambda \iint_0^z z\theta_{(z)}dz + C_1 z + C_0 \tag{1086}$$

When evaluating the boundary conditions, the constants C_0, C_1, C_2 and C_3 are determined and the new polynomial to be used in the second iteration is determined.

$$\phi_2^2 = \frac{1}{\alpha^2}\lambda \iint_0^z z\theta''_{(z)}dz - \lambda \iint_0^z z\theta_{(z)}dz + C_1 z + C_0 \tag{1087}$$

Taking a linear combination of both terms:

$$\theta_{(z)} = A\phi_1^2 + B\phi_2^2 = A\phi_2^1 + B\phi_2^2 \tag{1088}$$

A closer approximation to the exact value can be achieved by repeating the two iteration steps, resulting in polynomials of higher and higher degree. Numerically it can be seen that with a fourth iteration the approximation can be considered exact.

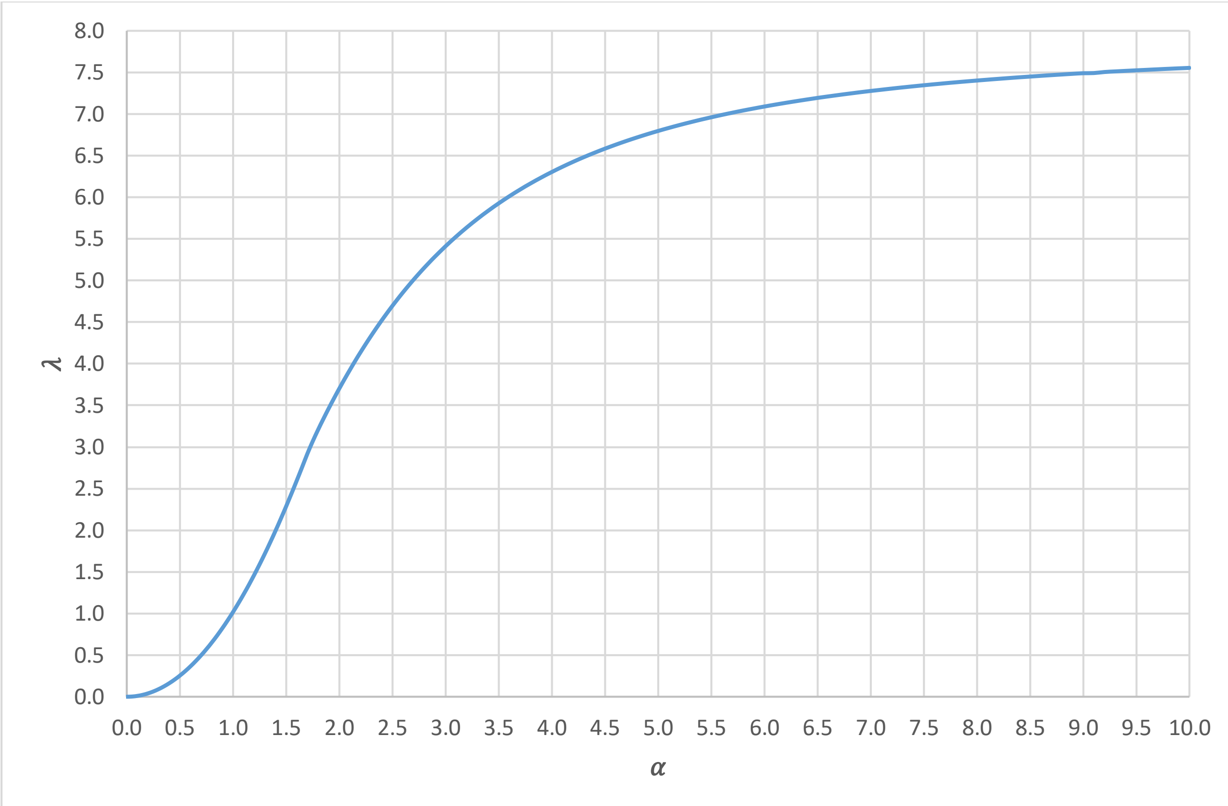


Figure 93. Eigenvalue as a function of the parameter $\alpha \leq 10$ for the case of a uniformly distributed axial load.

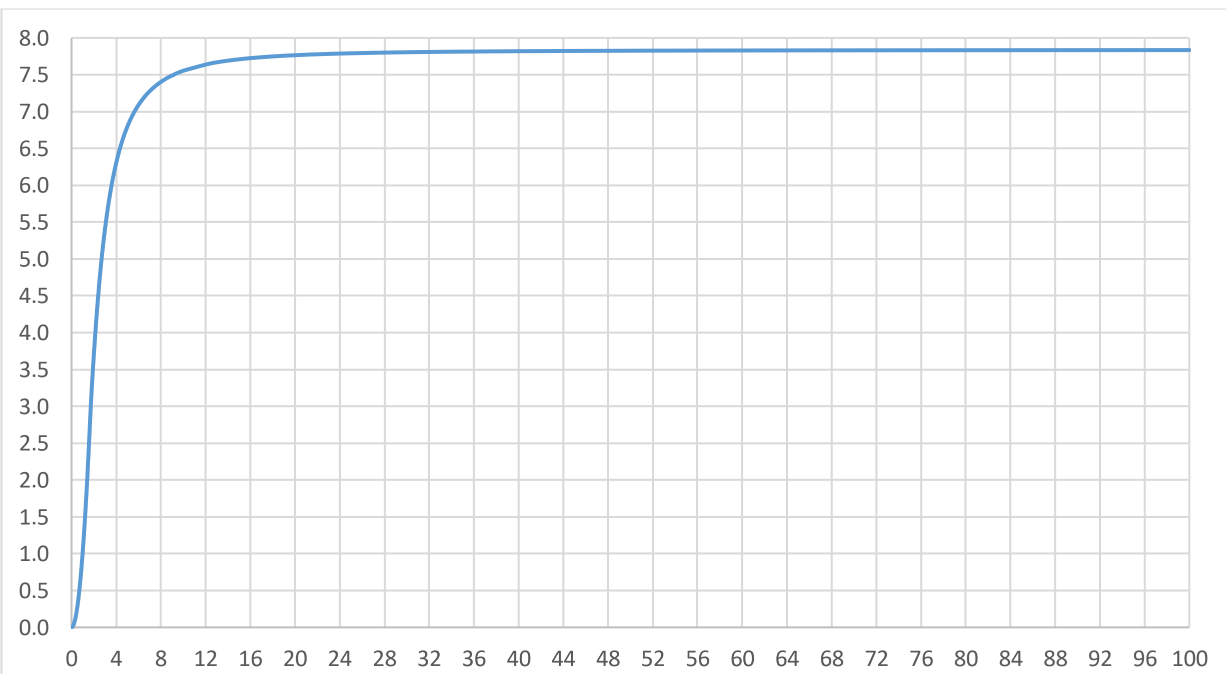


Figure 94. Eigenvalue as a function of the parameter $\alpha \leq 100$ for the case of a uniformly distributed axial load.

Tabla.4 *Eigenvalue as a function of the parameter α≤100 for the case of a uniformly distributed axial load*

α	δ	α	δ	α	δ	α	δ
0.00	0.00000	1.95	3.5845	4.90	6.7587	8.80	7.4754
0.05	0.00254	2.00	3.7024	5.00	6.7968	8.90	7.4832
0.10	0.01017	2.05	3.8166	5.10	6.8331	9.00	7.4907
0.15	0.02289	2.10	3.9274	5.20	6.8675	9.10	7.4925
0.20	0.04069	2.15	4.0352	5.30	6.9003	9.20	7.5051
0.25	0.06357	2.20	4.1390	5.40	6.9315	9.30	7.5120
0.30	0.09155	2.25	4.2395	5.50	6.9612	9.40	7.5187
0.35	0.12461	2.30	4.3369	5.60	6.9895	9.50	7.5251
0.40	0.16276	2.35	4.4312	5.70	7.0165	9.60	7.5314
0.45	0.20600	2.40	4.5225	5.80	7.0423	9.70	7.5375
0.50	0.25433	2.45	4.6108	5.90	7.0669	9.80	7.5434
0.55	0.30775	2.50	4.6963	6.00	7.0904	9.90	7.5491
0.60	0.36626	2.55	4.7790	6.10	7.1129	10.00	7.5547
0.65	0.42986	2.60	4.8589	6.20	7.1344	12.50	7.6546
0.70	0.49857	2.65	4.9363	6.30	7.1549	15.00	7.7097
0.75	0.57236	2.70	5.0112	6.40	7.1747	17.50	7.7432
0.80	0.65126	2.75	5.0836	6.50	7.1937	20.00	7.7651
0.85	0.73525	2.80	5.1537	6.60	7.2118	22.50	7.7802
0.90	0.82435	2.85	5.2214	6.70	7.2291	25.00	7.7910
0.95	0.91855	2.90	5.2870	6.80	7.2458	27.50	7.7990
1.00	1.01784	3.00	5.4118	6.90	7.2619	30.00	7.8051
1.05	1.12224	3.10	5.5286	7.00	7.2773	32.50	7.8099
1.10	1.23173	3.20	5.6381	7.10	7.2917	35.00	7.8136
1.15	1.34630	3.30	5.7408	7.20	7.3063	37.50	7.8167
1.20	1.46595	3.40	5.8370	7.30	7.3199	40.00	7.8192
1.25	1.59065	3.50	5.9273	7.40	7.3331	42.50	7.8213
1.30	1.72038	3.60	6.0122	7.50	7.3458	45.00	7.8230
1.35	1.85507	3.70	6.0919	7.60	7.3580	47.50	7.8245
1.40	1.99467	3.80	6.1669	7.70	7.3698	50.00	7.8257
1.45	2.1390	3.90	6.2374	7.80	7.3812	52.50	7.8268
1.50	2.28800	4.00	6.3039	7.90	7.3921	55.00	7.8277
1.55	2.44121	4.10	6.3666	8.00	7.4027	57.50	7.8286
1.60	2.59809	4.20	6.4257	8.10	7.4129	60.00	7.8293
1.65	2.75783	4.30	6.4816	8.20	7.4228	65.00	7.8305
1.70	2.92053	4.40	6.5343	8.30	7.4323	70.00	7.8314
1.75	3.06883	4.50	6.5842	8.40	7.4415	80.00	7.8328
1.80	3.20672	4.60	6.6314	8.50	7.4504	90.00	7.8338
1.85	3.33738	4.70	6.6761	8.60	7.4590	100.00	7.8344
1.90	3.46299	4.80	6.7185	8.70	7.4673	∞	7.8373

- **Point Load at x=0 (z=0)**

For the case of a point load applied at x=0 (z=0), the function $\alpha_{(z)}$:

$$\alpha_{(z)} = 1 \rightarrow f_{(z)} = q \tag{1089}$$

Substituting into the differential equation:

$$\theta''_{(z)} + \left(\frac{\frac{\lambda}{H}}{1 - \frac{1}{\alpha^2}\frac{\lambda}{H}}\right)\theta_{(z)} = 0 \tag{1090}$$

The expression for $\theta_{(z)}$ can be derived as:

$$\theta_{(z)} = C_1 \cos\left(\sqrt{\beta}z\right) + C_2 \sin\left(\sqrt{\beta}z\right) \tag{1091}$$

Where:

$$\beta = \frac{\frac{\lambda}{H}}{1 - \frac{1}{\alpha^2}\frac{\lambda}{H}} \tag{1092}$$

The linear algebraic system resulting from the boundary conditions, written in matrix form, is:

$$\begin{bmatrix} 0 & \sqrt{\beta} \\ \cos\sqrt{\beta} & \sin\sqrt{\beta} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1093}$$

Which has a solution different from the trivial one if the determinant is equal to zero (the matrix of coefficients is singular), that is, for:

$$Cos\sqrt{\beta} = 0 \rightarrow \sqrt{\beta} = (2n-1)\frac{\pi}{2} \; / \; n = 1, 2, 3 \ldots \tag{1094}$$

i.e.,

$$\frac{\frac{\lambda}{H}}{1 - \frac{1}{\alpha^2}\frac{\lambda}{H}} = (2n-1)^2\frac{\pi^2}{4} \tag{1095}$$

After some simple manipulations:

$$\frac{\lambda}{H} = \frac{1}{\frac{4}{(2n-1)^2\pi^2} + \frac{1}{\alpha^2}} \tag{1096}$$

Replacing by its characteristic rigidities:

$$q_{cr} = \frac{1}{\frac{4H^2}{(2n-1)^2\pi^2 K_b} + \frac{1}{K_s}} \tag{1097}$$

Sorting out:

$$q_{cr} = \left\{\left[(2n-1)^2 \frac{\pi^2 K_b}{4H^2}\right]^{-1} + K_s{}^{-1}\right\}^{-1} \tag{1098}$$

For the case when n=1, we have:

$$q_{cr} = \left[\left(\frac{\pi^2 K_b}{4H^2}\right)^{-1} + K_{s1}{}^{-1}\right]^{-1} = \left[q_{cr,flexión\ global}{}^{-1} + q_{cr,corte}{}^{-1}\right]^{-1} \tag{1099}$$

Since the resulting critical load is independent of some approximation function, it can be considered exact and identical to the one that would be obtained by applying Föppl's theorem.

4.3.3.2 Case 2

- **Calculation of the Transfer Matrix**

According to the coupled differential equations:

$$\begin{Bmatrix} K_b\theta''_{(x)} + K_s\left[u'_{(x)} - \theta_{(x)}\right] = 0 \\ K_s\left[u''_{(x)} - \theta'_{(x)}\right] - f_{(x)}u''_{(x)} = 0 \end{Bmatrix} \tag{1100}$$

Using the method of coefficients:

$$\begin{bmatrix} K_s D & K_b D^2 - K_s \\ (K_s - q)D^2 & -K_s D \end{bmatrix} \begin{Bmatrix} u_{(x)} \\ \theta_{(x)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1101}$$

The expression for $u_{(z)}$ and $\theta_{(z)}$ is proposed:

$$\begin{Bmatrix} u_{(z)} = C_0 + C_1 z + C_2\cos\left(\sqrt{\xi}z\right) + C_3\sin\left(\sqrt{\xi}z\right) \\ \theta_{(z)} = C_4 + C_5\cos\left(\sqrt{\xi}z\right) + C_6\sin\left(\sqrt{\xi}z\right) \end{Bmatrix} \tag{1102}$$

Where:

$$\left\{\xi = \frac{qK_s}{(K_s - q)K_b}, \alpha^* = \sqrt{\frac{K_s}{K_b}}\right\} \tag{1103}$$

Expressing the coefficients of the function $\theta_{(z)}$ as a function of the coefficients of $u_{(z)}$:

$$\theta_{(z)} = C_1 - \left[\frac{K_s\sqrt{\xi}\sin\left(\sqrt{\xi}z\right)}{\frac{\xi}{\alpha^{*2}} + 1}\right] C_2 + \left[\frac{\sqrt{\xi}\cos\left(\sqrt{\xi}z\right)}{\frac{\xi}{\alpha^{*2}} + 1}\right] C_3 \tag{1104}$$

Internal forces such as bending moment and shear force associated with lateral displacement result in:

$$\begin{Bmatrix} M_{(z)} = K_b\theta'_{(z)} = -\left[\frac{K_b\xi}{\frac{\xi}{\alpha^{*2}} + 1}\cos\left(\sqrt{\xi}z\right)\right] C_2 - \left[\frac{K_b\xi}{\frac{\xi}{\alpha^{*2}} + 1}\sin\left(\sqrt{\xi}z\right)\right] C_3 \\ V_{(z)} = (q - K_s)u'_{(x)} + K_s\theta_{(x)} = (q)C_1 + \left[\left(\frac{K_b\xi}{\frac{\xi}{\alpha^{*2}} + 1} - q\right) r\sin\left(\sqrt{\xi}z\right)\right] C_2 - \left[\left(\frac{K_b\xi}{\frac{\xi}{\alpha^{*2}} + 1} - q\right)\sqrt{\xi}\cos\left(\sqrt{\xi}z\right)\right] C_3 \end{Bmatrix} \tag{1105}$$

Writing the equations in matrix form:

$$\begin{Bmatrix} u_i(z_i) \\ \theta_i(z_i) \\ M_i(z_i) \\ V_i(z_i) \end{Bmatrix} = K_i(z_i) \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{Bmatrix} \tag{1106}$$

Where:

$$K_i(z_i) = \begin{bmatrix} 1 & z_i & \cos\left(\sqrt{\xi}z\right) & \sin\left(\sqrt{\xi}z\right) \\ 0 & 1 & -\frac{K_s\sqrt{\xi}\sin\left(\sqrt{\xi}z\right)}{\frac{\xi}{\alpha^{*2}}+1} & -\frac{K_b\xi}{\frac{\xi}{\alpha^{*2}}+1}\sin\left(\sqrt{\xi}z\right) \\ 0 & 0 & -\frac{K_b\xi}{\frac{\xi}{\alpha^{*2}}+1}\cos\left(\sqrt{\xi}z\right) & -\frac{K_b\xi}{\frac{\xi}{\alpha^{*2}}+1}\sin\left(\sqrt{\xi}z\right) \\ 0 & q & \left(\frac{K_b\xi}{\frac{\xi}{\alpha^{*2}}+1}-q\right)\sqrt{\xi}\sin\left(\sqrt{\xi}z\right) & -\left(\frac{K_b\xi}{\frac{\xi}{\alpha^{*2}}+1}-q\right)\sqrt{\xi}\cos\left(\sqrt{\xi}z\right) \end{bmatrix}_i \tag{1107}$$

- **Stability Analysis Under Point Loads Applied at Floor Level**

Applying sequentially from the base to the top of the beam and expressing the equation between the product symbol:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_k(0) \begin{Bmatrix} u_1(h_1) \\ \theta_1(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ \theta_1(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1108}$$

Where:

$$\mathrm{t} = \prod_{k=1}^{n} T_k(0) \tag{1109}$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 4x4 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ \theta_{(1)} = 0 \\ K_b\theta'_{(0)} = 0 \\ (q-K_s)u'_{(0)} + K_s\theta_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1(h_1) = 0 \\ \theta_1(h_1) = 0 \\ M_n(0) = 0 \\ V_n(0) = 0 \end{Bmatrix} \tag{1110}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1111}$$

Solving for bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1112}$$

Which has a different solution than the trivial if the determinant is equal to zero (the coefficient matrix is singular). Solving the critical loads of the beam.

4.3.4 Parallel Coupling of Bending Beam and Shear Beam of a Field (CTB) - Translational Behavior

4.3.4.1 Case 1

The potential energy of the CTB model of a field is:

$$V = \frac{1}{2}\int_0^H \left[K_{b2}{u''_{(x)}}^2 + K_{s1}{u'_{(x)}}^2\right] dx \tag{1113}$$

Where:

$$\left\{K_b = \sum_{i=1}^{n} rEI_i \ , K_s = \left(K_b^{-1} + K_c^{-1}\right)^{-1}, K_c = \sum_{i=1}^{n} \frac{\pi^2 EI_i}{h^2} \ , K_b = \sum_{i=1}^{n-1} \frac{12EI_b}{lh} \ , r = \frac{K_c}{K_c + K_b}\right\} \tag{1114}$$

The work done by the external force is expressed as:

$$W = -f_{(x)}dl = -\frac{1}{2}\int_0^H f_{(x)}{u'_{(x)}}^2 \, dx \tag{1115}$$

Consequently, the total potential energy of the model is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H \left[K_{b2}{u''_{(x)}}^2 + K_{s1}{u'_{(x)}}^2\right] dx - \frac{1}{2}\int_0^H f(x){u'_{(x)}}^2 \, dx \tag{1116}$$

Closed-form solutions of the model are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\delta\mathcal{U} = \int_0^H \left[K_{b2}u''_{(x)}\delta u''_{(x)} + K_{s1}u'_{(x)}\delta u'_{(x)} - f_{(x)}u'_{(x)}\delta u'_{(x)}\right]dx - \frac{1}{2}\int_0^H f(x)u'^2_{(x)}\delta f_{(x)}dx \tag{1117}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\begin{aligned}\delta\mathcal{U} = {} & \left[K_{b2}u''_{(x)}\delta u'_{(x)}\right]_0^H + \left\{\left[K_{s1}u'_{(x)} - K_{b2}u'''_{(x)} - f_{(x)}u'_{(x)}\right]\delta u_{(x)}\right\}_0^H \\ & + \int_0^H \left[K_{b2}u''''_{(x)} - K_{s1}u''_{(x)} + f'_{(x)}u'_{(x)} + f_{(x)}u''_{(x)}\right]\delta u_{(x)}dx \\ & - \frac{1}{2}\int_0^H f(x)u'^2_{(x)}\delta f_{(x)}dx\end{aligned} \tag{1118}$$

Setting the terms equal to zero, the following equation results:

$$K_{b2}u''''_{(x)} - K_{s1}u''_{(x)} + f'_{(x)}u'_{(x)} + f_{(x)}u''_{(x)} = 0 \tag{1119}$$

And boundary conditions:

$$\left\{\begin{array}{c} u_{(1)} = 0 \\ u'_{(1)} = 0 \\ u''_{(0)} = 0 \\ \left[f_{(0)} - K_{s1}\right]u'_{(0)} + K_{b2}u'''_{(0)} = 0 \end{array}\right\} \tag{1120}$$

Integrating the equation once and evaluating at x=0:

$$u'''_{(x)} - \frac{K_{s1}}{K_{b2}}u'_{(x)} + \frac{f_{(x)}}{K_{b2}}u'_{(x)} = 0 \tag{1121}$$

A third order differential equation is obtained, where the critical load results from the smallest eigenvalue. Normalizing the differential equation by the variable $z = x/H$:

$$u'''_{(z)} - \frac{K_{s1}}{K_{b2}}H^2u'_{(z)} + \frac{f_{(z)}}{K_{b2}}H^3u'_{(z)} = 0 \tag{1122}$$

The equation can be rewritten as:

$$u'''_{(z)} - \alpha^2u'_{(z)} + \lambda\alpha_{(z)}u'_{(z)} = 0 \tag{1123}$$

Where:

$$\left\{\alpha = H\sqrt{\frac{K_{s1}}{K_{b2}}}, \lambda = \frac{qH^3}{K_{b2}}\right\} \tag{1124}$$

- **Uniformly Distributed Load**

The stability of the sandwich beam (CTB), the governing differential equation is of the form:

$$\left(\frac{d^3}{dz^3} - \alpha^2\frac{d}{dz}\right)u_{(z)} - \lambda\left[-\alpha_{(z)}\frac{d}{dz}\right]u_{(z)} = 0 \tag{1125}$$

Multiplying the equation by $[u'_{(z)}]$ and integrating from 0 to 1:

$$\int_0^1 [u'_{(z)}u'''_{(z)} - \alpha^2 u'^2_{(z)}]dz + \lambda \int_0^1 \alpha_{(z)}[u'_{(z)}]^2 dz = 0 \tag{1126}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\int_0^1 [-u''^2_{(z)} - \alpha^2 u'^2_{(z)}]dz + \lambda \int_0^1 \alpha_{(z)}[u'_{(z)}]^2 dz = 0 \tag{1127}$$

Clearing the parameter λ:

$$\lambda = \frac{\int_0^1 [u''^2_{(z)} + \alpha^2 u'^2_{(z)}]dz}{\int_0^1 \alpha_{(z)} u'^2_{(z)} dz} \tag{1128}$$

Where λ is the Rayleigh quotient.

For the case of a uniformly distributed load, the function $\alpha_{(z)}$ results in:

$$\alpha_{(z)} = z \rightarrow f_{(z)} = qz \tag{1129}$$

The Rayleigh quotient becomes:

$$\lambda = \frac{\int_0^1 [u''^2_{(z)} + \alpha^2 u'^2_{(z)}]dz}{\int_0^1 z u'^2_{(z)} dz} \tag{1130}$$

Taking into account the boundary conditions. We consider two simple polynomials of different degrees that satisfy the boundary condition:

$$\phi_1^1 = 1 - \frac{4}{3}z + \frac{1}{3}z^4, \phi_2^1 = 1 - \frac{5}{4}z + \frac{1}{4}z^5 \tag{1131}$$

Taking a linear combination of both terms:

$$u_{(z)} = A\phi_1^1 + B\phi_2^1 = A\left(1 - \frac{4}{3}z + \frac{1}{3}z^4\right) + B\left(1 - \frac{5}{4}z + \frac{1}{4}z^5\right) \tag{1132}$$

We expand the integrals and substitute into the Rayleigh quotient:

$$\mathcal{U} = \int_0^1 \left[u''^2_{(z)} + \alpha^2 u'^2_{(z)}\right] dz - \lambda \int_0^1 z\left[u'_{(z)}\right]^2 dz \tag{1133}$$

Expanding the integrals and grouping common terms:

$$\begin{aligned}\mathcal{U} = A^2[(3.2 + 1.1429\alpha^2) - 0.4\lambda] + B^2[(3.5714 + 1.1111\alpha^2) - 0.4167\lambda] \\ + AB[(6.6667 + 2.25\alpha^2) - 0.8148\lambda]\end{aligned} \tag{1134}$$

The condition for the critical load to be the minimum is expressed as:

$$\left\{\begin{aligned} \frac{\partial \mathcal{U}}{\partial A} = 0 \rightarrow [(6.4 + 2.2858\alpha^2) - 0.8\lambda]A + [(6.6667 + 2.25\alpha^2) - 0.8148\lambda]B = 0 \\ \frac{\partial \mathcal{U}}{\partial B} = 0 \rightarrow [(6.6667 + 2.25\alpha^2) - 0.8148\lambda]A + [(7.1428 + 2.2222\alpha^2) - 0.8334\lambda]B = 0 \end{aligned}\right\} \tag{1135}$$

Expressing in matrix form:

$$\begin{bmatrix} (6.4 + 2.2858\alpha^2) - 0.8\lambda & (6.6667 + 2.25\alpha^2) - 0.8148\lambda \\ (6.6667 + 2.25\alpha^2) - 0.8148\lambda & (7.1428 + 2.2222\alpha^2) - 0.8334\lambda \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1136}$$

For a nontrivial solution (a and b cannot be equal to zero simultaneously), the determinant of the coefficient matrix for a and b must be equal to zero. Operating the determinant:

$$\lambda^2 - (66.8571 + 5.7857\alpha^2)\lambda + (6.1473\alpha^4 + 200.0205\alpha^2 + 462.4561) = 0 \tag{1137}$$

The minimum eigenvalue is obtained from the minimum root of the quadratic equation.

$$\left\{\begin{aligned} \lambda_1 = (33.4286 + 2.8929\alpha^2) - \sqrt{2.2213\alpha^4 - 6.6123\alpha^2 + 655.0133} \\ q_{cr}H = \lambda_1 \frac{K_b}{H^2} \rightarrow q_{cr}H = \lambda_1 \frac{K_b}{H^2} \end{aligned}\right\} \tag{1138}$$

Which is the first approximation to the value of the critical load of the beam.

- 2nd Iteration:

The first polynomial to be considered will be the one with the highest degree of the previous iteration:

$$\phi_1^2 = 1 - \frac{5}{4}z + \frac{1}{4}z^5 \tag{1139}$$

To obtain a new polynomial of higher degree and that takes into account the eigenvalue calculated in the previous iteration, we will integrate the differential equation resulting from the beam model three times:

$$u_{(z)} = \iint_0^z \alpha^2 u_{(z)} dz - \lambda \iiint_0^z \alpha_{(z)} u'_{(z)} dzdz + C_2 z^2 + C_1 z + C_0 \tag{1140}$$

For the case of a uniform load:

$$u_{(z)} = \iint_0^z \alpha^2 u_{(z)} dz - \lambda \iiint_0^z z u'_{(z)} dzdz + C_2 z^2 + C_1 z + C_0 \tag{1141}$$

When evaluating the boundary conditions, the constants C_0, C_1, C_2 and C_3 are determined and the new polynomial ϕ_2^2 to be used in the second iteration is determined.

Taking a linear combination of both terms:

$$u_{(z)} = A\phi_1^2 + B\phi_2^2 \tag{1142}$$

Solving similarly to iteration 1, the new eigenvalue λ_2 is obtained. A closer approximation to the exact value can be achieved by repeating the two iteration steps, resulting in polynomials of higher and higher degree. Numerically it is observed that with a third iteration the approximation can be considered exact.

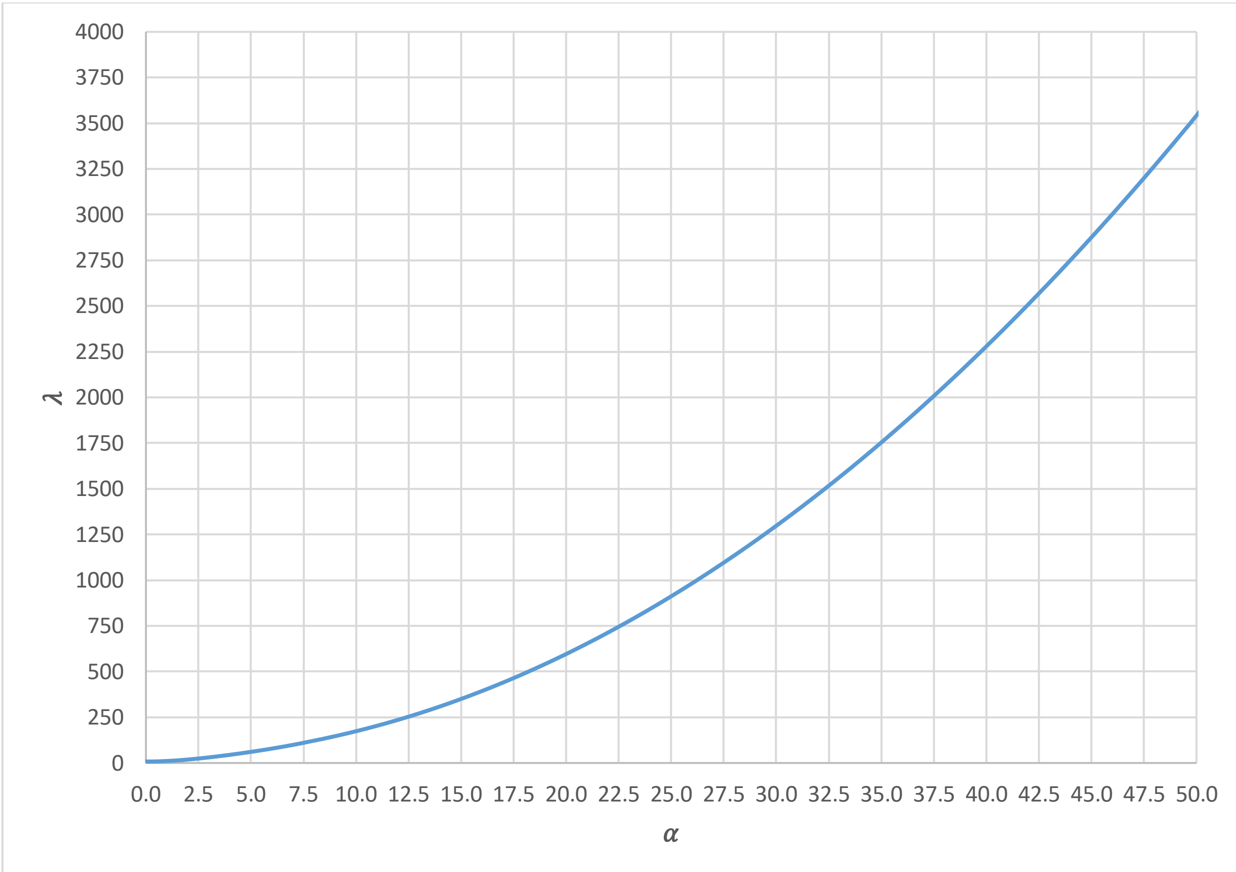


Figure 95. Eigenvalue as a function of the parameter α≤50 for the case of a uniformly distributed axial load.

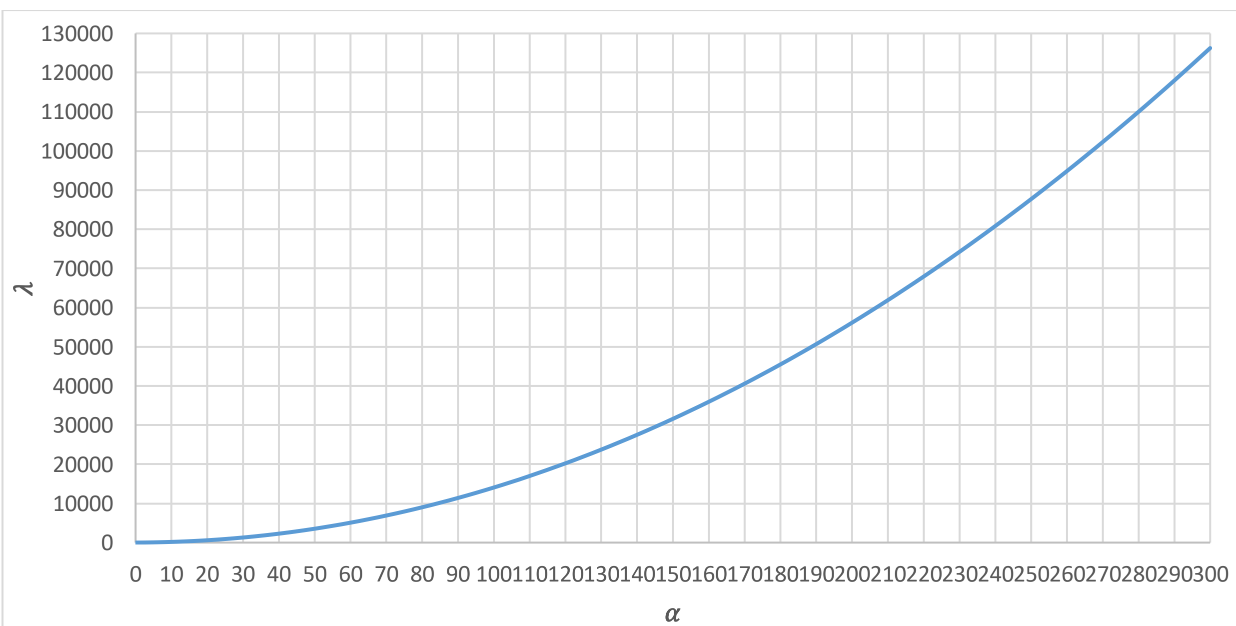


Figure 96. Eigenvalue as a function of the parameter α≤300 for the case of a uniformly distributed axial load.

Tabla.5 *Eigenvalue as a function of the parameter $\alpha \leq 15.5$ for the case of a uniformly distributed axial load*

α	δ	α	δ	α	δ	α	δ
0.00	7.837	3.90	43.195	7.80	117.373	11.70	226.025
0.10	7.867	4.00	44.695	7.90	119.664	11.80	229.348
0.20	7.957	4.10	46.214	8.00	121.980	11.90	232.699
0.30	8.107	4.20	47.753	8.10	124.321	12.00	236.076
0.40	8.316	4.30	49.310	8.20	126.687	12.10	239.481
0.50	8.583	4.40	50.887	8.30	129.079	12.20	242.914
0.60	8.909	4.50	52.483	8.40	131.207	12.30	246.374
0.70	9.291	4.60	54.099	8.50	133.938	12.40	249.861
0.80	9.730	4.70	55.733	8.60	136.406	12.50	253.376
0.90	10.224	4.80	57.387	8.70	138.899	12.60	256.919
1.00	10.772	4.90	59.061	8.80	141.419	12.70	260.489
1.10	11.372	5.00	60.755	8.90	143.964	12.80	264.086
1.20	12.023	5.10	62.468	9.00	146.535	12.90	267.711
1.30	12.724	5.20	64.202	9.10	149.133	13.00	271.364
1.40	13.472	5.30	65.955	9.20	151.757	13.10	275.044
1.50	14.267	5.40	67.729	9.30	154.406	13.20	278.752
1.60	15.106	5.50	69.523	9.40	157.083	13.30	282.487
1.70	15.988	5.60	71.337	9.50	159.785	13.40	286.250
1.80	16.911	5.70	73.173	9.60	162.514	13.50	290.041
1.90	17.873	5.80	75.032	9.70	165.270	13.60	293.859
2.00	18.873	5.90	76.918	9.80	168.052	13.70	297.705
2.10	19.908	6.00	78.815	9.90	170.861	13.80	301.579
2.20	20.978	6.10	80.734	10.00	173.696	13.90	305.480
2.30	22.080	6.20	82.675	10.10	176.558	14.00	309.409
2.40	23.214	6.30	84.638	10.20	179.447	14.10	313.366
2.50	24.377	6.40	86.624	10.30	182.363	14.20	317.351
2.60	25.568	6.50	88.633	10.40	185.306	14.30	321.363
2.70	26.786	6.60	90.667	10.50	188.276	14.40	325.403
2.80	28.030	6.70	92.725	10.60	191.272	14.50	329.471
2.90	29.300	6.80	94.810	10.70	194.296	14.60	333.566
3.00	30.593	6.90	96.922	10.80	197.347	14.70	337.689
3.10	31.909	7.00	99.063	10.90	200.424	14.80	341.840
3.20	33.247	7.10	101.233	11.00	203.529	14.90	346.019
3.30	34.607	7.20	103.435	11.10	206.661	15.00	350.225
3.40	35.988	7.30	105.672	11.20	209.821	15.10	354.460
3.50	37.389	7.40	107.945	11.30	213.007	15.20	358.722
3.60	38.811	7.50	110.257	11.40	216.221	15.30	363.012
3.70	40.253	7.60	112.612	11.50	219.462	15.40	367.329
3.80	41.714	7.70	115.014	11.60	222.730	15.50	371.675

Tabla.6 *Eigenvalue as a function of the parameter α≤300 for the case of a uniformly distributed axial load.*

α	δ	α	δ	α	δ	α	δ
15.6	376.048	19.50	568.355	28.50	1174.521	46.5	3068.003
15.7	380.449	19.60	573.844	28.75	1194.599	47.0	3133.569
15.8	384.878	19.70	579.362	29.00	1214.851	47.5	3199.837
15.9	389.335	19.80	584.907	29.25	1235.279	48.0	3266.807
16.0	393.820	19.90	590.481	29.5	1255.882	48.5	3334.477
16.1	398.332	20.00	596.082	29.75	1276.660	49.0	3402.848
16.2	402.873	20.25	610.207	30.0	1297.613	49.5	3471.921
16.3	407.441	20.50	624.508	30.5	1340.045	50.0	3541.695
16.4	412.037	20.75	638.983	31.0	1383.178	55.0	4277.999
16.5	416.661	21.00	653.633	31.5	1427.012	60.0	5084.421
16.6	421.313	21.25	668.458	32.0	1471.547	65.0	5960.964
16.7	425.992	21.50	683.458	32.5	1516.782	70.0	6907.628
16.8	430.700	21.75	698.633	33.0	1562.719	75.0	7924.413
16.9	435.435	22.00	713.983	33.5	1609.357	80.0	9011.319
17.0	440.199	22.25	729.508	34.0	1656.695	85.0	10168.348
17.1	444.990	22.50	745.208	34.5	1704.735	90.0	11395.499
17.2	449.809	22.75	761.082	35.0	1753.475	95.0	12692.772
17.3	454.656	23.00	777.132	35.5	1802.917	100	14060.167
17.4	459.531	23.25	793.357	36.0	1853.059	105	15497.684
17.5	464.434	23.50	809.757	36.5	1903.903	110	17005.324
17.6	469.365	23.75	826.331	37.0	1955.447	120	20230.972
17.7	474.323	24.00	843.081	37.5	2007.693	130	23737.109
17.8	479.310	24.25	860.006	38.0	2060.640	140	27523.737
17.9	484.324	24.50	877.106	38.5	2114.287	150	31590.856
18.0	489.367	24.75	894.381	39.0	2168.636	160	35938.465
18.1	494.437	25.00	911.831	39.5	2223.686	170	40566.565
18.2	499.535	25.25	929.456	40.0	2279.437	180	45475.155
18.3	504.661	25.50	947.257	40.5	2335.889	190	50664.236
18.4	509.816	25.75	965.232	41.0	2393.043	200	56133.808
18.5	514.998	26.00	983.383	41.5	2450.897	210	61883.871
18.6	520.208	26.25	1001.708	42.0	2509.452	220	67914.424
18.7	525.445	26.50	1020.209	42.5	2568.709	230	74225.468
18.8	530.711	26.75	1038.885	43.0	2628.667	240	80817.003
18.9	536.005	27.00	1057.736	43.5	2689.325	250	87689.029
19.0	541.327	27.25	1076.763	44.0	2750.685	260	94841.546
19.1	546.677	27.50	1095.964	44.5	2812.747	270	102274.553
19.2	552.054	27.75	1115.341	45.0	2875.509	280	109988.051
19.3	557.460	28.00	1134.892	45.5	2938.972	290	117982.040
19.4	562.893	28.25	1154.619	46.0	3003.137	300	126256.520

- **Point Load at x=0 (z=0)**

For the case of a point load applied at x=0 (z=0), the function $\alpha_{(z)}$:

$$\alpha_{(z)} = 1 \rightarrow f_{(z)} = q \tag{1143}$$

Substituting into the differential equation:

$$u'''_{(z)} + (\lambda/H - \alpha^2)u'_{(z)} = 0 \tag{1144}$$

The expression for $u_{(z)}$ can be derived as:

$$u_{(z)} = C_0 + C_1 Cos\left(\sqrt{\lambda/H - \alpha^2}z\right) + C_2 Sen\left(\sqrt{\lambda/H - \alpha^2}z\right) \tag{1145}$$

The linear algebraic system resulting from the boundary conditions, written in matrix form, is:

$$\begin{bmatrix} 1 & \cos\left(\sqrt{\lambda/H - \alpha^2}\right) & \sin\left(\sqrt{\lambda/H - \alpha^2}\right) \\ 0 & -\sin\left(\sqrt{\lambda/H - \alpha^2}\right) & \cos\left(\sqrt{\lambda/H - \alpha^2}\right) \\ 0 & \cos\left(\sqrt{\lambda/H - \alpha^2}\right) & 0 \end{bmatrix} \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \end{Bmatrix} = 0 \tag{1146}$$

Which has a different solution than the trivial one if the determinant is equal to zero (the matrix of coefficients is singular), that is:

$$Cos\left(\sqrt{\lambda/H - \alpha^2}\right) = 0 \rightarrow \sqrt{\lambda/H - \alpha^2} = (2n-1)\frac{\pi}{2} \; / \; n = 1, 2, 3 \; \dots \tag{1147}$$

Solving, it is found that the critical load is:

$$q_{cr} = K_s + (2n-1)^2 \frac{\pi^2}{4} \frac{K_b}{H^2} \tag{1148}$$

For the case when n=1, we have:

$$q_{cr} = K_s + \frac{\pi^2 K_b}{4H^2} = q_{cr,flexiòn} + q_{cr,corte} \tag{1149}$$

Since the resulting critical load is independent of some approximation function, it can be considered exact and identical to the one that would be obtained by applying Föppl's theorem.

4.3.4.2 Case 2

- **Calculation of the Transfer Matrix**

According to fourth degree differential equations:

$$K_{b2}u''''_{(x)} + (q - K_{s1})u''_{(x)} = 0 \tag{1150}$$

Using the method of coefficients:

$$D^2(D^2 + r^2) = 0 \tag{1151}$$

The expression for $u_{(z)}$ and $u'_{(z)}$ is proposed:

$$\begin{Bmatrix} u_{(z)} = C_0 + C_1 z + C_2 \cos\left(\sqrt{\xi} z\right) + C_3 \sin\left(\sqrt{\xi} z\right) \\ u'_{(z)} = C_1 - C_2\sqrt{\xi}\sin\left(\sqrt{\xi} z\right) + C_3\sqrt{\xi}\cos\left(\sqrt{\xi} z\right) \end{Bmatrix} \tag{1152}$$

Where:

$$\left\{ \xi = \frac{q - K_{s1}}{K_{b2}}, \alpha^* = \sqrt{\frac{K_{s1}}{K_{b2}}} \right\} \tag{1153}$$

Internal forces such as bending moment and shear force associated with lateral displacement result in:

$$\begin{Bmatrix} M_{(z)} = K_{b2}u''_{(x)} = -\left[\xi K_{b2}\cos\left(\sqrt{\xi} z\right)\right]C_2 - \left[\xi K_{b2}\sin\left(\sqrt{\xi} z\right)\right]C_3 \\ V_{(z)} = K_{b2}u'''_{(x)} + (q - K_{s1})u'_{(x)} = (q - K_{s1})C_1 \end{Bmatrix} \tag{1154}$$

Writing the equations in matrix form:

$$\begin{Bmatrix} u_i(z_i) \\ u'_i(z_i) \\ M_i(z_i) \\ V_i(z_i) \end{Bmatrix} = K_i(z_i) \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{Bmatrix} \tag{1155}$$

Where:

$$K_i(z_i) = \begin{bmatrix} 1 & z_i & \cos(\sqrt{\xi}z) & \sin(\sqrt{\xi}z) \\ 0 & 1 & -\sqrt{\xi}\sin(\sqrt{\xi}z) & \sqrt{\xi}\cos(\sqrt{\xi}z) \\ 0 & 0 & -\xi K_{b2}\cos(\sqrt{\xi}z) & -\xi K_{b2}\sin(\sqrt{\xi}z) \\ 0 & q-K_{s1} & 0 & 0 \end{bmatrix}_i \tag{1156}$$

- **Stability Analysis Under Point Loads Applied at Floor Level**

Applying sequentially from the base to the top of the beam and expressing the equation between the product symbol:

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_k(0) \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1157}$$

Where:

$$\mathrm{t} = \prod_{k=1}^{n} T_k(0) \tag{1158}$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 4x4 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ u'_{(1)} = 0 \\ K_{b2}u''_{(0)} = 0 \\ K_{b2}u'''_{(0)} + (q-K_{s1})u'_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1(h_1) = 0 \\ u_1'(h_1) = 0 \\ M_n(0) = 0 \\ V_n(0) = 0 \end{Bmatrix} \tag{1159}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1160}$$

Solving for bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1161}$$

Which has a different solution than the trivial if the determinant is equal to zero (the coefficient matrix is singular). Solving the critical loads of the beam.

4.3.5 Parallel Coupling of Bending Beam and Shear Beam of a Field (CTB) - Torsional Behavior

Just as the torsional displacement analysis can be derived based on the analogy that exists between the stresses of thin-walled structures in bending and torsion, the torsional stability analysis of a structural core can also be extended using this analogy. The model to be used is a thin-walled open cross-section equivalent cantilever having an effective Saint Venant stiffness (GJ_e) and deformation stiffness (EI_w).

4.3.5.1 Case 1

When analyzing the balance of an elementary section of the structural core, its differential equation is:

$$EI_w\varphi'''' - GJ^*\varphi'' + m'_{(x)}\varphi'_{(x)} + m_{(x)}\varphi''_{(x)} = 0 \tag{1162}$$

Where:

$$\left\{\begin{matrix} J^* = J + \bar{J} \\ J = \dfrac{1}{3}\displaystyle\sum_{i=1}^{m} h_i v_i^3\,(Sec. abierta\,), J = \dfrac{4A_0^2}{\sum_{i=1}^{m}\dfrac{h_i}{v_i}}(Sec. cerrada), \\ \bar{J} = \dfrac{4A_0^2}{\dfrac{l^3 sG}{12EI_b} + \dfrac{1.2ls}{A_b}}, A_b = t_b d, I_b = \dfrac{t_b d^3}{12} \end{matrix}\right\} \tag{1163}$$

Boundary conditions:

$$\left\{\begin{matrix} \varphi_{(1)} = 0 \\ \varphi'_{(1)} = 0 \\ \varphi''_{(0)} = 0 \\ \left(f_{(0)} - GJ^*\right)\varphi'_{(0)} + EI_w\varphi'''_{(0)} = 0 \end{matrix}\right\} \tag{1164}$$

The differential equation is identical to the equation presented for the case of the stability analysis of a CTB beam, with the difference that only the nomenclature of its stiffnesses changes; furthermore, the same boundary conditions hold, so the solution given in the previous section is completely valid for the pure torsional analysis of a structural core. To solve it, it is necessary to use the equivalent stiffnesses:

$$\begin{Bmatrix} K_b \rightarrow EI_w \\ K_s \rightarrow GJ \\ \rho A \rightarrow \rho I \end{Bmatrix} \tag{1165}$$

4.3.5.2 Case 2

- **Calculation of the Transfer Matrix**

According to fourth degree differential equations:

$$EI_w\varphi'''' - GJ^*\varphi'' + m_{(x)}\varphi''_{(x)} = 0 \tag{1166}$$

Using the method of coefficients:

$$D^2(D^2 + r^2) = 0 \tag{1167}$$

The expression for $\varphi_{(z)}$ and $\varphi'_{(z)}$ is proposed:

$$\begin{Bmatrix} \varphi_{(z)} = C_0 + C_1 z + C_2 \cos\left(\sqrt{\xi}z\right) + C_3 \sin\left(\sqrt{\xi}z\right) \\ \varphi'_{(z)} = C_1 - C_2 r \sin\left(\sqrt{\xi}z\right) + C_3 r \cos\left(\sqrt{\xi}z\right) \end{Bmatrix} \tag{1168}$$

Where:

$$\begin{Bmatrix} \xi = \dfrac{m - GJ^*}{EI_w}, \alpha^* = \sqrt{\dfrac{GJ^*}{EI_w}} \end{Bmatrix} \tag{1169}$$

Internal forces such as bending moment and shear force associated with lateral displacement result in:

$$\begin{Bmatrix} M_{(z)} = EI_w\varphi''_{(x)} = -\left[\xi EI_w \cos\left(\sqrt{\xi}z\right)\right]C_2 - \left[\xi EI_w \sin\left(\sqrt{\xi}z\right)\right]C_3 \\ V_{(z)} = EI_w\varphi'''_{(x)} + (m - GJ^*)\varphi'_{(x)} = (q - GJ^*)C_1 \end{Bmatrix} \tag{1170}$$

Writing the equations in matrix form:

$$\begin{Bmatrix} u_i(z_i) \\ u_i'(z_i) \\ M_i(z_i) \\ V_i(z_i) \end{Bmatrix} = K_i(z_i) \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{Bmatrix} \tag{1171}$$

Where:

$$K_i(z_i) = \begin{bmatrix} 1 & z_i & \cos(\sqrt{\xi} z) & \sin(\sqrt{\xi} z) \\ 0 & 1 & -\sqrt{\xi}\sin(\sqrt{\xi} z) & \sqrt{\xi}\cos(\sqrt{\xi} z) \\ 0 & 0 & -\xi E I_w \cos(\sqrt{\xi} z) & -\xi E I_w \sin(\sqrt{\xi} z) \\ 0 & q - GJ^* & 0 & 0 \end{bmatrix}_i \tag{1172}$$

- **Stability Analysis Under Point Loads Applied at Floor Level**

Applying sequentially from the base to the top of the beam and expressing the equation between the product symbol:

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_k(0) \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1173}$$

Where:

$$\mathrm{t} = \prod_{k=1}^{n} T_k(0) \tag{1174}$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 4x4 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ u'_{(1)} = 0 \\ EI_w u''_{(0)} = 0 \\ EI_w u'''_{(0)} + (q - GJ^*) u'_{(0)} = 0 \end{Bmatrix} \to \begin{Bmatrix} u_1(h_1) = 0 \\ u_1'(h_1) = 0 \\ M_n(0) = 0 \\ V_n(0) = 0 \end{Bmatrix} \tag{1175}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1176}$$

Solving for bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1177}$$

Which has a different solution than the trivial if the determinant is equal to zero (the coefficient matrix is singular). Solving the critical loads of the beam.

4.3.6 Sandwich Beam of Two Field (SWB)

4.3.6.1 Case 1

The potential energy of the two-field SWB model is expressed as follows:

$$V = \frac{1}{2}\int_0^H \left\{K_{b1}{\theta'_{(x)}}^2 + K_{s1}\left[\theta_{(x)} - u'_{(x)}\right]^2\right\} dx + \frac{1}{2}\int_0^H K_{b2}{u''_{(x)}}^2\, dx \tag{1178}$$

- Coupled shear wall:

$$\begin{Bmatrix} K_{b1} = \sum_{i=1}^{w} EA_{w,i}c_i^2\,, K_{b2} = \sum_{i=1}^{w} rEI_{wi}\,, K_{s1} = \left(K_b^{-1} + K_w^{-1}\right)^{-1} \\ K_b = \sum_{i=1}^{b} \dfrac{6EI_v[(l^* + S_1)^2 + (l^* + S_2)^2]}{l^{*3}h\left(1 + 12\dfrac{\rho EI_b}{l^{*2}GA_b}\right)}\,, K_w = \sum_{i=1}^{w} \dfrac{\pi^2 EI_w}{h^2}, r = \dfrac{K_c}{K_c + K_b} \end{Bmatrix} \tag{1179}$$

- Frame:

$$\begin{Bmatrix} K_{b1} = \sum_{i=1}^{c} EA_{c,i}c_i^2\,, K_{b2} = \sum_{i=1}^{c} rEI_{c,i}\,, K_{s1} = \left(K_b^{-1} + K_c^{-1}\right)^{-1} \\ K_b = \sum_{i=1}^{b} \dfrac{12EI_{b,i}}{lh}, K_c = \sum_{i=1}^{c} \dfrac{\pi^2 EI_{c,i}}{h^2}, r = \dfrac{K_c}{K_c + K_b} \end{Bmatrix} \tag{1180}$$

The work done by the external force is expressed as:

$$W = -f_{(x)}dl = -\frac{1}{2}\int_0^H f_{(x)}{u'_{(x)}}^2\, dx \tag{1181}$$

Consequently, the total potential energy of the model is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H \left\{K_{b1}{\theta'_{(x)}}^2 + K_{s1}\left[\theta_{(x)} - u'_{(x)}\right]^2 + K_{b2}{u''_{(x)}}^2\right\} dx - \frac{1}{2}\int_0^H f(x){u'_{(x)}}^2\, dx \tag{1182}$$

Closed-form solutions of the model are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\delta \mathcal{U} = \int_0^H \{K_{b1}\theta'_{(x)}\delta\theta'_{(x)} + K_{s1}[\theta_{(x)} - u'_{(x)}]\delta\theta_{(x)} - K_{s1}[\theta_{(x)} - u'_{(x)}]\delta u'_{(x)} + K_{b2}u''_{(x)}\delta u''_{(x)}$$
$$- f_{(x)}u'_{(x)}\delta u'_{(x)}\}dx - \frac{1}{2}\int_0^H f(x)u'^2_{(x)}\delta f_{(x)}dx \tag{1183}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\delta \mathcal{U} = \left[K_{b1}\theta'_{(x)}\delta\theta_{(x)}\right]_0^H + \left[K_{b2}u''_{(x)}\delta u'_{(x)}\right]_0^H - \left\{\{K_{b2}u'''_{(x)} + K_{s1}[\theta_{(x)} - u'_{(x)}] + f_{(x)}u'_{(x)}\}\delta u_{(x)}\right\}_0^H$$
$$+ \int_0^H \{K_{s1}[\theta_{(x)} - u'_{(x)}] - K_{b1}\theta''_{(x)}\}\delta\theta_{(x)}dx$$
$$+ \int_0^H \{K_{b2}u''''_{(x)} + [f_{(x)} - K_{s1}]u''_{(x)} + K_{s1}\theta'_{(x)} + f'_{(x)}u'_{(x)}\}\delta u_{(x)}dx$$
$$- \frac{1}{2}\int_0^H f(x)u'^2_{(x)}\delta f_{(x)}dx \tag{1184}$$

Setting the terms equal to zero, the following equations result:

$$\begin{Bmatrix} K_{s1}[\theta_{(x)} - u'_{(x)}] - K_{b1}\theta''_{(x)} = 0 \\ K_{b2}u''''_{(x)} + [f_{(x)} - K_{s1}]u''_{(x)} + K_{s1}\theta'_{(x)} + f'_{(x)}u'_{(x)} = 0 \end{Bmatrix} \tag{1185}$$

And boundary conditions:

$$\begin{Bmatrix} \theta'_{(0)} = 0 \\ u''_{(0)} = 0 \\ K_{b2}u'''_{(0)} + K_{s1}[\theta_{(0)} - u'_{(0)}] + f_{(0)}u'_{(0)} = 0 \end{Bmatrix} \tag{1186}$$

Integrating the equation once and evaluating at x=0:

$$K_{b2}u''''_{(x)} - K_{s1}u''_{(x)} + K_{s1}\theta'_{(x)} + f_{(x)}u'_{(x)} = 0 \tag{1187}$$

We have a new system of coupled differential equations:

$$\begin{Bmatrix} K_{s1}[\theta_{(x)} - u'_{(x)}] - K_{b1}\theta''_{(x)} = 0 \\ K_{b2}u'''_{(x)} - K_{s1}u'_{(x)} + K_{s1}\theta_{(x)} + f_{(x)}u'_{(x)} = 0 \end{Bmatrix} \tag{1188}$$

Using the method of coefficients for the solution of the system of equations:

$$\begin{bmatrix} -K_{s1}D & K_{s1} - K_{b1}D^2 \\ K_{b2}D^3 - \left(K_{s1} + f_{(x)}\right)D & K_{s1}D \end{bmatrix} \begin{Bmatrix} u_{(x)} \\ \theta_{(x)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1189}$$

The determinant is equal to zero (the coefficient matrix is singular):

$$\frac{K_{b1}K_{b2}}{K_{s1}} u_{(x)}^{'''''} - (K_{b1} + K_{b2}) u_{(x)}^{'''} + f_{(x)} \left[\frac{K_{b1}}{K_{s1}} u_{(x)}^{'''} - u_{(x)}' \right] = 0 \tag{1190}$$

Reordering:

$$u_{(x)}^{'''''} - K_{s1} \left(\frac{1}{K_{b1}} + \frac{1}{K_{b2}} \right) u_{(x)}^{'''} + f_{(x)} \left[\frac{1}{K_{b2}} u_{(x)}^{'''} - \frac{K_{s1}}{K_{b1}K_{b2}} u_{(x)}' \right] = 0 \tag{1191}$$

A fourth order differential equation is obtained, where the critical load results from the smallest eigenvalue. Normalizing the differential equation by the variable $z = x/H$:

$$u_{(z)}^{'''''} - K_{s1} \left(\frac{1}{K_{b1}} + \frac{1}{K_{b2}} \right) H^2 u_{(z)}^{'''} + f_{(z)} \left[\frac{H^2}{K_{b2}} u_{(z)}^{'''} - \frac{K_{s1}H^4}{K_{b1}K_{b2}} u_{(z)}' \right] = 0 \tag{1192}$$

Where:

$$f_{(z)} = q\alpha_{(z)} \tag{1193}$$

We define:

$$\left\{ \alpha = H\sqrt{\frac{K_{s1}}{K_{b2}}}, \kappa = \sqrt{1 + \frac{K_{b2}}{K_{b1}}}, \lambda = \frac{qH^3}{K_{b2}} \right\} \tag{1194}$$

Rewriting:

$$u_{(z)}^{'''''} - (\alpha\kappa)^2 u_{(z)}^{'''} + \lambda\alpha_{(z)} \left[u_{(z)}^{'''} - \alpha^2(\kappa^2 - 1) u_{(z)}' \right] = 0 \tag{1195}$$

However, the rotation function is of a lower degree:

$$\theta_{(z)}^{''''} - (\alpha\kappa)^2 \theta_{(z)}'' + \lambda\alpha_{(z)} \left[\theta_{(z)}'' - \alpha^2(\kappa^2 - 1) \theta_{(z)} \right] = 0 \tag{1196}$$

Expressing the boundary conditions as a function of $\theta_{(z)}$:

$$\begin{Bmatrix} \theta_{(1)} = 0 \\ \theta'_{(0)} = 0 \\ \theta''_{(1)} = 0 \\ \theta'''_{(0)} = 0 \end{Bmatrix} \tag{1197}$$

- **Uniformly Distributed Load**

The stability of the sandwich beam (SWB), the governing differential equation is of the form:

$$\left[\frac{d^4}{dz^4} - (\alpha\kappa)^2 \frac{d^2}{dz^2}\right]\theta_{(z)} - \lambda\left\{-\alpha_{(z)}\left[\frac{d^2}{dz^2} - \alpha^2(\kappa^2 - 1)\right]\right\}\theta_{(z)} = 0 \tag{1198}$$

Multiplying the equation by $\left[\theta''_{(z)} - \alpha^2(\kappa^2 - 1)\theta_{(z)}\right]$ and integrating from 0 to 1:

$$\int_0^1 \left[\theta''_{(z)}\theta''''_{(z)} - (\alpha\kappa)^2\theta''^2_{(z)} - \alpha^2(\kappa^2 - 1)\theta_{(z)}\theta''''_{(z)} + \alpha^2(\kappa^2 - 1)(\alpha\kappa)^2\theta_{(z)}\theta''_{(z)}\right]dz + \lambda\int_0^1 \alpha_{(z)}\left[\theta''_{(z)} - \alpha^2(\kappa^2 - 1)\theta_{(z)}\right]^2 dz = 0 \tag{1199}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$-\int_0^1 \left[\theta'''^2_{(z)} + \alpha^2(2\kappa^2 - 1)\theta''^2_{(z)} + \alpha^2(\kappa^2 - 1)(\alpha\kappa)^2\theta'^2_{(z)}\right]dz + \lambda\int_0^1 \alpha_{(z)}\left[\theta''_{(z)} - \alpha^2(\kappa^2 - 1)\theta_{(z)}\right]^2 dz = 0 \tag{1200}$$

Solving the parameter γ:

$$\lambda = \frac{\int_0^1 \left[\theta'''^2_{(z)} + \alpha^2(2\kappa^2 - 1)\theta''^2_{(z)} + \alpha^2(\kappa^2 - 1)(\alpha\kappa)^2\theta'^2_{(z)}\right]dz}{\int_0^1 \alpha_{(z)}\left[\theta''_{(z)} - \alpha^2(\kappa^2 - 1)\theta_{(z)}\right]^2 dz} \tag{1201}$$

This Rayleigh ratio represents an approximation of the upper limit of the critical load, and it is exact if and only if the exact equilibrium curve $\theta_{(z)}$ is used to calculate λ.

For the case of a uniformly distributed load, the function $\alpha_{(z)}$ results in:

$$\alpha_{(z)} = z \rightarrow f_{(z)} = qz \tag{1202}$$

The Rayleigh quotient becomes:

$$\lambda = \frac{\int_0^1 \left[\theta'''^2_{(z)} + \alpha^2(2\kappa^2 - 1)\theta''^2_{(z)} + \alpha^2(\kappa^2 - 1)(\alpha\kappa)^2\theta'^2_{(z)}\right] dz}{\int_0^1 z\left[\theta''_{(z)} - \alpha^2(\kappa^2 - 1)\theta_{(z)}\right]^2 dz} \tag{1203}$$

Taking into account the boundary conditions. We consider two simple polynomials of different degrees that satisfy the boundary condition:

$$\phi_1^1 = 1 - \frac{6}{5}z^2 + \frac{1}{5}z^4, \phi_2^1 = 1 - \frac{10}{9}z^2 + \frac{1}{9}z^5 \tag{1204}$$

Taking a linear combination of both terms:

$$\theta_{(z)} = A\phi_1^1 + B\phi_2^1 = A\left(1 - \frac{6}{5}z^2 + \frac{1}{5}z^4\right) + B\left(1 - \frac{10}{9}z^2 + \frac{1}{9}z^5\right) \tag{1205}$$

We expand the integrals and substitute into the Rayleigh quotient:

$$\begin{aligned} U = \int_0^1 \left[\theta'''^2_{(z)} + \alpha^2(2\kappa^2 - 1)\theta''^2_{(z)} + \alpha^2(\kappa^2 - 1)(\alpha\kappa)^2\theta'^2_{(z)}\right] dz \\ - \lambda \int_0^1 z\left[\theta''_{(z)} - \alpha^2(\kappa^2 - 1)\theta_{(z)}\right]^2 dz \end{aligned} \tag{1206}$$

Expanding the integrals and joining common terms:

$$U = A^2(a_1 - \lambda a_2) + B^2(b_1 - \lambda b_2) + AB[(ab)_1 - \lambda(ab)_2] \tag{1207}$$

Where:

$$\left\{\begin{array}{c} a_1 = 7.68 + 3.072[\alpha^2(2\kappa^2 - 1)] + 1.2434[\alpha^2(\kappa^2 - 1)(\alpha\kappa)^2] \\ a_2 = 0.96 + 0.1507[\alpha^2(\kappa^2 - 1)]^2 - 1.3166[\alpha^2(\kappa^2 - 1)] \\ b_1 = 8.8889 + 3.1746[\alpha^2(2\kappa^2 - 1)] + 1.2689[\alpha^2(\kappa^2 - 1)(\alpha\kappa)^2] \\ b_2 = 1.1111 + 0.1555[\alpha^2(\kappa^2 - 1)]^2 - 1.5089[\alpha^2(\kappa^2 - 1)] \\ (ab)_1 = 16 + 6.2222[\alpha^2(2\kappa^2 - 1)] + 2.5111[\alpha^2(\kappa^2 - 1)(\alpha\kappa)^2] \\ (ab)_2 = 2.0571 + 0.3062[\alpha^2(\kappa^2 - 1)]^2 - 3.1030[\alpha^2(\kappa^2 - 1)] \end{array}\right\} \tag{1208}$$

The condition for the critical load to be the minimum is expressed as:

$$\begin{Bmatrix} \dfrac{\partial U}{\partial A} = 0 \rightarrow 2(a_1 - \lambda a_2)A + [(ab)_1 - \lambda(ab)_2]B \\ \dfrac{\partial U}{\partial B} = 0 \rightarrow [(ab)_1 - \lambda(ab)_2]A + 2(b_1 - \lambda b_2)B \end{Bmatrix} \tag{1209}$$

Expressing in matrix form:

$$\begin{bmatrix} 2(a_1 - \lambda a_2) & [(ab)_1 - \lambda(ab)_2] \\ [(ab)_1 - \lambda(ab)_2] & 2(b_1 - \lambda b_2) \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1210}$$

For a nontrivial solution (a and b cannot be equal to zero simultaneously), the determinant of the coefficient matrix for a and b must be equal to zero; namely:

$$\begin{vmatrix} 2(a_1 - \lambda a_2) & [(ab)_1 - \lambda(ab)_2] \\ [(ab)_1 - \lambda(ab)_2] & 2(b_1 - \lambda b_2) \end{vmatrix} = 0 \tag{1211}$$

Operating the determinant, we have:

$$[4a_2 b_2 - (ab)_2^2]\lambda^2 + [2(ab)_1(ab)_2 - 4(a_1 b_2 + a_2 b_1)]\lambda + [4a_1 b_1 - (ab)_1^2] = 0 \tag{1212}$$

The minimum eigenvalue is obtained from the minimum root of the quadratic equation.

$$\lambda = \frac{qH^3}{K_{b2}} \rightarrow q_{cr}H = \lambda \frac{K_{b2}}{H^2} \tag{1213}$$

Which is the first approximation to the value of the critical load of the SWB beam. For most practical cases the resulting critical load is accurate enough; In order to obtain a better approximation to the exact critical load, it is necessary to repeat the previous procedure with two new higher degree polynomials.

The first polynomial to be considered will be the one with the highest degree of the previous iteration:

$$\phi_1^2 = 1 - \frac{10}{9}z^2 + \frac{1}{9}z^5 \tag{1214}$$

To obtain a new polynomial of higher degree and that takes into account the eigenvalue calculated in the previous iteration, we will integrate the resulting differential equation of the SWB beam model four times:

$$\theta_{(z)} = (\alpha\kappa)^2 \iint \theta_{(z)} dz - \lambda \iiiint \alpha_{(z)} \theta''_{(z)} dz + \lambda\alpha^2(\kappa^2 - 1) \iiiint \alpha_{(z)} \theta_{(z)} dz + C_3 z^3 + C_2 z^2 + C_1 z + C_0 \quad (1215)$$

For the case of a uniform load:

$$\theta_{(z)} = (\alpha\kappa)^2 \iint \theta_{(z)} dz - \lambda \iiiint z \theta''_{(z)} dz + \lambda\alpha^2(\kappa^2 - 1) \iiiint z \theta_{(z)} dz + C_3 z^3 + C_2 z^2 + C_1 z + C_0 \quad (1216)$$

When evaluating the boundary conditions, the constants C_0, C_1, C_2 and C_3 are determined and the new polynomial to be used in the second iteration is determined.

Taking a linear combination of both terms:

$$\theta_{(z)} = A\phi_1^2 + B\phi_2^2 = A\phi_2^1 + B\phi_2^2 \quad (1217)$$

A closer approximation to the exact value can be achieved by repeating the two iteration steps, resulting in polynomials of higher and higher degree. Numerically it can be seen that with a fourth iteration the approximation can be considered exact.

- **Point Load at x=0 (z=0)**

For the case of a point load applied at x=0 (z=0), the function $\alpha_{(z)}$:

$$\alpha_{(z)} = 1 \rightarrow f_{(z)} = q \quad (1218)$$

Substituting into the differential equation:

$$\theta''''_{(z)} - (\alpha\kappa)^2 \theta''_{(z)} + \frac{\lambda}{H}\left[\theta''_{(z)} - \alpha^2(\kappa^2 - 1)\theta_{(z)}\right] = 0 \quad (1219)$$

The expression for $\theta_{(z)}$ can be derived as:

$$\theta_{(z)} = C_1 \cosh\left(\sqrt{\xi} z\right) + C_2 \sinh\left(\sqrt{\xi} z\right) + C_3 \cos\left(\sqrt{\beta} z\right) + C_4 \sin\left(\sqrt{\beta} z\right) \quad (1220)$$

Where:

$$\left\{\begin{array}{l} \xi = \dfrac{-\left[\frac{\lambda}{H} - (\alpha\kappa)^2\right] + \sqrt{\left[\frac{\lambda}{H} - (\alpha\kappa)^2\right]^2 + 4\frac{\lambda}{H}\alpha^2(\kappa^2 - 1)}}{2} \\ \beta = \dfrac{\left[\frac{\lambda}{H} - (\alpha\kappa)^2\right] + \sqrt{\left[\frac{\lambda}{H} - (\alpha\kappa)^2\right]^2 + 4\frac{\lambda}{H}\alpha^2(\kappa^2 - 1)}}{2} \end{array}\right\} \quad (1221)$$

The linear algebraic system resulting from the boundary conditions, written in matrix form, is:

$$\begin{bmatrix} \cosh\sqrt{\xi} & \sinh\sqrt{\xi} & \cos\sqrt{\beta} & \sin\sqrt{\beta} \\ 0 & \xi^{1/2} & 0 & \beta^{1/2} \\ \xi\cosh\sqrt{\xi} & \xi\sinh\sqrt{\xi} & -\beta\cos\sqrt{\beta} & -\beta\sin\sqrt{\beta} \\ 0 & \xi^{3/2} & 0 & -\beta^{3/2} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = 0 \qquad (1222)$$

Which has a solution different from the trivial one if the determinant is equal to zero (the matrix of coefficients is singular), that is, for:

$$Cos\sqrt{\beta} = 0 \rightarrow \sqrt{\beta} = (2n-1)\frac{\pi}{2} \ / \ n = 1, 2, 3 \ldots \qquad (1223)$$

i.e.,

$$\frac{\left[\frac{\lambda}{H} - (\alpha\kappa)^2\right] + \sqrt{\left[\frac{\lambda}{H} - (\alpha\kappa)^2\right]^2 + 4\frac{\lambda}{H}\alpha^2(\kappa^2 - 1)}}{2} = (2n-1)^2\frac{\pi^2}{4} \qquad (1224)$$

After some simple manipulations:

$$\frac{\lambda}{H} = (2n-1)^2\frac{\pi^2}{4} + \frac{1}{\frac{4(\kappa^2-1)}{(2n-1)^2\pi^2} + \frac{1}{\alpha^2}} \qquad (1225)$$

Replacing by its characteristic rigidities:

$$q_{cr} = (2n-1)^2\frac{\pi^2 K_{b2}}{4H^2} + \frac{1}{\frac{4H^2}{(2n-1)^2\pi^2 K_{b1}} + \frac{1}{K_{s1}}} \qquad (1226)$$

Sorting properly:

$$q_{cr} = (2n-1)^2\frac{\pi^2 K_{b2}}{4H^2} + \left\{\left[(2n-1)^2\frac{\pi^2 K_{b1}}{4H^2}\right]^{-1} + K_{s1}{}^{-1}\right\}^{-1} \qquad (1227)$$

Tabla.7 *Eigenvalue as a function of the parameter* $\alpha \leq 300$ *and* $1.0000 \leq \kappa \leq 1.0010$ *for the case of a uniformly distributed axial load*

α \ κ	1.0000	1.0001	1.0002	1.0003	1.0004	1.0005	1.00075	1.001
0.00	7.837	7.837	7.837	7.837	7.837	7.837	7.837	7.837
0.25	8.025	8.025	8.025	8.025	8.025	8.025	8.025	8.025
0.50	8.583	8.583	8.583	8.583	8.583	8.583	8.583	8.583
0.75	9.504	9.504	9.504	9.504	9.504	9.504	9.503	9.503
1.0	10.772	10.772	10.771	10.771	10.771	10.771	10.770	10.770
2.5	24.377	24.370	24.363	24.356	24.349	24.342	24.325	24.308
5.0	60.763	60.706	60.649	60.593	60.536	60.480	60.339	60.199
7.5	110.356	110.184	110.011	109.839	109.667	109.494	109.063	108.633
10	180.103	179.710	179.315	178.918	178.517	178.114	177.096	175.908
15	375.170	372.847	370.536	368.238	365.954	363.682	358.062	352.530
20	645.789	638.861	631.998	625.201	618.474	611.818	595.498	579.650
25	993.453	977.024	960.837	944.903	929.231	913.829	876.547	841.067
30	1418.270	1384.802	1352.054	1320.066	1288.872	1258.498	1186.249	1119.331
35	1920.277	1859.052	1799.654	1742.207	1686.797	1633.482	1509.471	1398.434
40	2499.489	2396.104	2296.840	2202.001	2111.775	2026.232	1832.554	1665.809
45	3155.914	2991.815	2836.189	2689.694	2552.612	2424.909	2144.534	1913.202
50	3889.557	3641.615	3409.868	3195.521	2998.794	2819.166	2437.723	2136.428
60	5588.502	5083.452	4628.145	4224.991	3871.677	3563.355	2952.339	2508.135
70	7596.334	6678.947	5889.347	5226.932	4676.364	4218.316	3367.608	2791.331
80	9913.056	8382.991	7138.713	6157.556	5385.894	4772.494	3695.166	3005.458
90	12538.671	10150.636	8333.911	6993.577	5994.334	5231.631	3952.173	3168.370
100	15473.180	11939.666	9446.823	7728.042	6508.082	5608.691	4154.620	3293.825
110	18716.583	13712.812	10462.460	8364.508	6938.873	5917.955	4315.436	3391.129
120	22268.880	15439.279	11376.149	8911.991	7299.585	6172.402	4444.229	3468.208
130	26130.072	17095.467	12190.267	9381.527	7602.212	6382.901	4548.098	3530.191
140	30300.158	18664.880	12911.369	9784.197	7857.111	6558.206	4633.366	3580.663
150	34779.140	20137.434	13548.049	10130.167	8072.896	6704.513	4704.066	3622.238
160	39567.016	21508.394	14109.526	10428.331	8256.589	6827.771	4763.239	3656.847
170	44663.788	22777.179	14604.806	10686.262	8412.684	6932.745	4813.197	3685.935
180	50069.455	23946.227	15042.236	10910.313	8547.089	7022.746	4855.716	3710.600
190	55784.017	25020.007	15429.316	11105.138	8663.585	7100.401	4892.174	3731.681
200	61807.473	26004.228	15772.654	11274.990	8765.098	7167.805	4923.651	3749.832
220	74781.073	27729.609	16350.315	11556.851	8932.253	7278.266	4974.902	3779.289
240	88990.253	29174.101	16810.104	11778.956	9062.854	7364.123	5014.457	3801.942
260	104435.013	30385.836	17181.134	11956.612	9166.629	7432.069	5045.589	3819.722
280	121115.354	31406.366	17485.079	12100.663	9250.332	7486.698	5070.512	3833.924
300	139031.276	32270.223	17736.669	12218.916	9318.754	7531.238	5090.761	3845.444

Tabla.8 *Eigenvalue as a function of the parameter* $\alpha \leq 300$ *and* $1.0020 \leq \kappa \leq 1.0090$ *for the case of a uniformly distributed axial load*

α \ κ	1.0020	1.0030	1.0040	1.0050	1.0060	1.0070	1.00800	1.009
0.00	7.837	7.837	7.837	7.837	7.837	7.837	7.837	7.837
0.25	8.025	8.025	8.025	8.025	8.025	8.025	8.025	8.025
0.50	8.583	8.583	8.583	8.582	8.582	8.582	8.582	8.582
0.75	9.502	9.502	9.501	9.500	9.499	9.499	9.498	9.497
1.0	10.767	10.765	10.763	10.760	10.758	10.756	10.754	10.751
2.5	24.239	24.171	24.104	24.037	23.970	23.904	23.839	23.773
5.0	59.642	59.091	58.547	58.010	57.479	56.956	56.439	55.929
7.5	106.912	105.199	103.498	101.813	100.149	98.507	96.892	95.305
10	170.792	167.394	161.063	156.460	152.035	147.786	143.712	139.808
15	331.323	311.656	293.595	276.961	261.805	247.974	235.353	223.826
20	521.202	470.650	427.370	390.397	358.728	331.463	307.840	287.234
25	717.227	619.617	542.848	481.808	432.524	392.098	358.441	330.039
30	901.314	746.529	634.334	550.391	485.639	434.348	392.799	358.499
35	1063.505	849.154	704.230	600.774	523.570	463.886	416.406	377.738
40	1200.903	930.252	757.193	637.918	551.001	484.852	432.951	391.162
45	1314.864	993.976	797.555	665.649	571.079	500.132	444.966	400.854
50	1408.542	1408.542	828.665	686.591	586.216	511.576	453.917	408.044
60	1548.741	1116.161	871.951	715.593	606.981	527.166	466.047	417.747
70	1644.792	1163.229	899.916	734.088	620.107	536.959	473.629	423.790
80	1711.917	1195.568	918.873	746.521	628.882	543.479	478.663	427.792
90	1760.522	1218.605	932.253	755.249	635.018	548.026	482.166	430.572
100	1796.693	1235.532	942.021	761.595	639.468	551.318	484.698	432.580
110	1824.243	1248.305	949.357	766.348	642.794	553.774	486.586	434.076
120	1845.659	1258.165	954.999	769.995	645.343	555.655	488.030	435.219
130	1862.610	1265.925	959.429	772.853	647.339	557.126	489.159	436.113
140	1876.240	1272.139	962.967	775.134	648.929	558.298	490.058	436.824
150	1887.353	1277.188	965.837	776.981	650.217	559.246	490.785	437.399
160	1896.528	1281.344	968.196	778.499	651.274	560.024	491.381	437.870
170	1904.186	1284.805	970.158	779.760	652.152	560.671	491.877	438.262
180	1910.643	1287.717	971.808	780.820	652.890	561.213	492.292	438.590
190	1916.134	1290.190	973.207	781.719	653.515	561.673	492.645	438.869
200	1920.843	1292.308	974.405	782.487	654.049	562.066	492.946	439.107
220	1928.446	1295.721	976.334	783.724	654.910	562.698	493.430	439.489
240	1934.261	1298.327	977.805	784.668	655.565	563.180	493.799	439.781
260	1938.807	1300.361	978.952	785.403	656.076	563.556	494.086	440.008
280	1942.425	1301.978	979.864	785.987	656.482	563.854	494.314	440.188
300	1945.352	1303.286	980.601	786.459	656.810	564.094	494.499	440.333

Tabla.9 *Eigenvalue as a function of the parameter $\alpha \leq 300$ and $1.01 \leq \kappa \leq 1.25$ for the case of a uniformly distributed axial load*

α \ κ	1.01	1.02	1.03	1.04	1.05	1.1	1.15	1.2	1.25
0.00	7.837	7.837	7.837	7.837	7.837	7.837	7.837	7.837	7.837
0.25	8.025	8.024	8.024	8.024	8.024	8.024	8.023	8.023	8.022
0.50	8.582	8.580	8.579	8.577	8.576	8.568	8.560	8.552	8.544
0.75	9.497	9.489	9.482	9.474	9.467	9.429	9.392	9.355	9.318
1.0	10.749	10.726	10.704	10.681	10.659	10.549	10.443	10.341	10.243
2.5	23.709	23.085	22.501	21.954	21.440	19.286	17.652	16.376	15.355
5.0	55.426	50.781	46.791	43.371	40.431	30.522	24.975	21.473	19.074
7.5	93.748	79.980	69.294	61.107	54.687	36.520	28.176	23.426	20.371
10	136.070	106.509	87.118	73.751	64.051	39.667	29.696	24.303	20.933
15	213.283	144.020	108.748	87.647	73.698	42.469	30.974	25.016	21.380
20	269.138	164.906	119.357	94.042	77.952	43.591	31.467	25.286	21.547
25	305.781	176.628	125.025	97.355	80.115	44.139	31.704	25.415	21.627
30	329.726	183.686	128.336	99.260	81.347	44.444	31.835	25.486	21.671
35	345.695	188.203	130.417	100.447	82.109	44.631	31.916	25.530	21.697
40	356.801	191.246	131.803	101.232	82.612	44.754	31.968	25.558	21.715
45	364.781	193.385	132.770	101.778	82.961	44.838	32.004	25.577	21.726
50	370.682	194.942	133.471	102.173	83.213	44.899	32.030	25.591	21.735
60	378.619	197.004	134.393	102.691	83.543	44.979	32.064	25.609	21.746
70	383.546	198.267	134.956	103.006	83.743	45.027	32.084	25.620	21.753
80	386.803	199.095	135.323	103.212	83.874	45.058	32.097	25.628	21.757
90	389.063	199.666	135.576	103.353	83.964	45.079	32.107	25.632	21.760
100	390.694	200.076	135.758	103.454	84.028	45.095	32.113	25.636	21.762
110	391.908	200.381	135.892	103.530	84.076	45.106	32.118	25.639	21.764
120	392.835	200.613	135.995	103.587	84.112	45.115	32.122	25.641	21.765
130	393.560	200.794	136.075	103.631	84.141	45.122	32.124	25.642	21.766
140	394.136	200.938	136.138	103.667	84.163	45.127	32.127	25.643	21.767
150	394.602	201.054	136.190	103.695	84.181	45.131	32.129	25.644	21.768
160	394.984	201.149	136.232	103.719	84.196	45.135	32.130	25.645	21.768
170	395.302	201.228	136.266	103.738	84.208	45.138	32.131	25.646	21.768
180	395.568	201.294	136.295	103.754	84.219	45.140	32.132	25.646	21.769
190	395.793	201.350	136.320	103.768	84.227	45.142	32.133	25.647	21.769
200	395.986	201.398	136.341	103.780	84.235	45.144	32.134	25.647	21.769
220	396.296	201.475	136.375	103.799	84.247	45.147	32.135	25.648	21.770
240	396.532	201.533	136.401	103.813	84.256	45.149	32.136	25.648	21.770
260	396.715	201.579	136.421	103.824	84.263	45.151	32.137	25.649	21.770
280	396.861	201.615	136.437	103.833	84.269	45.152	32.137	25.649	21.770
300	396.979	201.644	136.450	103.840	84.273	45.153	32.138	25.649	21.771

For the case when $n = 1$, we have:

$$q_{cr} = \frac{\pi^2 K_{b2}}{4H^2} + \left[\left(\frac{\pi^2 K_{b1}}{4H^2}\right)^{-1} + K_{s1}{}^{-1}\right]^{-1} = q_{cr,flexiòn\ local} + \left[q_{cr,flexiòn\ global}{}^{-1} + q_{crmcorte}{}^{-1}\right]^{-1} \quad (1228)$$

Since the resulting critical load is independent of some approximation function, it can be considered exact and identical to the one that would be obtained by applying Föppl's theorem.

4.3.6.2 Case 2

- **Calculation of the Transfer Matrix**

According to the coupled differential equations:

$$\begin{Bmatrix} K_{s1}\left[\theta_{(x)} - u'_{(x)}\right] - K_{b1}\theta''_{(x)} = 0 \\ K_{b2}u''''_{(x)} + (q - K_{s1})u''_{(x)} + K_{s1}\theta'_{(x)} = 0 \end{Bmatrix} \quad (1229)$$

Using the method of coefficients:

$$\begin{bmatrix} -K_{s1}D & -K_{b1}D^2 + K_{s1} \\ K_{b2}D^4 - (K_{s1} - q)D^2 & K_{s1}D \end{bmatrix} \begin{Bmatrix} u_{(x)} \\ \theta_{(x)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (1230)$$

To avoid trivial solutions, the determinant must be equal to zero, that is:

$$D^2\left\{D^4 - \left[\frac{K_{s1}(K_{b1} + K_{b2})}{K_{b1}K_{b2}} - \frac{q}{K_{b2}}\right]D^2 - \left(\frac{K_{s1}q}{K_{b1}K_{b2}}\right)\right\} = 0 \quad (1231)$$

Rewriting:

$$D^2\left\{D^4 - [(\alpha^*\kappa)^2 - \lambda]D^2 - \left[\alpha^{*2}(\kappa^2 - 1)\lambda\right]\right\} = 0 \quad (1232)$$

Where:

$$\left\{\alpha^* = \sqrt{\frac{K_{s1}}{K_{b2}}}, \kappa = \sqrt{1 + \frac{K_{b2}}{K_{b1}}}, \lambda = \frac{q}{K_{b2}}\right\} \quad (1233)$$

The expression for $u_{(z)}$ and $\theta_{(z)}$ is proposed:

$$\begin{cases} u_{(z)} = C_0 + C_1 z + C_2 \cosh\left(\sqrt{\xi} z\right) + C_3 \sinh\left(\sqrt{\xi} z\right) + C_4 \cos\left(\sqrt{\beta} z\right) + C_5 \sin\left(\sqrt{\beta} z\right) \\ \theta_{(z)} = C_6 + C_7 \cosh\left(\sqrt{\xi} z\right) + C_8 \sinh\left(\sqrt{\xi} z\right) + C_9 \cos\left(\sqrt{\beta} z\right) + C_{10} \sin\left(\sqrt{\beta} z\right) \end{cases} \quad (1234)$$

Where:

$$\begin{cases} \xi = \dfrac{\left[(\alpha^* \kappa)^2 - \lambda\right] + \sqrt{[(\alpha^* \kappa)^2 - \lambda]^2 + 4\alpha^{*2}(\kappa^2 - 1)\lambda}}{2} \\ \beta = \dfrac{-\left[(\alpha^* \kappa)^2 - \lambda\right] + \sqrt{[(\alpha^* \kappa)^2 - \lambda]^2 + 4\alpha^{*2}(\kappa^2 - 1)\lambda}}{2} \end{cases} \quad (1235)$$

Expressing the coefficients of $\theta_{(z)}$ as a function of the coefficients of $u_{(z)}$:

$$\begin{cases} u_{(z)} = C_0 + C_1 \cosh\left(\sqrt{\xi} z\right) + C_2 \sinh\left(\sqrt{\xi} z\right) + C_3 \cos\left(\sqrt{\beta} z\right) + C_4 \sin\left(\sqrt{\beta} z\right) \\ \theta_{(z)} = C_1 + C_2\left[R_\xi \sinh\left(\sqrt{\xi} z\right)\right] + C_3\left[R_\xi \cosh\left(\sqrt{\xi} z\right)\right] + C_4\left[-R_\beta \sin\left(\sqrt{\beta} z\right)\right] + C_5\left[R_\beta \cos\left(\sqrt{\beta} z\right)\right] \end{cases} \quad (1236)$$

Where:

$$\left\{ R_\xi = \frac{K_{s1}\sqrt{\xi}}{K_{s1} - \xi K_{b1}}, R_\beta = \frac{K_{s1}\sqrt{\beta}}{K_{s1} - \beta K_{b1}} \right\} \quad (1237)$$

Internal forces such as bending moment and shear force associated with lateral displacement result in:

$$\begin{cases} \begin{Bmatrix} M_{1(z)} = K_{b1}\theta'_{(z)} = C_2\left[K_{b1} R_\xi \sqrt{\xi} \cosh\left(\sqrt{\xi} z\right)\right] + C_3\left[K_{b1} R_\xi \sqrt{\xi} \sinh\left(\sqrt{\xi} z\right)\right] \\ +C_4\left[-K_{b1} R_\beta \sqrt{\beta} \cos\left(\sqrt{\beta} z\right)\right] + C_4\left[-K_{b1} R_\beta \sqrt{\beta} \sin\left(\sqrt{\beta} z\right)\right] \end{Bmatrix} \\ \begin{Bmatrix} M_2 = K_{b2} u''_{(x)} = C_2\left[K_{b2} \xi \cosh\left(\sqrt{\xi} z\right)\right] + C_3\left[K_{b2} \xi \sinh\left(\sqrt{\xi} z\right)\right] \\ +C_4\left[-K_{b2} \beta \cos\left(\sqrt{\beta} z\right)\right] + C_5\left[-K_{b2} \beta \sin\left(\sqrt{\beta} z\right)\right] \end{Bmatrix} \\ \begin{Bmatrix} V_{(z)} = K_{s1}\left[\theta_{(x)} - u'_{(x)}\right] + K_{b2} u'''_{(x)} = qC_1 + C_2\left[P_\xi \sinh\left(\sqrt{\xi} z\right)\right] \\ +C_3\left[P_\xi \cosh\left(\sqrt{\xi} z\right)\right] + C_4\left[P_\beta \sin\left(\sqrt{\beta} z\right)\right] + C_5\left[-P_\beta \cos\left(\sqrt{\beta} z\right)\right] \end{Bmatrix} \end{cases} \quad (1238)$$

Where:

$$\left\{ P_\xi = \left(\frac{K_{b1} K_{s1}}{K_{s1} - \xi K_{b1}} + K_{b2}\right) \xi \sqrt{\xi}, P_\beta = \left(-\frac{K_{b1} K_{s1}}{K_{s1} - \beta K_{b1}} + K_{b2}\right) \beta \sqrt{\beta} \right\} \quad (1239)$$

Writing the equations in matrix form:

$$\begin{Bmatrix} u_i(z_i) \\ u_i'(z_i) \\ \theta_i(z_i) \\ M_{\mathrm{l,i}}(z_i) \\ M_{\mathrm{r,i}}(z_i) \\ V_i(z_i) \end{Bmatrix} = K_i(z_i) \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{Bmatrix} \tag{1240}$$

Where:

$K_i(z_i)$

$$= \begin{bmatrix} 1 & z & \cosh(\sqrt{\xi}z) & \sinh(\sqrt{\xi}z) & \cos(\sqrt{\beta}z) & \sin(\sqrt{\beta}z) \\ 0 & 1 & \sqrt{\xi}\sinh(\sqrt{\xi}z) & \sqrt{\xi}\cosh(\sqrt{\xi}z) & -\sqrt{\beta}\sin(\sqrt{\beta}z) & \sqrt{\beta}\cos(\sqrt{\beta}z) \\ 0 & 1 & R_\xi \sinh(\sqrt{\xi}z) & R_\xi \cosh(\sqrt{\xi}z) & -R_\beta \sin(\sqrt{\beta}z) & R_\beta \cos(\sqrt{\beta}z) \\ 0 & 0 & K_{b1}R_\xi\sqrt{\xi}\cosh(\sqrt{\xi}z) & K_{b1}R_\xi\sqrt{\xi}\sinh(\sqrt{\xi}z) & -K_{b1}R_\beta\sqrt{\beta}\cos(\sqrt{\beta}z) & -K_{b1}R_\beta\sqrt{\beta}\sin(\sqrt{\beta}z) \\ 0 & 0 & K_{b2}\xi\cosh(\sqrt{\xi}z) & K_{b2}\xi\sinh(\sqrt{\xi}z) & -K_{b2}\beta\cos(\sqrt{\beta}z) & -K_{b2}\beta\sin(\sqrt{\beta}z) \\ 0 & q & (P_\xi + q\sqrt{\xi})\sinh(\sqrt{\xi}z) & (P_\xi + q\sqrt{\xi})\cosh(\sqrt{\xi}z) & (P_\beta - q\sqrt{\xi})\sin(\sqrt{\beta}z) & -(P_\beta - q\sqrt{\xi})\cos(\sqrt{\beta}z) \end{bmatrix}_i \tag{1241}$$

- **Stability Analysis Under Point Loads Applied at Floor Level**

Applying sequentially from the base to the top of the beam and expressing the equation between the product symbol:

$$\begin{Bmatrix} u_n(0) \\ u'_n(0) \\ \theta_n(0) \\ M_{\mathrm{ln}}(0) \\ M_{\mathrm{rn}}(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_k(0) \begin{Bmatrix} u_1(h_1) \\ u'_1(h_1) \\ \theta_1(h_1) \\ M_{\mathrm{l1}}(h_1) \\ M_{\mathrm{r1}}(h_1) \\ V_1(h_1) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ u'_1(h_1) \\ \theta_1(h_1) \\ M_{\mathrm{l1}}(h_1) \\ M_{\mathrm{r1}}(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1242}$$

Where:

$$\mathrm{t} = \prod_{k=1}^{n} T_k(0) \tag{1243}$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 6x6 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ u'_{(1)} = 0 \\ \theta_{(1)} = 0 \\ \psi'_{(0)} = 0 \\ \theta'_{(0)} = 0 \\ (K_{s1} + K_{s2} - q)u'_{(0)} - K_{s1}\psi_{(0)} - K_{s2}\theta_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1(h_1) = 0 \\ u'_1(h_1) = 0 \\ \theta_1(h_1) = 0 \\ M_{1n\,(0)} = 0 \\ M_{2n\,(0)} = 0 \\ V_{n(0)} = 0 \end{Bmatrix} \qquad (1244)$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ u'_n(0) \\ \theta_n(0) \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & t_{1,5} & t_{1,6} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} & t_{2,5} & t_{2,6} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} & t_{3,5} & t_{3,6} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,1} & t_{5,2} & t_{5,3} & t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,1} & t_{6,2} & t_{6,3} & t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ M_1(h_1) \\ M_2(h_1) \\ V_1(h_1) \end{Bmatrix} \qquad (1245)$$

Solving for bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ M_2(h_1) \\ V_1(h_1) \end{Bmatrix} \qquad (1246)$$

Which has a different solution than the trivial if the determinant is equal to zero (the matrix of coefficients is singular). Solving the critical loads of the beam.

4.3.7 Generalized Sandwich Beam of Three Field (GSB1)

4.3.7.1 Case 1

The potential energy of the three-field GSB1 model is expressed as follows:

$$V = \frac{1}{2}\int_0^H \left\{K_{b1}{\psi'}^2 + K_{s1}\left[u'_{(x)} - \psi_{(x)}\right]^2 + K_{b2}{\theta'_{(x)}}^2 + K_{s2}\left[u'_{(x)} - \theta_{(x)}\right]^2\right\} dx \tag{1247}$$

- Coupled shear wall:

$$\left\{\begin{matrix} K_{b1} = \sum_{i=1}^{w} EA_{w,i}c_i^2\,, K_{b2} = \sum_{i=1}^{w} rEI_{w,i}\,, K_{s1} = \left(K_b^{-1} + K_w^{-1}\right)^{-1}, K_{s2} = \sum_{i=1}^{c} GA_{c,i} \\ K_b = \sum_{i=1}^{b} \frac{6EI_{b,i}[(l^* + S_1)^2 + (l^* + S_2)^2]}{l^{*3}h\left(1 + 12\frac{kEI_{b.i}}{l^{*2}GA_{b.i}}\right)}\,, K_w = \sum_{i=1}^{w} \frac{\pi^2 EI_w}{h^2}, r = \frac{K_c}{K_c + K_b} \end{matrix}\right\} \tag{1248}$$

- Frame:

$$\left\{\begin{matrix} K_{b1} = \sum_{i=1}^{c} EA_{ci}c_i^2\,, K_{b2} = \sum_{i=1}^{c} rEI_{ci}\,, K_{s1} = \left(K_b^{-1} + K_c^{-1}\right)^{-1} \\ K_{s2} = \sum_{i=1}^{c} GA_{ci}\,, K_b = \sum_{i=1}^{b} \frac{12EI_b}{lh}, K_c = \sum_{i=1}^{c} \frac{\pi^2 EI_c}{h^2}, r = \frac{K_c}{K_c + K_b} \end{matrix}\right\} \tag{1249}$$

- Dual (frame + shear wall):

$$\left\{\begin{matrix} K_{b1} = \sum_{i=1}^{c} EA_{c,i}c_i^2\,, K_{b2} = \sum_{i=1}^{c} rEI_{c,i} + \sum_{i=1}^{w} rEI_{w,i}\,, K_{s1} = \left(K_b^{-1} + K_c^{-1}\right)^{-1} \\ K_{s2} = \sum_{i=1}^{w} GA_{w,i}\;\;;\;\; K_b = \sum_{i=1}^{b} \frac{12EI_b}{lh}\;\;;\;\; K_c = \sum_{i=1}^{c} \frac{\pi^2 EI_c}{h^2}\;\;;\;\; r = \frac{K_c}{K_c + K_b} \end{matrix}\right\} \tag{1250}$$

The work done by the external force is expressed as:

$$W = -f_{(x)}dl = -\frac{1}{2}\int_0^H f_{(x)}{u'_{(x)}}^2\, dx \tag{1251}$$

Consequently, the total potential energy of the model is expressed as:

$$
\begin{aligned}
\mathcal{U} = \frac{1}{2}\int_0^H \left\{K_{b1}{\psi'_{(x)}}^2 + K_{s1}\left[u'_{(x)} - \psi_{(x)}\right]^2 + K_{b2}{\theta'_{(x)}}^2 + K_{s2}\left[u'_{(x)} - \theta_{(x)}\right]^2\right\} dx \\
- \frac{1}{2}\int_0^H f(x){u'_{(x)}}^2\, dx
\end{aligned} \tag{1252}
$$

Closed-form solutions of the model are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$
\begin{aligned}
\delta\mathcal{U} = \int_0^H \{K_{b1}\psi'_{(x)}\delta\psi'_{(x)} + K_{s1}\left[u'_{(x)} - \psi_{(x)}\right]\left[\delta u'_{(x)} - \delta\psi_{(x)}\right] + K_{b2}\theta'_{(x)}\delta\theta'_{(x)} \\
+ K_{s2}\left[u'_{(x)} - \theta_{(x)}\right]\left[\delta u'_{(x)} - \delta\theta_{(x)}\right] - f_{(x)}u'_{(x)}\delta u'_{(x)}\}dx \\
- \frac{1}{2}\int_0^H f(x)u'^2_{(x)}\delta f_{(x)}dx
\end{aligned} \tag{1253}
$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$
\begin{aligned}
\delta\mathcal{U} = \left[K_{b1}\psi'_{(x)}\delta\psi_{(x)}\right]_0^H + \left\{[K_{s1} + K_{s2} - f(x)]u'_{(x)} - K_{s1}\psi_{(x)} - K_{s2}\psi_{(x)}\right\}\delta u_{(x)}\Big|_0^H \\
+ \left[K_{b2}\theta'_{(x)}\delta\theta_{(x)}\right]_0^H - \int_0^H \left\{K_{b1}\psi''_{(x)} + K_{s1}\left[u'_{(x)} - \psi_{(x)}\right]\right\}\delta\theta_{(x)} \\
- \int_0^H \left\{[K_{s1} + K_{s2} - f(x)]u''_{(x)} - K_{s1}\psi'_{(x)} - K_{s2}\theta'_{(x)} - f'_{(x)}u'_{(x)}\right\}\delta u_{(x)} \\
- \int_0^H \left\{K_{b2}\theta''_{(x)} + K_{s2}\left[u'_{(x)} - \theta_{(x)}\right]\right\}\delta\psi_{(x)} - \frac{1}{2}\int_0^H f(x)u'^2_{(x)}\delta f_{(x)}dx
\end{aligned} \tag{1254}
$$

Setting the terms equal to zero, the following equations result:

$$
\left\{
\begin{array}{c}
K_{b1}\psi''_{(x)} + K_{s1}\left[u'_{(x)} - \psi_{(x)}\right] = 0 \\
K_{b2}\theta''_{(x)} + K_{s2}\left[u'_{(x)} - \theta_{(x)}\right] = 0 \\
[K_{s1} + K_{s2} - f(x)]u''_{(x)} - K_{s1}\psi'_{(x)} - K_{s2}\theta'_{(x)} - f'_{(x)}u'_{(x)} = 0
\end{array}
\right\} \tag{1255}
$$

And boundary conditions:

$$
\left\{
\begin{array}{c}
\theta'_{(0)} = 0 \\
\psi'_{(0)} = 0 \\
[K_{s1} + K_{s2} - f(0)]u'_{(0)} - K_{s1}\psi_{(0)} - K_{s2}\theta_{(0)} = 0
\end{array}
\right\} \tag{1256}
$$

Integrating the equation once and evaluating at x=0:

$$[K_{s1} + K_{s2} - f(x)]u'_{(x)} - K_{s1}\psi_{(x)} - K_{s2}\theta_{(x)} = 0 \tag{1257}$$

We have a new system of coupled differential equations:

$$\left\{\begin{matrix} \theta'_{(0)} = 0 \\ \psi'_{(0)} = 0 \\ [K_{s1} + K_{s2} - f(x)]u'_{(x)} - K_{s1}\psi_{(x)} - K_{s2}\theta_{(x)} = 0 \end{matrix}\right\} \tag{1258}$$

Using the method of coefficients for the solution of the system of equations:

$$\begin{bmatrix} K_{s1}D & 0 & K_{b1}D^2 - K_{s1} \\ K_{s2}D & K_{b2}D^2 - K_{s2} & 0 \\ [K_{s1} + K_{s2} - f(x)]D & -K_{s2} & -K_{s1} \end{bmatrix} \begin{Bmatrix} u_{(x)} \\ \theta_{(x)} \\ \psi_{(x)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Which has a solution other than the trivial one if the determinant is equal to zero:

$$\begin{aligned} u'''''_{(x)} - \left[\frac{K_{s1}K_{s2}(K_{b1} + K_{b2})}{K_{b1}K_{b2}(K_{s1} + K_{s2})}\right] u'''_{(x)} \\ + f(x)\left\{-\left(\frac{1}{K_{s1} + K_{s2}}\right) u'''''_{(x)} + \left[\frac{K_{b1}K_{s2} + K_{s1}K_{b2}}{K_{b1}K_{b2}(K_{s1} + K_{s2})}\right] u'''_{(x)} \right. \\ \left. - \left[\frac{K_{s1}K_{s2}}{K_{b1}K_{b2}(K_{s1} + K_{s2})}\right] u'_{(x)}\right\} = 0 \end{aligned} \tag{1259}$$

A fourth order differential equation is obtained, where the critical load results from the smallest eigenvalue. Normalizing the differential equation by the variable $z{=}x/H$:

$$\begin{aligned} u'''''_{(z)} - \left[\frac{K_{s1}K_{s2}(K_{b1} + K_{b2})}{K_{b1}K_{b2}(K_{s1} + K_{s2})}H^2\right] u'''_{(z)} \\ + f(z)\left\{-\left(\frac{1}{K_{s1} + K_{s2}}\right) u'''''_{(z)} + \left[\frac{K_{b1}K_{s2} + K_{s1}K_{b2}}{K_{b1}K_{b2}(K_{s1} + K_{s2})}H^2\right] u'''_{(z)} \right. \\ \left. - \left[\frac{K_{s1}K_{s2}}{K_{b1}K_{b2}(K_{s1} + K_{s2})}H^4\right] u'_{(z)}\right\} = 0 \end{aligned} \tag{1260}$$

Where:

$$f_{(z)} = q\alpha_{(z)} \tag{1261}$$

The equation can be rewritten as:

$$u_{(z)}''''' - a_0 u_{(z)}''' + q\alpha_{(z)}\left[-a_1 u_{(z)}''''' + a_2 u_{(z)}''' - a_3 u_{(z)}'\right] = 0 \tag{1262}$$

Where:

$$\left\{\begin{matrix} a_0 = \dfrac{K_{s1}K_{s2}(K_{b1}+K_{b2})}{K_{b1}K_{b2}(K_{s1}+K_{s2})}H^2, a_1 = \dfrac{1}{K_{s1}+K_{s2}} \\ a_2 = \dfrac{K_{b1}K_{s2}+K_{s1}K_{b2}}{K_{b1}K_{b2}(K_{s1}+K_{s2})}H^2, a_3 = \dfrac{K_{s1}K_{s2}}{K_{b1}K_{b2}(K_{s1}+K_{s2})}H^4 \end{matrix}\right\} \tag{1263}$$

Expressing the boundary conditions as a function of $u_{(z)}$:

$$\left\{\begin{matrix} u_{(1)} = 0 \\ u_{(1)}' = 0 \\ u_{(0)}'' = 0 \\ u_{(1)}''' = 0 \\ u_{(0)}'''' = 0 \end{matrix}\right\} \tag{1264}$$

- **Uniformly Distributed Load**

For beam stability, the governing differential equation is of the form:

$$\left(\frac{d^5}{dz^5} - a_0\frac{d^3}{dz^3}\right)u_{(z)} - q\left[\alpha_{(z)}\left(a_1\frac{d^5}{dz^5} - a_2\frac{d^3}{dz^3} + a_3\frac{d}{dz}\right)\right]u_{(z)} = 0 \tag{1265}$$

Multiplying the equation by $\left(a_1\frac{d^5}{dz^5} - a_2\frac{d^3}{dz^3} + a_3\frac{d}{dz}\right)$, integrating from 0 to 1 and clearing:

$$\lambda = \frac{\int_0^1\left\{a_1 u'''''^2_{(z)} + (a_0a_1 + a_2)u''''^2_{(z)} + (a_0a_2 + a_3)u'''^2_{(z)} + a_0a_3u''^2_{(z)}\right\}dz}{\int_0^1 \alpha_{(z)}\left[a_1u_{(z)}''''' - a_2u_{(z)}''' + a_3u_{(z)}'\right]^2 dz} \tag{1266}$$

This Rayleigh ratio represents an approximation of the upper limit of the critical load, and it is exact if and only if the exact equilibrium curve $u_{(z)}$ is used to calculate λ.

For the case of a uniformly distributed load, the function $\alpha_{(z)}$ results in:

$$\alpha_{(z)} = z \rightarrow f_{(z)} = qz \tag{1267}$$

The Rayleigh quotient becomes:

$$\lambda = \frac{\int_0^1 \{a_1 u''''^2_{(z)} + (a_0 a_1 + a_2) u''''^2_{(z)} + (a_0 a_2 + a_3) u'''^2_{(z)} + a_0 a_3 u''^2_{(z)}\} dz}{\int_0^1 z \left[a_1 u'''''_{(z)} - a_2 u'''_{(z)} + a_3 u'_{(z)}\right]^2 dz} \tag{1268}$$

- **Point Load at x=0 (z=0)**

For the case of a point load applied at x=H (z=1), the function $\alpha_{(z)}$:

$$\alpha_{(z)} = 1 \rightarrow f_{(z)} = q \tag{1269}$$

Substituting into the differential equation:

$$u'''''_{(z)} - a_0 u'''_{(z)} + q\left[-a_1 u'''''_{(z)} + a_2 u'''_{(z)} - a_3 u'_{(z)}\right] = 0 \tag{1270}$$

The expression for $u_{(z)}$ can be derived as:

$$u_{(z)} = C_0 + C_1 \cosh(\sqrt{\xi} z) + C_2 \sinh(\sqrt{\xi} z) + C_3 \cos(\sqrt{\beta} z) + C_4 \sin(\sqrt{\beta} z) \tag{1271}$$

Where:

$$\begin{Bmatrix} \xi = \dfrac{(a_0 - q a_2) + \sqrt{(a_0 - q a_2)^2 + 4 q a_3 (1 - q a_1)}}{2(1 - q a_1)} \\ \beta = \dfrac{-(a_0 - q a_2) + \sqrt{(a_0 - q a_2)^2 + 4 q a_3 (1 - q a_1)}}{2(1 - q a_1)} \end{Bmatrix} \tag{1272}$$

The linear algebraic system resulting from the boundary conditions, written in matrix form, is:

$$\begin{bmatrix} 1 & \cosh(\sqrt{\xi}) & \sinh(\sqrt{\xi}) & \cos(\sqrt{\beta}) & \sin(\sqrt{\beta}) \\ 0 & \xi^{1/2} \sinh(\sqrt{\xi}) & \xi^{1/2} \cosh(\sqrt{\xi}) & -\beta^{1/2} \sin(\sqrt{\beta}) & \beta^{1/2} \cos(\sqrt{\beta}) \\ 0 & \xi & 0 & -\beta & 0 \\ 0 & \xi^{3/2} \sinh(\sqrt{\xi}) & \xi^{3/2} \cosh(\sqrt{\xi}) & \beta^{3/2} \sin(\sqrt{\beta}) & -\beta^{3/2} \cos(\sqrt{\beta}) \\ 0 & \xi^2 & 0 & \beta^2 & 0 \end{bmatrix} \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = 0 \tag{1273}$$

Which has a different solution than the trivial one if the determinant is equal to zero (the matrix of coefficients is singular), that is, for:

$$Cos\sqrt{\beta} = 0 \rightarrow \sqrt{\beta} = (2n - 1)\frac{\pi}{2} \ / \ n = 1, 2, 3 \ldots \tag{1274}$$

i.e.,

$$\frac{-(a_0 - qa_2) + \sqrt{(a_0 - qa_2)^2 + 4qa_3(1 - qa_1)}}{2(1 - qa_1)} = (2n-1)^2\frac{\pi^2}{4} \qquad (1275)$$

Solving:

$$q_{cr} = \frac{\frac{(2n-1)^4\pi^4}{4} + a_0(2n-1)^2\pi^2}{4a_0 + a_2(2n-1)^2\pi^2 + a_1\frac{(2n-1)^4\pi^4}{4}} \qquad (1276)$$

Replacing the coefficients and after some simple manipulations:

$$q_{cr} = \frac{1}{\frac{4H^2}{(2n-1)^2\pi^2K_{b1}} + \frac{1}{K_{s1}}} + \frac{1}{\frac{4H^2}{(2n-1)^2\pi^2K_{b2}} + \frac{1}{K_{s2}}} \qquad (1277)$$

Sorting properly:

$$q_{cr} = \left\{\left[(2n-1)^2\frac{\pi^2K_{b1}}{4H^2}\right]^{-1} + {K_{s1}}^{-1}\right\}^{-1} + \left\{\left[(2n-1)^2\frac{\pi^2K_{b2}}{4H^2}\right]^{-1} + {K_{s2}}^{-1}\right\}^{-1} \qquad (1278)$$

For the case when $n = 1$, we have:

$$q_{cr} = \left[\left(\frac{\pi^2K_{b1}}{4H^2}\right)^{-1} + {K_{s1}}^{-1}\right]^{-1} + \left[\left(\frac{\pi^2K_{b1}}{4H^2}\right)^{-1} + {K_{s1}}^{-1}\right]^{-1} \qquad (1279)$$

i.e.,

$$q_{cr} = \left[{q_{cr,flexiòn\,global}}^{-1} + {q_{cr,corte\,global}}^{-1}\right]^{-1} + \left[{q_{cr,flexiòn\,local}}^{-1} + {q_{cr,corte\,local}}^{-1}\right]^{-1} \qquad (1280)$$

Since the resulting critical load is independent of some approximation function, it can be considered exact and identical to the one that would be obtained by applying Föppl's theorem.

4.3.7.2 Case 2

- **Calculation of the Transfer Matrix**

According to the coupled differential equations:

$$\begin{Bmatrix} K_{b1}\psi''_{(x)} + K_{s1}\left[u'_{(x)} - \psi_{(x)}\right] = 0 \\ K_{b2}\theta''_{(x)} + K_{s2}\left[u'_{(x)} - \theta_{(x)}\right] = 0 \\ \left[K_{s1} + K_{s2} - f_{(x)}\right]u''_{(x)} - K_{s1}\psi'_{(x)} - K_{s2}\theta'_{(x)} = 0 \end{Bmatrix} \qquad (1281)$$

Using the method of coefficients:

$$\begin{bmatrix} K_{s1}D & 0 & K_{b1}D^2 - K_{s1} \\ K_{s2}D & K_{b2}D^2 - K_{s2} & 0 \\ (K_{s1} + K_{s2} - q)D^2 & -K_{s2}D & -K_{s1}D \end{bmatrix} \begin{Bmatrix} u_{(x)} \\ \theta_{(x)} \\ \psi_{(x)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (1282)$$

To avoid trivial solutions, the determinant must be equal to zero, that is:

$$D^2\left\{D^4 - \left[\frac{K_{s1}K_{s2}(K_{b1} + K_{b2}) - q(K_{b1}K_{s2} + K_{s1}K_{b2})}{K_{b1}K_{b2}(K_{s1} + K_{s2} - q)}\right]D^2 - \left[\frac{K_{s1}K_{s2}q}{K_{b1}K_{b2}(K_{s1} + K_{s2} - q)}\right]\right\} = 0 \qquad (1283)$$

Rewriting:

$$D^2(D^4 - r_1D^2 - r_2) = 0 \qquad (1284)$$

Where:

$$\begin{Bmatrix} r_1 = \dfrac{K_{s1}K_{s2}(K_{b1} + K_{b2}) - q(K_{b1}K_{s2} + K_{s1}K_{b2})}{K_{b1}K_{b2}(K_{s1} + K_{s2} - q)} \\ r_2 = \dfrac{K_{s1}K_{s2}q}{K_{b1}K_{b2}(K_{s1} + K_{s2} - q)} \end{Bmatrix} \qquad (1285)$$

The expression for $u_{(z)}$, $\psi_{(z)}$ and $\theta_{(z)}$ is proposed:

$$\begin{Bmatrix} u_{(z)} = C_0 + C_1z + C_2\cosh\left(\sqrt{\xi}z\right) + C_3\sinh\left(\sqrt{\xi}z\right) + C_4\cos\left(\sqrt{\beta}z\right) + C_5\sin\left(\sqrt{\beta}z\right) \\ \psi_{(z)} = C_6 + C_7\cosh\left(\sqrt{\xi}z\right) + C_8\sinh\left(\sqrt{\xi}z\right) + C_9\cos\left(\sqrt{\beta}z\right) + C_{10}\sin\left(\sqrt{\beta}z\right) \\ \theta_{(z)} = C_{11} + C_{12}\cosh\left(\sqrt{\xi}z\right) + C_{13}\sinh\left(\sqrt{\xi}z\right) + C_{14}\cos\left(\sqrt{\beta}z\right) + C_{15}\sin\left(\sqrt{\beta}z\right) \end{Bmatrix} \qquad (1286)$$

Where:

$$\left\{\begin{aligned} \xi &= \frac{r_1 + \sqrt{r_1^2 + 4r_2}}{2} \\ \beta &= \frac{-r_1 + \sqrt{r_1^2 + 4r_2}}{2} \end{aligned}\right\} \tag{1287}$$

Expressing the coefficients of $\psi_{(z)}$ and $\theta_{(z)}$ as a function of the coefficients of $u_{(z)}$:

$$\left\{\begin{gathered} u_{(z)} = C_0 + C_1 z + C_2 \cosh(\sqrt{\xi} z) + C_3 \sinh(\sqrt{\xi} z) + C_4 \cos(\sqrt{\beta} z) + C_5 \sin(\sqrt{\beta} z) \\ \psi_{(z)} = C_1 + [R_{\psi 1} \sinh(\sqrt{\xi} z)] C_2 + [R_{\psi 1} \cosh(\sqrt{\xi} z)] C_3 - [R_{\psi 2} \sin(\sqrt{\beta} z)] C_4 + [R_{\psi 2} \cos(\sqrt{\beta} z)] C_5 \\ \theta_{(z)} = C_1 + [R_{\theta 1} \sinh(\sqrt{\xi} z)] C_2 + [R_{\theta 1} \cosh(\sqrt{\xi} z)] C_3 - [R_{\theta 2} \sin(\sqrt{\beta} z)] C_4 + [R_{\theta 2} \cos(\sqrt{\beta} z)] C_5 \end{gathered}\right\} \tag{1288}$$

Where:

$$\left\{R_{\psi 1} = \frac{K_{s1}\sqrt{\xi}}{K_{s1} - \xi K_{b1}}, R_{\psi 2} = \frac{K_{s1}\sqrt{\xi}}{K_{s1} - \beta K_{b1}}, R_{\theta 1} = \frac{K_{s2}\sqrt{\beta}}{K_{s2} - \beta K_{b2}}, R_{\theta 2} = \frac{K_{s2}\sqrt{\beta}}{K_{s2} - \beta K_{b2}}\right\} \tag{1289}$$

Internal forces such as bending moment and shear force associated with lateral displacement result in:

$$\left\{\begin{gathered} \left\{\begin{gathered} M_{1(z)} = K_{b1}\psi'_{(x)} = \left[R_{\psi 1}\sqrt{\xi} K_{b1} \cosh\left(\sqrt{\xi} z\right)\right] C_2 + \left[R_{\psi 1}\sqrt{\xi} K_{b1} \sinh\left(\sqrt{\xi} z\right)\right] C_3 \\ - \left[R_{\psi 2}\sqrt{\beta} K_{b1} \cos\left(\sqrt{\beta} z\right)\right] C_4 - \left[R_{\psi 2}\sqrt{\beta} K_{b1} \sin\left(\sqrt{\beta} z\right)\right] C_5 \end{gathered}\right\} \\ \left\{\begin{gathered} M_{2(z)} = K_{b2}\theta'_{(x)} = \left[R_{\theta 1}\sqrt{\xi} K_{b1} \cosh\left(\sqrt{\xi} z\right)\right] C_2 + \left[R_{\theta 1}\sqrt{\xi} K_{b1} \sinh\left(\sqrt{\xi} z\right)\right] C_3 \\ - \left[R_{\theta 2}\sqrt{\beta} K_{b1} \cos\left(\sqrt{\beta} z\right)\right] C_4 - \left[R_{\theta 2}\sqrt{\beta} K_{b1} \sin\left(\sqrt{\beta} z\right)\right] C_5 \end{gathered}\right\} \\ \left\{\begin{gathered} V_{(z)} = (K_{s1} + K_{s2} - q) u'_{(x)} - K_{s1}\psi_{(x)} - K_{s2}\theta_{(x)} = -qC_1 + R_1 \sinh\left(\sqrt{\xi} z\right) C_2 \\ + R_1 \cosh\left(\sqrt{\xi} z\right) C_3 - R_2 \sin\left(\sqrt{\beta} z\right) C_4 + R_2 \cos\left(\sqrt{\beta} z\right) C_5 \end{gathered}\right\} \end{gathered}\right\} \tag{1290}$$

Where:

$$\left\{\begin{aligned} R_1 &= (K_{s1} + K_{s2} - q)\sqrt{\xi} - K_{s1}R_{\psi 1} - K_{s2}R_{\theta 1} \\ R_2 &= (K_{s1} + K_{s2} - q)\sqrt{\beta} - K_{s1}R_{\psi 2} - K_{s2}R_{\theta 2} \end{aligned}\right\} \tag{1291}$$

Writing the equations in matrix form:

$$\begin{Bmatrix} u_i(z_i) \\ \psi_i(z_i) \\ \theta_i(z_i) \\ M_{1\mathrm{i}}(z_i) \\ M_{2\mathrm{i}}(z_i) \\ V_i(z_i) \end{Bmatrix} = K_i(z_i) \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{Bmatrix} \tag{1292}$$

Where:

$$K_i(z_i) = \begin{bmatrix} 1 & z & \cosh(\sqrt{\xi}z) & \sinh(\sqrt{\xi}z) & \cos(\sqrt{\beta}z) & \sin(\sqrt{\beta}z) \\ 0 & 1 & R_{\psi 1}\sinh(\sqrt{\xi}z) & R_{\psi 1}\cosh(\sqrt{\xi}z) & -R_{\psi 2}\sin(\sqrt{\beta}z) & R_{\psi 2}\cos(\sqrt{\beta}z) \\ 0 & 1 & R_{\theta 1}\sinh(\sqrt{\xi}z) & R_{\theta 1}\cosh(\sqrt{\xi}z) & -R_{\theta 2}\sin(\sqrt{\beta}z) & R_{\theta 2}\cos(\sqrt{\beta}z) \\ 0 & 0 & R_{\psi 1}\sqrt{\xi}K_{b1}\cosh(\sqrt{\xi}z) & R_{\psi 1}\sqrt{\xi}K_{b1}\sinh(\sqrt{\xi}z) & -R_{\psi 2}\sqrt{\beta}K_{b1}\cos(\sqrt{\beta}z) & -R_{\psi 2}\sqrt{\beta}K_{b1}\sin(\sqrt{\beta}z) \\ 0 & 0 & R_{\theta 1}\sqrt{\xi}K_{b1}\cosh(\sqrt{\xi}z) & R_{\theta 1}\sqrt{\xi}K_{b1}\sinh(\sqrt{\xi}z) & -R_{\theta 2}\sqrt{\beta}K_{b1}\cos(\sqrt{\beta}z) & -R_{\theta 2}\sqrt{\beta}K_{b1}\sin(\sqrt{\beta}z) \\ 0 & -q & R_1\sinh(\sqrt{\xi}z) & R_1\cosh(\sqrt{\xi}z) & -R_2\sin(\sqrt{\beta}z) & R_2\cos(\sqrt{\beta}z) \end{bmatrix}_i \tag{1293}$$

- **Stability Analysis Under Point Loads Applied at Floor Level**

Applying sequentially from the base to the top of the beam and expressing the equation between the product symbol:

$$\begin{Bmatrix} u_n(0) \\ \psi_n(0) \\ \theta_n(0) \\ M_{1\mathrm{n}}(0) \\ M_{2\mathrm{n}}(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_k(0) \begin{Bmatrix} u_1(h_1) \\ \psi_1(h_1) \\ \theta_1(h_1) \\ M_{11}(h_1) \\ M_{21}(h_1) \\ V_1(h_1) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ \psi_1(h_1) \\ \theta_1(h_1) \\ M_{11}(h_1) \\ M_{21}(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1294}$$

Where:

$$\mathrm{t} = \prod_{k=1}^{n} T_k(0) \tag{1295}$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 6x6 and remains constant across all floors.Según las condiciones de contorno definidas en el caso 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ \psi_{(1)} = 0 \\ \theta_{(1)} = 0 \\ \psi'_{(0)} = 0 \\ \theta'_{(0)} = 0 \\ (K_{s1} + K_{s2} - q)u'_{(0)} - K_{s1}\psi_{(0)} - K_{s2}\theta_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1(h_1) = 0 \\ \psi_1(h_1) = 0 \\ \theta_1(h_1) = 0 \\ M_{1n\,(0)} = 0 \\ M_{2n\,(0)} = 0 \\ V_{n\,(0)} = 0 \end{Bmatrix} \tag{1296}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ \psi_n(0) \\ \theta_n(0) \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & t_{1,5} & t_{1,6} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} & t_{2,5} & t_{2,6} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} & t_{3,5} & t_{3,6} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,1} & t_{5,2} & t_{5,3} & t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,1} & t_{6,2} & t_{6,3} & t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ M_{11}(h_1) \\ M_{21}(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1297}$$

Solving for bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ M_2(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1298}$$

Which has a different solution than the trivial if the determinant is equal to zero (the matrix of coefficients is singular). Solving the critical loads of the beam.

4.3.8 Generalized Sandwich Beam of Three Field (GSB2)

4.3.8.1 Case 1

The potential energy of the three-field GSB2 model is expressed as follows:

$$V = \frac{1}{2}\int_0^H \left\{K_{b1}{\theta'_{(x)}}^2 + K_{s1}\left[\theta_{(x)} - \psi_{(x)}\right]^2 + K_{b2}{\psi'_{(x)}}^2\right\}dx + \frac{1}{2}\int_0^H K_{s2}\left[\psi_{(x)} - u'_{(x)}\right]^2 dx \tag{1299}$$

- Coupled shear wall:

$$\left\{\begin{matrix} K_{b1} = \sum_{i=1}^{w} EA_{wi}c_i^2\,, K_{b2} = r\sum_{i=1}^{w} EI_{wi} \\ K_{s1} = \sum_{i=1}^{b}\left[\frac{h}{L}\left(\frac{L^2}{12EI_b} + \frac{1}{GA'_b}\right)\right]^{-1}, K_{s2} = \sum_{i=1}^{w}\left[\frac{1}{2}\left(\frac{h^2}{\pi^2 EI_w} + \frac{1}{GA'_w}\right)\right]^{-1} \end{matrix}\right\} \tag{1300}$$

- Frame:

$$\left\{\begin{matrix} K_{b1} = \sum_{i=1}^{c} EA_{c,i}c_i^2\,, K_{b2} = \sum_{i=1}^{c} rEI_{c,i} \\ K_{s1} = \sum_{i=1}^{b}\left[\frac{h}{L}\left(\frac{L^2}{12EI_b} + \frac{1}{GA'_b}\right)\right]^{-1}, K_{s2} = \sum_{i=1}^{c}\left[\frac{1}{2}\left(\frac{h^2}{\pi^2 EI_c} + \frac{1}{GA'_c}\right)\right]^{-1} \end{matrix}\right\} \tag{1301}$$

- Dual (frame + shear wall):

$$\left\{\begin{matrix} K_{b1} = \sum_{i=1}^{c} EA_{c,i}c_i^2\,, K_{b2} = \sum_{i=1}^{c} rEI_{c,i} + \sum_{i=1}^{w} rEI_{w,i}\,, K_{s1} = \sum_{i=1}^{b}\left[\frac{h}{L}\left(\frac{L^2}{12EI_b} + \frac{1}{GA'_b}\right)\right]^{-1} \\ K_{s2} = \sum_{i=1}^{c}\left[\frac{1}{2}\left(\frac{h^2}{\pi^2 EI_c} + \frac{1}{GA'_c}\right)\right]^{-1} + \sum_{i=1}^{w}\left[\frac{1}{2}\left(\frac{h^2}{\pi^2 EI_w} + \frac{1}{GA'_w}\right)\right]^{-1} \end{matrix}\right\} \tag{1302}$$

The work done by the external force is expressed as:

$$W = -f_{(x)}dl = -\frac{1}{2}\int_0^H f_{(x)}{u'_{(x)}}^2\,dx \tag{1303}$$

Consequently, the total potential energy of the model is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H \left\{K_{b1}\theta_{(x)}^{'}{}^2 + K_{s1}\left[\theta_{(x)} - \psi_{(x)}\right]^2 + K_{b2}\psi_{(x)}^{'}{}^2 + K_{s2}\left[\psi_{(x)} - u_{(x)}^{'}\right]^2\right\} dx$$
$$- \frac{1}{2}\int_0^H f(x) u_{(x)}^{\prime}{}^2 \, dx \tag{1304}$$

Closed-form solutions of the model are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\delta\mathcal{U} = \int_0^H \left\{K_{b1}\theta_{(x)}^{\prime}\delta\theta_{(x)}^{\prime} + K_{s1}\left[\theta_{(x)} - \psi_{(x)}\right]\left[\delta\theta_{(x)} - \delta\psi_{(x)}\right] + K_{b2}\psi_{(x)}^{\prime}\delta\psi_{(x)}^{\prime}\right.$$
$$\left. + K_{s2}\left[\psi_{(x)} - u_{(x)}^{'}\right]\left[\delta\psi_{(x)} - \delta u_{(x)}^{'}\right] - f_{(x)}u_{(x)}^{\prime}\delta u_{(x)}^{\prime}\right\} dx$$
$$- \frac{1}{2}\int_0^H f(x) u_{(x)}^{\prime 2}\delta f_{(x)} dx \tag{1305}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\delta\mathcal{U} = \left\{[K_{s2} - f(x)]u_{(x)}^{\prime} - K_{s2}\psi_{(x)}\right\}\delta u_{(x)}\Big|_0^H + \left[K_{b1}\theta_{(x)}^{\prime}\delta\theta_{(x)}\right]_0^H + \left[K_{b2}\psi_{(x)}^{\prime}\delta\psi_{(x)}\right]_0^H$$
$$- \int_0^H \left\{[K_{s2} - f(x)]u_{(x)}^{\prime\prime} - K_{s2}\psi_{(x)}^{\prime} - f_{(x)}^{\prime}u_{(x)}^{\prime}\right\}\delta u_{(x)}$$
$$- \int_0^H \left\{K_{b1}\theta_{(x)}^{\prime\prime} - K_{s1}\left[\theta_{(x)} - \psi_{(x)}\right]\right\}\delta\theta_{(x)}$$
$$- \int_0^H \left[K_{b2}\psi_{(x)}^{\prime\prime} - (K_{s1} + K_{s2})\psi_{(x)} + K_{s1}\theta_{(x)} + K_{s2}u_{(x)}^{\prime}\right]\delta\psi_{(x)}$$
$$- \frac{1}{2}\int_0^H f(x) u_{(x)}^{\prime 2}\delta f_{(x)} dx \tag{1306}$$

Setting the terms equal to zero, the following equations result:

$$\left\{\begin{matrix} [K_{s2} - f(x)]u_{(x)}^{\prime\prime} - K_{s2}\psi_{(x)}^{\prime} - f_{(x)}^{\prime}u_{(x)}^{\prime} = 0 \\ K_{b1}\theta_{(x)}^{\prime\prime} - K_{s1}\left[\theta_{(x)} - \psi_{(x)}\right] = 0 \\ K_{b2}\psi_{(x)}^{\prime\prime} - (K_{s1} + K_{s2})\psi_{(x)} + K_{s1}\theta_{(x)} + K_{s2}u_{(x)}^{\prime} = 0 \end{matrix}\right\} \tag{1307}$$

And boundary conditions:

$$\begin{Bmatrix} [K_{s2}-f(0)]u'_{(0)}-K_{s2}\psi_{(0)}=0 \\ \theta'_{(0)}=0 \\ \psi'_{(0)}=0 \end{Bmatrix} \tag{1308}$$

Integrating the equation once and evaluating at x=0:

$$[K_{s2}-f(x)]u'_{(x)}-K_{s2}\psi_{(x)}=0 \tag{1309}$$

We have a new system of coupled differential equations:

$$\begin{Bmatrix} [K_{s2}-f(x)]u'_{(x)}-K_{s2}\psi_{(x)}=0 \\ K_{b1}\theta''_{(x)}-K_{s1}\left[\theta_{(x)}-\psi_{(x)}\right]=0 \\ K_{b2}\psi''_{(x)}-(K_{s1}+K_{s2})\psi_{(x)}+K_{s1}\theta_{(x)}+K_{s2}u'_{(x)}=0 \end{Bmatrix} \tag{1310}$$

Using the method of coefficients for the solution of the system of equations:

$$\begin{bmatrix} [K_{s2}-f(x)]D & 0 & -K_{s2} \\ 0 & K_{b1}D^2-K_{s1} & K_{s1} \\ K_{s2}D & K_{s1} & K_{b2}D^2-(K_{s1}+K_{s2}) \end{bmatrix} \begin{Bmatrix} u_{(x)} \\ \theta_{(x)} \\ \psi_{(x)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{1311}$$

Which has a solution other than the trivial one if the determinant is equal to zero:

$$u'''''_{(x)}-\left[\frac{K_{s1}(K_{b1}+K_{b2})}{K_{b1}K_{b2}}\right]u'''_{(x)}+f(x)\left\{-\frac{1}{K_{s2}}u'''''_{(x)}+\left(\frac{K_{s1}K_{b2}+K_{s2}K_{b1}+K_{s1}K_{b1}}{K_{b1}K_{b2}K_{s2}}\right)u'''_{(x)}-\frac{K_{s1}}{K_{b1}K_{b2}}u'_{(x)}\right\}=0 \tag{1312}$$

Or its equivalent:

$$\frac{K_{b1}K_{b2}}{K_{s1}}u'''''_{(x)}-(K_{b1}+K_{b2})u'''_{(x)}+f(x)\left\{-\frac{K_{b1}K_{b2}}{K_{s1}K_{s2}}u'''''_{(x)}+\left[\frac{K_{b2}}{K_{s2}}+K_{b1}\left(\frac{1}{K_{s1}}+\frac{1}{K_{s2}}\right)\right]u'''_{(x)}-u'_{(x)}\right\}=0 \tag{1313}$$

A fourth order differential equation is obtained, where the critical load results from the smallest eigenvalue. Normalizing the differential equation by the variable z=x/H:

$$\begin{aligned} u'''''_{(z)}-\left[\frac{K_{s1}(K_{b1}+K_{b2})}{K_{b1}K_{b2}}H^2\right]u'''_{(z)} \\ +f(z)\left\{-\frac{1}{K_{s2}}u'''''_{(z)}+\left[\left(\frac{K_{s1}K_{b2}+K_{s2}K_{b1}+K_{s1}K_{b1}}{K_{b1}K_{b2}K_{s2}}\right)H^2\right]u'''_{(z)}-\left(\frac{K_{s1}}{K_{b1}K_{b2}}H^4\right)u'_{(z)}\right\}=0 \end{aligned}$$

$$ \tag{1314} $$

Where:

$$f_{(z)} = q\alpha_{(z)} \tag{1315}$$

The equation can be rewritten as:

$$u_{(z)}''''' - a_0 u_{(z)}''' + q\alpha_{(z)}\left[-a_1 u_{(z)}''''' + a_2 u_{(z)}''' - a_3 u_{(z)}'\right] = 0 \tag{1316}$$

Where:

$$a_0 = \frac{K_{s1}(K_{b1} + K_{b2})}{K_{b1}K_{b2}} H^2, a_1 = \frac{1}{K_{s2}}$$
$$a_2 = \left(\frac{K_{s1}K_{b2} + K_{s2}K_{b1} + K_{s1}K_{b1}}{K_{b1}K_{b2}K_{s2}}\right) H^2, a_3 = \frac{K_{s1}}{K_{b1}K_{b2}} H^4 \tag{1317}$$

Expressing the boundary conditions as a function of $u_{(z)}$:

$$\left\{\begin{matrix} u_{(1)} = 0 \\ u_{(1)}' = 0 \\ u_{(0)}'' = 0 \\ u_{(1)}''' = 0 \\ u_{(0)}'''' = 0 \end{matrix}\right\} \tag{1318}$$

- **Uniformly Distributed Load**

For beam stability, the governing differential equation is of the form:

$$\left(\frac{d^5}{dz^5} - a_0 \frac{d^3}{dz^3}\right) u_{(z)} - q\left[\alpha_{(z)}\left(a_1 \frac{d^5}{dz^5} - a_2 \frac{d^3}{dz^3} + a_3 \frac{d}{dz}\right)\right] u_{(z)} = 0 \tag{1319}$$

Multiplying the equation by $\left(a_1 \frac{d^5}{dz^5} - a_2 \frac{d^3}{dz^3} + a_3 \frac{d}{dz}\right)$, integrating from 0 to 1 and solving:

$$\lambda = \frac{\int_0^1 \{a_1 u'''''^2_{(z)} + (a_0 a_1 + a_2) u''''^2_{(z)} + (a_0 a_2 + a_3) u'''^2_{(z)} + a_0 a_3 u''^2_{(z)}\} dz}{\int_0^1 \alpha_{(z)} \left[a_1 u_{(z)}''''' - a_2 u_{(z)}''' + a_3 u_{(z)}'\right]^2 dz} \tag{1320}$$

This Rayleigh ratio represents an approximation of the upper limit of the critical load, and it is exact if and only if the exact equilibrium curve $u_{(z)}$ is used to calculate λ.

For the case of a uniformly distributed load, the function $\alpha_{(z)}$ results in:

$$\alpha_{(z)} = z \rightarrow f_{(z)} = qz \tag{1321}$$

The Rayleigh quotient becomes:

$$\lambda = \frac{\int_0^1 \{a_1 u''''^2_{(z)} + (a_0 a_1 + a_2) u''''^2_{(z)} + (a_0 a_2 + a_3) u'''^2_{(z)} + a_0 a_3 u''^2_{(z)}\} dz}{\int_0^1 z \left[a_1 u'''''_{(z)} - a_2 u'''_{(z)} + a_3 u'_{(z)}\right]^2 dz} \tag{1322}$$

- **Point Load at x=0 (z=0)**

For the case of a point load applied at x=H (z=1), the function $\alpha_{(z)}$:

$$\alpha_{(z)} = 1 \rightarrow f_{(z)} = q \tag{1323}$$

Substituting into the differential equation:

$$u'''''_{(z)} - a_0 u'''_{(z)} + q\left[-a_1 u'''''_{(z)} + a_2 u'''_{(z)} - a_3 u'_{(z)}\right] = 0 \tag{1324}$$

The expression for $u_{(z)}$ can be derived as:

$$u_{(z)} = C_0 + C_1 \cosh\left(\sqrt{\xi} z\right) + C_2 \sinh\left(\sqrt{\xi} z\right) + C_3 \cos\left(\sqrt{\beta} z\right) + C_4 \sin\left(\sqrt{\beta} z\right) \tag{1325}$$

Where:

$$\begin{Bmatrix} \xi = \dfrac{(a_0 - q a_2) + \sqrt{(a_0 - q a_2)^2 + 4 q a_3 (1 - q a_1)}}{2(1 - q a_1)} \\ \beta = \dfrac{-(a_0 - q a_2) + \sqrt{(a_0 - q a_2)^2 + 4 q a_3 (1 - q a_1)}}{2(1 - q a_1)} \end{Bmatrix} \tag{1326}$$

The linear algebraic system resulting from the boundary conditions, written in matrix form, is:

$$\begin{bmatrix} 1 & \cosh\left(\sqrt{\xi}\right) & \sinh\left(\sqrt{\xi}\right) & \cos\left(\sqrt{\beta}\right) & \sin\left(\sqrt{\beta}\right) \\ 0 & \xi^{1/2} \sinh\left(\sqrt{\xi}\right) & \xi^{1/2} \cosh\left(\sqrt{\xi}\right) & -\beta^{1/2} \sin\left(\sqrt{\beta}\right) & \beta^{1/2} \cos\left(\sqrt{\beta}\right) \\ 0 & \xi & 0 & -\beta & 0 \\ 0 & \xi^{3/2} \sinh\left(\sqrt{\xi}\right) & \xi^{3/2} \cosh\left(\sqrt{\xi}\right) & \beta^{3/2} \sin\left(\sqrt{\beta}\right) & -\beta^{3/2} \cos\left(\sqrt{\beta}\right) \\ 0 & \xi^2 & 0 & \beta^2 & 0 \end{bmatrix} \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = 0 \tag{1327}$$

Which has a different solution than the trivial one if the determinant is equal to zero (the matrix of coefficients is singular), that is, for:

$$Cos\sqrt{\beta} = 0 \rightarrow \sqrt{\beta} = (2n-1)\frac{\pi}{2} \;/\; n = 1, 2, 3 \ldots \tag{1328}$$

i.e.,

$$\frac{-(a_0 - qa_2) + \sqrt{(a_0 - qa_2)^2 + 4qa_3(1 - qa_1)}}{2(1 - qa_1)} = (2n-1)^2\frac{\pi^2}{4} \tag{1329}$$

Solving:

$$q_{cr} = \frac{\frac{(2n-1)^4\pi^4}{4} + a_0(2n-1)^2\pi^2}{4a_0 + a_2(2n-1)^2\pi^2 + a_1\frac{(2n-1)^4\pi^4}{4}} \tag{1330}$$

Replacing the coefficients and after some simple manipulations:

$$q_{cr} = \frac{1}{\left[\frac{1}{\frac{4H^2}{(2n-1)^2\pi^2 K_{b1}} + \frac{1}{K_{s1}}} + (2n-1)^2\frac{\pi^2 K_{b2}}{4H^2}\right]^{-1} + \frac{1}{K_{s2}}} \tag{1331}$$

Sorting properly:

$$q_{cr} = \left\{\left\{\left\{\left[(2n-1)^2\frac{\pi^2 K_{b1}}{4H^2}\right]^{-1} + K_{s1}^{-1}\right\}^{-1} + (2n-1)^2\frac{\pi^2 K_{b2}}{4H^2}\right\}^{-1} + K_{s2}^{-1}\right\}^{-1} \tag{1332}$$

For the case when $n = 1$, we have:

$$q_{cr} = \left\{\left\{\left[\left(\frac{\pi^2 K_{b1}}{4H^2}\right)^{-1} + K_{s1}^{-1}\right]^{-1} + \frac{\pi^2 K_{b2}}{4H^2}\right\}^{-1} + K_{s2}^{-1}\right\}^{-1} \tag{1333}$$

i.e.,

$$q_{cr} = \left\{\left[\left(q_{cr,flexión\ global}^{-1} + q_{cr,corte\ global}^{-1}\right)^{-1} + q_{cr,flexión\ local}\right]^{-1} + q_{cr,corte\ local}^{-1}\right\}^{-1} \quad (1334)$$

Since the resulting critical load is independent of some approximation function, it can be considered exact and identical to the one that would be obtained by applying Föppl's theorem.

4.3.8.2 Case 2

- **Calculation of the Transfer Matrix**

According to the coupled differential equations:

$$\begin{Bmatrix} K_{s2}u''_{(x)} - K_{s2}\psi'_{(x)} = 0 \\ K_{b1}\theta''_{(x)} - K_{s1}\left[\theta_{(x)} - \psi_{(x)}\right] = 0 \\ K_{b2}\psi''_{(x)} - (K_{s1} + K_{s2})\psi_{(x)} + K_{s1}\theta_{(x)} + K_{s2}u'_{(x)} = 0 \end{Bmatrix} \quad (1335)$$

Using the method of coefficients:

$$\begin{bmatrix} K_{s2}D^2 & 0 & -K_{s2}D \\ 0 & K_{b1}D^2 - K_{s1} & K_{s1} \\ K_{s2}D & K_{s1} & K_{b2}D^2 - (K_{s1} + K_{s2}) \end{bmatrix} \begin{Bmatrix} u_{(x)} \\ \theta_{(x)} \\ \psi_{(x)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (1336)$$

To avoid trivial solutions, the determinant must be equal to zero, that is:

$$D^2\left\{D^4 - \left[\frac{K_{s1}K_{s2}(K_{b1} + K_{b2}) - q(K_{s1}K_{b2} + K_{s2}K_{b1} + K_{s1}K_{b1})}{K_{b1}K_{b2}(K_{s2} - q)}\right]D^2 - \left[\frac{K_{s1}K_{s2}q}{K_{b1}K_{b2}(K_{s2} - q)}\right]\right\} = 0 \quad (1337)$$

Rewriting:

$$D^2(D^4 - r_1D^2 - r_2) = 0 \quad (1338)$$

Where:

$$\begin{Bmatrix} r_1 = \dfrac{K_{s1}K_{s2}(K_{b1} + K_{b2}) - q(K_{s1}K_{b2} + K_{s2}K_{b1} + K_{s1}K_{b1})}{K_{b1}K_{b2}(K_{s2} - q)} \\ r_2 = \dfrac{K_{s1}K_{s2}q}{K_{b1}K_{b2}(K_{s2} - q)} \end{Bmatrix} \quad (1339)$$

The expression for $u_{(z)}$, $\psi_{(z)}$ and $\theta_{(z)}$ is proposed:

$$\begin{cases} u_{(z)} = C_0 + C_1 z + C_2 \cosh\left(\sqrt{\xi} z\right) + C_3 \sinh\left(\sqrt{\xi} z\right) + C_4 \cos\left(\sqrt{\beta} z\right) + C_5 \sin\left(\sqrt{\beta} z\right) \\ \theta_{(z)} = C_6 + C_7 \cosh\left(\sqrt{\xi} z\right) + C_8 \sinh\left(\sqrt{\xi} z\right) + C_9 \cos\left(\sqrt{\beta} z\right) + C_{10} \sin\left(\sqrt{\beta} z\right) \\ \psi_{(z)} = C_{11} + C_{12} \cosh\left(\sqrt{\xi} z\right) + C_{13} \sinh\left(\sqrt{\xi} z\right) + C_{14} \cos\left(\sqrt{\beta} z\right) + C_{15} \sin\left(\sqrt{\beta} z\right) \end{cases} \quad (1340)$$

Where:

$$\left\{ \xi = \frac{r_1 + \sqrt{r_1^2 + 4r_2}}{2}, \beta = \frac{-r_1 + \sqrt{r_1^2 + 4r_2}}{2} \right\} \quad (1341)$$

Expressing the coefficients of $\psi_{(z)}$ and $\theta_{(z)}$ as a function of the coefficients of $u_{(z)}$:

$$\begin{cases} u_{(z)} = C_0 + C_1 z + C_2 \cosh\left(\sqrt{\xi} z\right) + C_3 \sinh\left(\sqrt{\xi} z\right) + C_4 \cos\left(\sqrt{\beta} z\right) + C_5 \sin\left(\sqrt{\beta} z\right) \\ \theta_{(z)} = C_1 + \left[p_1 \sqrt{\xi} \sinh\left(\sqrt{\xi} z\right)\right] C_2 + \left[p_1 \sqrt{\xi} \cosh\left(\sqrt{\xi} z\right)\right] C_3 - \left[p_3 \sqrt{\beta} \sin\left(\sqrt{\beta} z\right)\right] C_4 + \left[p_3 \sqrt{\beta} \cos\left(\sqrt{\beta} z\right)\right] C_5 \\ \psi_{(z)} = C_1 + \left[p_2 \sqrt{\xi} \sinh\left(\sqrt{\xi} z\right)\right] C_2 + \left[p_2 \sqrt{\xi} \cosh\left(\sqrt{\xi} z\right)\right] C_3 - \left[p_4 \sqrt{\beta} \sin\left(\sqrt{\beta} z\right)\right] C_4 + \left[p_4 \sqrt{\beta} \cos\left(\sqrt{\beta} z\right)\right] C_5 \end{cases} \quad (1342)$$

Where:

$$\begin{cases} p_1 = \dfrac{K_{s1} K_{s2}}{K_{b1} K_{b2} \xi^2 - (K_{s1} K_{b2} + K_{s2} K_{b1} + K_{s1} K_{b1}) \xi + K_{s1} K_{s2}} \\ p_2 = \dfrac{-(K_{b1} \xi - K_{s1}) K_{s1}}{K_{b1} K_{b2} \xi^2 - (K_{s1} K_{b2} + K_{s2} K_{b1} + K_{s1} K_{b1}) \xi + K_{s1} K_{s2}} \\ p_3 = \dfrac{K_{s1} K_{s2}}{K_{b1} K_{b2} \beta^2 - (K_{s1} K_{b2} + K_{s2} K_{b1} + K_{s1} K_{b1}) \beta + K_{s1} K_{s2}} \\ p_4 = \dfrac{(K_{b1} \beta + K_{s1}) K_{s1}}{K_{b1} K_{b2} \beta^2 - (K_{s1} K_{b2} + K_{s2} K_{b1} + K_{s1} K_{b1}) \beta + K_{s1} K_{s2}} \end{cases} \quad (1343)$$

Internal forces such as bending moment and shear force associated with lateral displacement result in:

$$\begin{cases} \begin{Bmatrix} M_{1(z)} = K_{b1} \theta'_{(x)} = \left[p_1 \xi K_{b1} \cosh\left(\sqrt{\xi} z\right)\right] C_2 + \left[p_1 \xi K_{b1} \sinh\left(\sqrt{\xi} z\right)\right] C_3 \\ -\left[p_3 \beta K_{b1} \cos\left(\sqrt{\beta} z\right)\right] C_4 - \left[p_3 \beta K_{b1} \sin\left(\sqrt{\beta} z\right)\right] C_5 \end{Bmatrix} \\ \begin{Bmatrix} M_{2(z)} = K_{b2} \psi'_{(x)} = \left[p_2 \xi K_{b2} \cosh\left(\sqrt{\xi} z\right)\right] C_2 + \left[p_2 \xi K_{b2} \sinh\left(\sqrt{\xi} z\right)\right] C_3 \\ -\left[p_4 \beta K_{b2} \cos\left(\sqrt{\beta} z\right)\right] C_4 - \left[p_4 \beta K_{b2} \sin\left(\sqrt{\beta} z\right)\right] C_5 \end{Bmatrix} \\ \begin{Bmatrix} V_{(z)} = (K_{s2} - q) u'_{(x)} - K_{s2} \psi_{(x)} = -q C_1 + R_1 \sqrt{\xi} \sinh\left(\sqrt{\xi} z\right) C_2 + R_1 \sqrt{\xi} \cosh\left(\sqrt{\xi} z\right) C_3 \\ -R_2 \sqrt{\beta} \sin\left(\sqrt{\beta} z\right) C_4 + R_2 \sqrt{\beta} \cos\left(\sqrt{\beta} z\right) C_5 \end{Bmatrix} \end{cases} \quad (1344)$$

Where:

$$\{R_1 = K_{s2}(1-p_2) - q, R_2 = K_{s2}(1-p_4) - q\} \tag{1345}$$

Writing the equations in matrix form:

$$\begin{Bmatrix} u_i(z_i) \\ \psi_i(z_i) \\ \theta_i(z_i) \\ M_{1\mathrm{i}}(z_i) \\ M_{2\mathrm{i}}(z_i) \\ V_i(z_i) \end{Bmatrix} = K_i(z_i) \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{Bmatrix} \tag{1346}$$

Where:

$$K_i(z_i) = \begin{bmatrix} 1 & z & \cosh(\sqrt{\xi}z) & \sinh(\sqrt{\xi}z) & \cos(\sqrt{\beta}z) & \sin(\sqrt{\beta}z) \\ 0 & 1 & p_1\sqrt{\xi}\sinh(\sqrt{\xi}z) & p_1\sqrt{\xi}\cosh(\sqrt{\xi}z) & -p_3\sqrt{\beta}\sin(\sqrt{\beta}z) & p_3\sqrt{\beta}\cos(\sqrt{\beta}z) \\ 0 & 1 & p_2\sqrt{\xi}\sinh(\sqrt{\xi}z) & p_2\sqrt{\xi}\cosh(\sqrt{\xi}z) & -p_4\sqrt{\beta}\sin(\sqrt{\beta}z) & p_4\sqrt{\beta}\cos(\sqrt{\beta}z) \\ 0 & 0 & p_1\xi K_{b1}\cosh(\sqrt{\xi}z) & p_1\xi K_{b1}\sinh(\sqrt{\xi}z) & -p_3\beta K_{b1}\cos(\sqrt{\beta}z) & -p_3\beta K_{b1}\sin(\sqrt{\beta}z) \\ 0 & 0 & p_2\xi K_{b2}\cosh(\sqrt{\xi}z) & p_2\xi K_{b2}\sinh(\sqrt{\xi}z) & -p_4\beta K_{b2}\cos(\sqrt{\beta}z) & -p_4\beta K_{b2}\sin(\sqrt{\beta}z) \\ 0 & -q & R_1\sqrt{\xi}\sinh(\sqrt{\xi}z) & R_1\sqrt{\xi}\cosh(\sqrt{\xi}z) & -R_2\sqrt{\beta}\sin(\sqrt{\beta}z) & R_2\sqrt{\beta}\cos(\sqrt{\beta}z) \end{bmatrix}_i \tag{1347}$$

- **Stability Analysis Under Point Loads Applied at Floor Level**

Applying sequentially from the base to the top of the beam and expressing the equation between the product symbol:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ \psi_n(0) \\ M_{1\mathrm{n}}(0) \\ M_{2\mathrm{n}}(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_k(0) \begin{Bmatrix} u_1(h_1) \\ \theta_1(h_1) \\ \psi_1(h_1) \\ M_{11}(h_1) \\ M_{21}(h_1) \\ V_1(h_1) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ \theta_1(h_1) \\ \psi_1(h_1) \\ M_{11}(h_1) \\ M_{21}(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1348}$$

Where:

$$\mathrm{t} = \prod_{k=1}^{n} T_k(0) \tag{1349}$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 6x6 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ \theta_{(1)} = 0 \\ \psi_{(1)} = 0 \\ \psi'_{(0)} = 0 \\ \theta'_{(0)} = 0 \\ [K_{s2} - q]u'_{(0)} - K_{s2}\psi_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1(h_1) = 0 \\ \theta_1(h_1) = 0 \\ \psi_1(h_1) = 0 \\ M_{1n\,(0)} = 0 \\ M_{2n\,(0)} = 0 \\ V_{n\,(0)} = 0 \end{Bmatrix} \tag{1350}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ \psi_n(0) \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & t_{1,5} & t_{1,6} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} & t_{2,5} & t_{2,6} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} & t_{3,5} & t_{3,6} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,1} & t_{5,2} & t_{5,3} & t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,1} & t_{6,2} & t_{6,3} & t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ M_{11}(h_1) \\ M_{21}(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1351}$$

Solving for bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ M_2(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1352}$$

Which has a different solution than the trivial if the determinant is equal to zero (the matrix of coefficients is singular). Solving the critical loads of the beam.

4.3.9 Modified Generalized Sandwich Beam of Two Field (MGSB1)

4.3.9.1 Case 1

The potential energy of the two-field MGSB1 model is expressed as follows:

$$V = \frac{1}{2}\int_0^H \left\{K_b^* {\theta'_{(x)}}^2 + K_s^*\left[u'_{(x)} - \theta_{(x)}\right]^2\right\} dx \tag{1353}$$

Where:

$$\left\{\begin{matrix} K_b^* = \eta \sum_{i=1}^{c} rEI_{c,i} + (1-\eta)\sum_{i=1}^{c} EA_{c,i}c_i^2 \,, K_s^* = \sum_{i=1}^{c} GA_{c,i} + \left(K_b^{-1} + K_c^{-1}\right)^{-1} \\ K_b = \sum_{i=1}^{b} \frac{6EI_{b,i}[(l^* + S_1)^2 + (l^* + S_2)^2]}{l^{*3}h\left(1 + 12\frac{kEI_{b,i}}{l^{*2}GA_{b,i}}\right)}, K_c = \sum_{i=1}^{c} \frac{\pi^2 EI_{c,i}}{h^2} \end{matrix}\right\} \tag{1354}$$

The equation can that allows to determine the critical load, according to the beam TB it results:

$$\theta''_{(z)} - \lambda\alpha_{(z)}\left[\frac{1}{\alpha^2}\theta''_{(z)} - \theta_{(z)}\right] = 0 \tag{1355}$$

Where:

$$\left\{\alpha = H\sqrt{\frac{K_s^*}{K_b^*}}, \lambda = \frac{qH^3}{K_b^*}\right\} \tag{1356}$$

Subject to boundary conditions:

$$\left\{\begin{matrix} \theta_{(1)} = 0 \\ \theta'_{(0)} = 0 \\ \theta''_{(0)} = 0 \end{matrix}\right\} \tag{1357}$$

- **Uniformly Distributed Load**

The Rayleigh quotient, according to the beam TB, is:

$$\lambda = \frac{\int_0^1 \left[\frac{1}{\alpha^2}\theta''^2_{(z)} + \theta'^2_{(z)}\right] dz}{\int_0^1 z\left[\frac{1}{\alpha^2}\theta''_{(z)} - \theta_{(z)}\right]^2 dz} \tag{1358}$$

- **Point Load at x=0 (z=0)**

The critical load, according to the beam TB, is:

$$q_{cr} = \left\{\left[(2n-1)^2 \frac{\pi^2 K_b^*}{4H^2}\right]^{-1} + K_s^{*-1}\right\}^{-1} \tag{1359}$$

For the case when $n = 1$, we have:

$$q_{cr} = \left[\left(\frac{\pi^2 K_b^*}{4H^2}\right)^{-1} + K_s^{*-1}\right]^{-1} = \left(q_{cr,flexiòn\,global}{}^{-1} + q_{cr,corte}{}^{-1}\right)^{-1} \tag{1360}$$

Since the resulting critical load is independent of some approximation function, it can be considered exact and identical to the one that would be obtained by applying Föppl's theorem.

4.3.9.2 Case 2

- **Calculation of the Transfer Matrix**

The transfer matrix, according to the beam TB, results:

$$K_i(z_i) = \begin{bmatrix} 1 & z_i & \cos(\sqrt{\xi}z) & \sin(\sqrt{\xi}z) \\ 0 & 1 & -\dfrac{K_s\sqrt{\xi}\sin(\sqrt{\xi}z)}{\dfrac{\xi}{\alpha^{*2}}+1} & -\dfrac{K_b\xi}{\dfrac{\xi}{\alpha^{*2}}+1}\sin(\sqrt{\xi}z) \\ 0 & 0 & -\dfrac{K_b\xi}{\dfrac{\xi}{\alpha^{*2}}+1}\cos(\sqrt{\xi}z) & -\dfrac{K_b\xi}{\dfrac{\xi}{\alpha^{*2}}+1}\sin(\sqrt{\xi}z) \\ 0 & q & \left(\dfrac{K_b\xi}{\dfrac{\xi}{\alpha^{*2}}+1} - q\right)\sqrt{\xi}\sin(\sqrt{\xi}z) & -\left(\dfrac{K_b\xi}{\dfrac{\xi}{\alpha^{*2}}+1} - q\right)\sqrt{\xi}\cos(\sqrt{\xi}z) \end{bmatrix}_i \tag{1361}$$

Where:

$$\left\{\xi = \frac{qK_s^*}{(K_s - q)K_b^*}, \alpha^* = \sqrt{\frac{K_s^*}{K_b^*}}\right\} \quad (1362)$$

- **Stability Analysis Under Point Loads Applied at Floor Level**

The critical load is obtained by setting the determinant equal to zero (the matrix of coefficients is singular):

$$\begin{vmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{vmatrix} = 0 \quad (1363)$$

Where:

$$\mathrm{t} = \prod_{k=1}^{n} T_k(0) \quad (1364)$$

4.3.10 Parallel Coupling of Shear Beam and Timoshenko Beam of Two Field (MCTB)

4.3.10.1 Case 1

The potential energy of the two-field MCTB model is expressed as follows:

$$V = \frac{1}{2}\int_0^H \left\{K_{b2}{\theta'_{(x)}}^2 + K_{s2}\left[\theta_{(x)} - u'_{(x)}\right]^2\right\} dx + \frac{1}{2}\int_0^H K_{s1}u'^2_{(x)}\, dx \tag{1365}$$

Where:

$$\left\{\begin{matrix} K_{s1} = \left(K_b^{-1} + K_c^{-1}\right)^{-1}, K_{b2} = \sum_{i=1}^{c} rEI_{c,i}\ , K_{s2} = \sum_{i=1}^{c} GA_{c,i}\ , \\ K_b = \sum_{i=1}^{b} \frac{12EI_{b,i}}{hL}\ , K_c = \sum_{i=1}^{c} \frac{\pi^2 EI_{c,i}}{h^2} \end{matrix}\right\} \tag{1366}$$

The work done by the external force is expressed as:

$$W = -f_{(x)}dl = -\frac{1}{2}\int_0^H f_{(x)}{u'_{(x)}}^2\, dx \tag{1367}$$

Consequently, the total potential energy of the model is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H \left\{K_{b2}{\theta'_{(x)}}^2 + K_{s2}\left[\theta_{(x)} - u'_{(x)}\right]^2 + K_{s1}u'^2_{(x)}\right\} dx - \frac{1}{2}\int_0^H f(x){u'_{(x)}}^2\, dx \tag{1368}$$

Closed-form solutions of the model are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\begin{aligned} \delta\mathcal{U} = \int_0^H \{K_{b2}\theta'_{(x)}\delta\theta'_{(x)} + K_{s2}\left[\theta_{(x)} - u'_{(x)}\right]\delta\theta_{(x)} - K_{s2}\left[\theta_{(x)} - u'_{(x)}\right]\delta u'_{(x)} + K_{s1}u'_{(x)}\delta u'_{(x)} \\ - f_{(x)}u'_{(x)}\delta u'_{(x)}\}dx - \frac{1}{2}\int_0^H f(x)u'^2_{(x)}\delta f_{(x)}dx \end{aligned} \tag{1369}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\delta \mathcal{U} = \left[K_{b2}\theta'_{(x)}\delta\theta_{(x)}\right]_0^H + \left\{\left[K_{s1} + K_{s2} - f_{(x)}\right]u'_{(x)} - K_{s2}\theta_{(x)}\right\}\delta u_{(x)}{}_0^H$$
$$- \int_0^H \left\{K_{b2}\theta''_{(x)} - K_{s2}\left[\theta_{(x)} - u'_{(x)}\right]\right\}\delta\theta_{(x)}dx$$
$$- \int_0^H \left\{\left[K_{s1} + K_{s2} - f_{(x)}\right]u''_{(x)} - K_{s2}\theta'_{(x)} - f'_{(x)}u'_{(x)}\right\}\delta u_{(x)}\,dx$$
$$- \frac{1}{2}\int_0^H f(x)u'^2_{(x)}\delta f_{(x)}dx \tag{1370}$$

Setting the terms equal to zero, the following equations result:

$$\begin{Bmatrix} K_{b2}\theta^{''}_{(x)} - K_{s2}\left[\theta_{(x)} - u^{'}_{(x)}\right] = 0 \\ \left[K_{s1} + K_{s2} - f^{'}_{(x)}\right]u^{''}_{(x)} - K_{s2}\theta^{'}_{(x)} - f_{(x)}u^{''}_{(x)} = 0 \end{Bmatrix} \tag{1371}$$

And boundary conditions:

$$\begin{Bmatrix} \theta^{'}_{(0)} = 0 \\ \left[K_{s1} + K_{s2} - f_{(0)}\right]u^{'}_{(0)} - K_{s2}\theta_{(0)} = 0 \end{Bmatrix} \tag{1372}$$

Integrating the equation once and evaluating at $x = 0$:

$$\left[K_{s1} + K_{s2} - f_{(x)}\right]u'_{(x)} - K_{s2}\theta_{(x)} = 0 \tag{1373}$$

We have a new system of coupled differential equations:

$$\begin{Bmatrix} K_{b2}\theta^{''}_{(x)} - K_{s2}\left[\theta_{(x)} - u^{'}_{(x)}\right] = 0 \\ \left[K_{s1} + K_{s2} - f_{(x)}\right]u^{'}_{(x)} - K_{s2}\theta_{(x)} = 0 \end{Bmatrix} \tag{1374}$$

Using the method of coefficients for the solution of the system of equations:

$$\begin{bmatrix} K_{s2}D & -K_{s2} + K_{b2}D^2 \\ \left[K_{s1} + K_{s2} - f_{(x)}\right]D & -K_{s2} \end{bmatrix}\begin{Bmatrix} u_{(x)} \\ \theta_{(x)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1375}$$

The determinant is equal to zero (the coefficient matrix is singular):

$$K_{b2}(K_{s1} + K_{s2})u'''_{(x)} - K_{s1}K_{s2}u'_{(x)} - f_{(x)}\left[K_{b2}u'''_{(x)} - K_{s2}u'_{(x)}\right] = 0 \tag{1376}$$

Reordering:

$$u'''_{(x)} - \frac{K_{s1}K_{s2}}{K_{b2}(K_{s1}+K_{s2})}u'_{(x)} - f_{(x)}\left[\frac{1}{K_{s1}+K_{s2}}u'''_{(x)} - \frac{K_{s2}}{K_{b2}(K_{s1}+K_{s2})}u'_{(x)}\right] = 0 \tag{1377}$$

A third order differential equation is obtained, where the critical load results from the smallest eigenvalue. Normalizing the differential equation by the variable $z = x/H$:

$$u'''_{(z)} - \frac{K_{s1}K_{s2}}{K_{b2}(K_{s1}+K_{s2})}H^2u'_{(z)} - f_{(z)}\left[\frac{1}{K_{s1}+K_{s2}}u'''_{(z)} - \frac{K_{s2}}{K_{b2}(K_{s1}+K_{s2})}H^2u'_{(z)}\right] = 0 \tag{1378}$$

Where:

$$f_{(z)} = q\alpha_{(z)} \tag{1379}$$

We define:

$$\left\{\alpha = H\sqrt{\frac{K_{s2}^2}{K_{b2}(K_{s1}+K_{s2})}}, \kappa = \sqrt{\frac{K_{s1}}{K_{s2}}}, \lambda = \frac{qH}{K_{s1}+K_{s2}}\right\} \tag{1380}$$

Rewriting:

$$u'''_{(z)} - (\alpha\kappa)^2u'_{(z)} - \lambda\alpha_{(z)}\left[u'''_{(z)} - \alpha^2(k^2+1)u'_{(z)}\right] = 0 \tag{1381}$$

However, the rotation function is of a lower degree:

$$\theta''_{(z)} - (\alpha\kappa)^2\theta_{(z)} - \lambda\alpha_{(z)}\left[\theta''_{(z)} - \alpha^2(k^2+1)\theta_{(z)}\right] = 0 \tag{1382}$$

Expressing the boundary conditions as a function of $\theta_{(z)}$:

$$\begin{Bmatrix} \theta_{(1)} = 0 \\ \theta'_{(0)} = 0 \end{Bmatrix} \tag{1383}$$

- **Uniformly Distributed Load**

The stability of the two-field MCTB, the governing differential equation is of the form:

$$\left[\frac{d^2}{dz^2}-(\alpha\kappa)^2\right]\theta_{(z)}-\lambda\left\{\alpha_{(z)}\left[\frac{d^2}{dz^2}-\alpha^2(\kappa^2+1)\right]\right\}\theta_{(z)}=0 \tag{1384}$$

Multiplying the equation by $\left[\theta''_{(z)}-\alpha^2(\kappa^2+1)\theta_{(z)}\right]$ and integrating from 0 to 1:

$$\int_0^1\left[\theta''^2_{(z)}-\alpha^2(2\kappa^2+1)\theta_{(z)}\theta''_{(z)}+\alpha^2(\kappa^2+1)(\alpha\kappa)^2\theta^2_{(x)}\right]dz \\ -\lambda\int_0^1\alpha_{(z)}\left[\theta''_{(z)}-\alpha^2(\kappa^2+1)\theta_{(z)}\right]^2dz=0 \tag{1385}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\int_0^1\left[\theta''^2_{(z)}+\alpha^2(2\kappa^2+1)\theta'^2_{(z)}+\alpha^2(\kappa^2+1)(\alpha\kappa)^2\theta^2_{(x)}\right]dz \\ -\lambda\int_0^1\alpha_{(z)}\left[\theta''_{(z)}-\alpha^2(\kappa^2+1)\theta_{(z)}\right]^2dz=0 \tag{1386}$$

Solving the parameter γ:

$$\lambda=\frac{\int_0^1\left[\theta''^2_{(z)}+\alpha^2(2\kappa^2+1)\theta'^2_{(z)}+\alpha^2(\kappa^2+1)(\alpha\kappa)^2\theta^2_{(x)}\right]dz}{\int_0^1\alpha_{(z)}\left[\theta''_{(z)}-\alpha^2(\kappa^2+1)\theta_{(z)}\right]^2dz} \tag{1387}$$

This Rayleigh ratio represents an approximation of the upper limit of the critical load, and it is exact if and only if the exact equilibrium curve $\theta_{(z)}$ is used to calculate λ.

For the case of a uniformly distributed load, the function $\alpha_{(z)}$ results in:

$$\alpha_{(z)}=z\rightarrow f_{(z)}=qz \tag{1388}$$

The Rayleigh quotient becomes:

$$\lambda=\frac{\int_0^1\left[\theta''^2_{(z)}+\alpha^2(2\kappa^2+1)\theta'^2_{(z)}+\alpha^2(\kappa^2+1)(\alpha\kappa)^2\theta^2_{(x)}\right]dz}{\int_0^1 z\left[\theta''_{(z)}-\alpha^2(\kappa^2+1)\theta_{(z)}\right]^2dz} \tag{1389}$$

Taking into account the boundary conditions. We consider two simple polynomials of different degrees that satisfy the boundary condition:

$$\phi_1^1 = 1 - z^3, \phi_2^1 = 1 - z^4 \tag{1390}$$

Taking a linear combination of both terms:

$$\theta_{(z)} = A\phi_1^1 + B\phi_2^1 = A(1 - z^3) + B(1 - z^4) \tag{1391}$$

We expand the integrals and substitute into the Rayleigh quotient:

$$U = \int_0^1 \left[\theta''^2_{(z)} + \alpha^2(2\kappa^2 + 1)\theta'^2_{(z)} + \alpha^2(\kappa^2 + 1)(\alpha\kappa)^2\theta^2_{(x)}\right] dz - \lambda \int_0^1 z\left[\theta''_{(z)} - \alpha^2(\kappa^2 + 1)\theta_{(z)}\right]^2 dz \tag{1392}$$

Expanding the integrals and joining common terms:

$$U = A^2(a_1 - \lambda a_2) + B^2(b_1 - \lambda b_2) + AB[(ab)_1 - \lambda(ab)_2] \tag{1393}$$

Where:

$$\begin{Bmatrix} a_1 = 12 + 1.8[\alpha^2(2\kappa^2 + 1)] + 0.6428[\alpha^2(\kappa^2 + 1)(\alpha\kappa)^2] \\ a_2 = 9 + 0.225[\alpha^2(\kappa^2 + 1)]^2 + 2[\alpha^2(\kappa^2 + 1)] \\ b_1 = 28.8 + 2.2857[\alpha^2(2\kappa^2 + 1)] + 0.7111[\alpha^2(\kappa^2 + 1)(\alpha\kappa)^2] \\ b_2 = 28.8 + 0.7111[\alpha^2(\kappa^2 + 1)]^2 + 3[\alpha^2(\kappa^2 + 1)] \\ (ab)_1 = 36 + 4[\alpha^2(2\kappa^2 + 1)] + 1.35[\alpha^2(\kappa^2 + 1)(\alpha\kappa)^2] \\ (ab)_2 = 28.8 + 0.4889[\alpha^2(\kappa^2 + 1)]^2 + 4.8571[\alpha^2(\kappa^2 + 1)] \end{Bmatrix} \tag{1394}$$

The condition for the critical load to be the minimum is expressed as:

$$\begin{Bmatrix} \dfrac{\partial U}{\partial A} = 0 \rightarrow 2(a_1 - \lambda a_2)A + [(ab)_1 - \lambda(ab)_2]B \\ \dfrac{\partial U}{\partial B} = 0 \rightarrow [(ab)_1 - \lambda(ab)_2]A + 2(b_1 - \lambda b_2)B \end{Bmatrix} \tag{1395}$$

Expressing in matrix form:

$$\begin{bmatrix} 2(a_1 - \lambda a_2) & [(ab)_1 - \lambda(ab)_2] \\ [(ab)_1 - \lambda(ab)_2] & 2(b_1 - \lambda b_2) \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1396}$$

For a nontrivial solution (a and b cannot be equal to zero simultaneously), the determinant of the coefficient matrix for a and b must be equal to zero; namely:

$$\begin{vmatrix} 2(a_1 - \lambda a_2) & [(ab)_1 - \lambda(ab)_2] \\ [(ab)_1 - \lambda(ab)_2] & 2(b_1 - \lambda b_2) \end{vmatrix} = 0 \tag{1397}$$

Operating the determinant, we have:

$$[4a_2b_2 - (ab)_2^2]\lambda^2 + [2(ab)_1(ab)_2 - 4(a_1b_2 + a_2b_1)]\lambda + [4a_1b_1 - (ab)_1^2] = 0 \tag{1398}$$

The minimum eigenvalue is obtained from the minimum root of the quadratic equation.

$$\lambda = \frac{qH}{K_{s1} + K_{s2}} \rightarrow q_{cr}H = \lambda(K_{s1} + K_{s2}) \tag{1399}$$

Which is the first approximation to the value of the critical load of the two-field MCTB beam. In order to obtain a better approximation to the exact critical load, it is necessary to repeat the previous procedure with two new higher degree polynomials.

The first polynomial to be considered will be the one with the highest degree of the previous iteration:

$$\phi_1^2 = 1 - z^4 \tag{1400}$$

To obtain a new polynomial of higher degree and that takes into account the eigenvalue calculated in the previous iteration, we will integrate the resulting differential equation of the two-field MCTB beam model twice:

$$\theta_{(z)} = (\alpha\kappa)^2 \iint \theta_{(z)} dz + \lambda \iint \alpha_{(z)} \theta''_{(z)} dz - \lambda\alpha^2(\kappa^2 + 1) \iint \alpha_{(z)} \theta_{(z)} dz + C_1 z + C_0 \tag{1401}$$

For the case of a uniform load:

$$\theta_{(z)} = (\alpha\kappa)^2 \iint \theta_{(z)} dz + \lambda \iint z\theta''_{(z)} dz - \lambda\alpha^2(\kappa^2 + 1) \iint z\theta_{(z)} dz + C_1 z + C_0 \tag{1402}$$

When evaluating the boundary conditions, the constants C_0 and C_1 are determined and the new polynomial to be used in the second iteration is determined.

Taking a linear combination of both terms:

$$\theta_{(z)} = A\phi_1^2 + B\phi_2^2 = A\phi_2^1 + B\phi_2^2 \tag{1403}$$

A closer approximation to the exact value can be achieved by repeating the two iteration steps, resulting in polynomials of higher and higher degree. Numerically it can be seen that with a fourth iteration the approximation can be considered exact.

- **Point Load at x=0 (z=0)**

For the case of a point load applied at x=0 (z=0), the function $\alpha_{(z)}$:

$$\alpha_{(z)} = 1 \rightarrow f_{(z)} = q \tag{1404}$$

Substituting into the differential equation:

$$\theta''_{(z)} - (\alpha\kappa)^2\theta_{(z)} - \frac{\lambda}{H}\left[\theta''_{(z)} - \alpha^2(k^2+1)\theta_{(z)}\right] = 0 \tag{1405}$$

The expression for $\theta_{(z)}$ can be derived as:

$$\theta_{(z)} = C_1\cos\left(\sqrt{\xi}z\right) + C_2\sin\left(\sqrt{\xi}z\right) \tag{1406}$$

Where:

$$\left\{\xi = \frac{\frac{\lambda}{H}\alpha^2(k^2+1) - (\alpha\kappa)^2}{\frac{\lambda}{H} - 1}\right\} \tag{1407}$$

The linear algebraic system resulting from the boundary conditions, written in matrix form, is:

$$\begin{bmatrix} \cosh\sqrt{\xi} & \sinh\sqrt{\xi} \\ 0 & \xi^{1/2} \end{bmatrix}\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = 0 \tag{1408}$$

Which has a solution different from the trivial one if the determinant is equal to zero (the matrix of coefficients is singular), that is, for:

$$Cos\sqrt{\xi} = 0 \rightarrow \sqrt{\beta} = (2n-1)\frac{\pi}{2}\ /\ n = 1,2,3\ldots \tag{1409}$$

i.e.,

$$\frac{\frac{\lambda}{H}\alpha^2(k^2+1)-(\alpha\kappa)^2}{1-\frac{\lambda}{H}} = (2n-1)^2\frac{\pi^2}{4} \tag{1410}$$

After some simple manipulations:

$$\frac{\lambda}{H} = \frac{(2n-1)^2\frac{\pi^2}{4}+(\alpha\kappa)^2}{(2n-1)^2\frac{\pi^2}{4}+\alpha^2(\kappa^2+1)} \tag{1411}$$

Replacing by its characteristic rigidities:

$$q_{cr} = K_{s1} + \frac{1}{\frac{4H^2}{(2n-1)^2\pi^2K_{b2}}+\frac{1}{K_{s2}}} \tag{1412}$$

Sorting properly:

$$q_{cr} = K_{s1} + \left\{\left[(2n-1)^2\frac{\pi^2K_{b2}}{4H^2}\right]^{-1} + K_{s2}{}^{-1}\right\}^{-1} \tag{1413}$$

For the case when $n = 1$, we have:

$$q_{cr} = K_{s1} + \left[\left(\frac{\pi^2K_{b2}}{4H^2}\right)^{-1} + K_{s2}{}^{-1}\right]^{-1} = q_{cr,corte\ global} + \left[q_{cr,flexión\ global}{}^{-1} + q_{cr,corte\ local}{}^{-1}\right]^{-1} \tag{1414}$$

Since the resulting critical load is independent of some approximation function, it can be considered exact and identical to the one that would be obtained by applying Föppl's theorem.

4.3.10.2 Case 2

- **Calculation of the Transfer Matrix**

According to the coupled differential equations:

$$\left\{\begin{matrix} K_{b2}\theta''_{(x)} - K_{s2}\left[\theta_{(x)} - u'_{(x)}\right] = 0 \\ [K_{s1}+K_{s2}-q]u'_{(x)} - K_{s2}\theta_{(x)} = 0 \end{matrix}\right\} \tag{1415}$$

Using the method of coefficients:

$$\begin{bmatrix} K_{s2}D & -K_{s2}+K_{b2}D^2 \\ [K_{s1}+K_{s2}-q]D & -K_{s2} \end{bmatrix} \begin{Bmatrix} u_{(x)} \\ \theta_{(x)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1416}$$

To avoid trivial solutions, the determinant must be equal to zero, that is:

$$D\left\{D^2 - \left[\frac{K_{s1}(K_{b1}+K_{b2})}{K_{b1}K_{b2}} - \frac{q}{K_{b2}}\right]\right\} = 0 \tag{1417}$$

Rewriting:

$$D\left\{D^2 + \frac{\lambda\alpha^{*2}(k^2+1)-(\alpha^*\kappa)^2}{1-\lambda}\right\} = 0 \tag{1418}$$

Where:

$$\left\{\alpha^* = \sqrt{\frac{K_{s2}^2}{K_{b2}(K_{s1}+K_{s2})}}, \kappa = \sqrt{\frac{K_{s1}}{K_{s2}}}, \lambda = \frac{q}{K_{s1}+K_{s2}}\right\} \tag{1419}$$

The expression for $u_{(z)}$ and $\theta_{(z)}$ is proposed:

$$\begin{Bmatrix} u_{(z)} = C_0 + C_1\cos\left(\sqrt{\xi}z\right) + C_2\sin\left(\sqrt{\xi}z\right) \\ \theta_{(z)} = C_3 + C_4\cos\left(\sqrt{\xi}z\right) + C_5\sin\left(\sqrt{\xi}z\right) \end{Bmatrix} \tag{1420}$$

Where:

$$\left\{\xi = \frac{\lambda\alpha^{*2}(k^2+1)-(\alpha^*\kappa)^2}{1-\lambda}\right\} \tag{1421}$$

Expressing the coefficients of $\theta_{(z)}$ as a function of the coefficients of $u_{(z)}$:

$$\begin{Bmatrix} u_{(z)} = C_0 + \cos\left(\sqrt{\xi}z\right)C_1 + \sin\left(\sqrt{\xi}z\right)C_2 \\ \theta_{(z)} = -\left[\frac{K_{S2}\sqrt{\xi}}{K_{S2}+\xi K_{b2}}\sin\left(\sqrt{\xi}z\right)\right]C_1 + \left[\frac{K_{S2}\sqrt{\xi}}{K_{S2}+\xi K_{b2}}\cos\left(\sqrt{\xi}z\right)\right]C_2 \end{Bmatrix} \tag{1422}$$

Internal forces such as bending moment and shear force associated with lateral displacement result in:

$$\left\{\begin{array}{c} M_{(z)} = K_{b2}\theta'_{(x)} = -\left[\frac{K_{S2}\xi}{K_{S2}+\xi K_{b2}}\cos\left(\sqrt{\xi}z\right)\right]C_1 - \left[\frac{K_{S2}\xi}{K_{S2}+\xi K_{b2}}\sin\left(\sqrt{\xi}z\right)\right]C_2 \\ \left\{\begin{array}{c} V_{(z)} = (K_{s1}+K_{s2}-q)u'_{(x)} - K_{s2}\theta_{(x)} = \left\{-(K_{s1}+K_{s2}-q) + \frac{K_{s2}^2\sqrt{\xi}}{K_{S2}+\xi K_{b2}}\right\}\sin\left(\sqrt{\xi}z\right)C_1 \\ -\left\{-(K_{s1}+K_{s2}-q) + \frac{K_{s2}^2\sqrt{\xi}}{K_{S2}+\xi K_{b2}}\right\}\cos\left(\sqrt{\xi}z\right)C_2 \end{array}\right\} \end{array}\right\} \quad (1423)$$

Where:

$$\left\{\begin{array}{l} R_1 = (K_{s1}+K_{s2}-q)\sqrt{\xi} - K_{s1}R_{\psi 1} - K_{s2}R_{\theta 1} \\ R_2 = (K_{s1}+K_{s2}-q)\sqrt{\beta} - K_{s1}R_{\psi 2} - K_{s2}R_{\theta 2} \end{array}\right\} \quad (1424)$$

Writing the equations in matrix form:

$$\left\{\begin{array}{c} u_i(z_i) \\ M_{\mathrm{i}}(z_i) \\ V_i(z_i) \end{array}\right\} = K_i(z_i)\left\{\begin{array}{c} C_0 \\ C_1 \\ C_2 \end{array}\right\} \quad (1425)$$

Where:

$$K_i(z_i) = \begin{bmatrix} 1 & \cos\left(\sqrt{\xi}z\right) & \sin\left(\sqrt{\xi}z\right) \\ 0 & -\frac{K_{S2}\xi}{K_{S2}+\xi K_{b2}}\cos\left(\sqrt{\xi}z\right) & -\frac{K_{S2}\xi}{K_{S2}+\xi K_{b2}}\sin\left(\sqrt{\xi}z\right) \\ 0 & \left[-(K_{s1}+K_{s2}-q) + \frac{K_{s2}^2\sqrt{\xi}}{K_{S2}+\xi K_{b2}}\right]\sin\left(\sqrt{\xi}z\right) & \left[(K_{s1}+K_{s2}-q) - \frac{K_{s2}^2\sqrt{\xi}}{K_{S2}+\xi K_{b2}}\right]\cos\left(\sqrt{\xi}z\right) \end{bmatrix}_i$$

$$(1426)$$

- **Stability Analysis Under Point Loads Applied at Floor Level**

Applying sequentially from the base to the top of the beam and expressing the equation between the product symbol:

$$\left\{\begin{array}{c} u_n(0) \\ M_{\mathrm{n}}(0) \\ V_n(0) \end{array}\right\} = \prod_{k=1}^{n} T_k(0)\left\{\begin{array}{c} u_1(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{array}\right\} = \mathrm{t}\left\{\begin{array}{c} u_1(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{array}\right\} \quad (1427)$$

Where:

$$\mathrm{t} = \prod_{k=1}^{n} T_k(0) \tag{1428}$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 3x3 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ \theta'_{(0)} = 0 \\ [K_{s1} + K_{s2} - q]u'_{(0)} - K_{s2}\theta_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1(h_1) = 0 \\ M_{n\,(0)} = 0 \\ V_{n(0)} = 0 \end{Bmatrix} \tag{1429}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} \\ t_{2,1} & t_{2,2} & t_{2,3} \\ t_{3,1} & t_{3,2} & t_{3,3} \end{bmatrix} \begin{Bmatrix} 0 \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1430}$$

Solving for bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{4,4} & t_{4,5} \\ t_{5,4} & t_{5,5} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1431}$$

Which has a different solution than the trivial one if the determinant is equal to zero (the matrix of coefficients is singular). Solving the critical loads of the beam.

4.3.11 Generalized Parallel Coupling of Two Beams and Three Field (GCTB)

4.3.11.1 Case 1

The potential energy of the three-field GCTB model is expressed as follows:

$$V = \frac{1}{2}\int_0^H \left\{K_{b1}{w'_{(x)}}^2 + K_{b2}{\theta'_{(x)}}^2 + K_{s2}\left[u'_{(x)} - \theta_{(x)}\right]^2\right\} dx + \frac{1}{2}\int_0^H K_{s1}\left[u'_{(x)} + m\theta_{(x)} - nw_{(x)}\right]^2 dx \tag{1432}$$

Where:

$$\left\{K_{b1} = E\left(A_2 + \frac{A_2^2}{A_1}\right), K_{b2} = E(I_1 + I_2), K_{s1} = G_{eq}t_w l_b, K_{s2} = G\kappa(A_1 + A_2)\right\} \tag{1433}$$

The work done by the external force is expressed as:

$$W = -f_{(x)}dl = -\frac{1}{2}\int_0^H f_{(x)}{u'_{(x)}}^2 dx \tag{1434}$$

Consequently, the total potential energy of the model is expressed as:

$$\begin{aligned}\mathcal{U} = \frac{1}{2}\int_0^H &\left\{K_{b1}{w'_{(x)}}^2 + K_{b2}{\theta'_{(x)}}^2 + K_{s2}\left[u'_{(x)} - \theta_{(x)}\right]^2 + K_{s1}\left[u'_{(x)} + m\theta_{(x)} - nw_{(x)}\right]^2\right\} dx \\ &- \frac{1}{2}\int_0^H f(x){u'_{(x)}}^2 dx\end{aligned} \tag{1435}$$

Closed-form solutions of the model are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\begin{aligned}\delta\mathcal{U} = \int_0^H &\{K_{b1}w'_{(x)}\delta w'_{(x)} + K_{b2}\theta'_{(x)}\delta\theta'_{(x)} + K_{s2}\left[u'_{(x)} - \theta_{(x)}\right]\left[\delta u'_{(x)} - \delta\theta_{(x)}\right] \\ &+ K_{s1}\left[u'_{(x)} + m\theta_{(x)} - nw_{(x)}\right]\left[\delta u'_{(x)} + m\delta\theta_{(x)} - n\delta w_{(x)}\right] \\ &- f_{(x)}u'_{(x)}\delta u'_{(x)}\}dx - \frac{1}{2}\int_0^H f(x)u'^2_{(x)}\delta f_{(x)}dx\end{aligned} \tag{1436}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\delta\mathcal{U} = K_{b1}\left[w'_{(x)}\delta w_{(x)}\right]_0^H + K_{b2}\left[\theta'_{(x)}\delta\theta_{(x)}\right]_0^H$$
$$+\left\{\left\{\left[K_{s1}+K_{s2}-f_{(x)}\right]u'_{(x)} - (K_{s2}-mK_{s1})\theta_{(x)} - nK_{s1}w_{(x)}\right\}\delta\theta_{(x)}\right\}_0^H$$
$$+\int_0^H\left\{-K_{b1}w''_{(x)} - nK_{s1}u'_{(x)} - mnK_{s1}\theta_{(x)} + n^2K_{s1}w_{(x)}\right\}\delta w_{(x)}$$
$$+\int_0^H\left\{-K_{b2}\theta''_{(x)} - (K_{s2}-mK_{s1})u'_{(x)} + (K_{s2}+m^2K_{s1})\theta_{(x)} - mnK_{s1}w_{(x)}\right\}\delta\theta_{(x)}$$
$$+\int_0^H\left\{-\left[K_{s1}+K_{s2}-f_{(x)}\right]u''_{(x)} + (K_{s2}-mK_{s1})\theta'_{(x)} + nK_{s1}w'_{(x)}\right.$$
$$\left.+f'_{(x)}u'_{(x)}\right\}\delta u_{(x)} - \int_0^H u_{(x)}\,\delta f(x)dx - \frac{1}{2}\int_0^H f(x)u'^2_{(x)}\delta f_{(x)}dx \quad (1437)$$

Setting the terms equal to zero, the following equations result:

$$\left\{\begin{array}{c} -K_{b1}w''_{(x)} - nK_{s1}u'_{(x)} - mnK_{s1}\theta_{(x)} + n^2K_{s1}w_{(x)} = 0 \\ -K_{b2}\theta''_{(x)} - (K_{s2}-mK_{s1})u'_{(x)} + (K_{s2}+m^2K_{s1})\theta_{(x)} - mnK_{s1}w_{(x)} = 0 \\ -\left[K_{s1}+K_{s2}-f_{(x)}\right]u''_{(x)} + (K_{s2}-mK_{s1})\theta'_{(x)} + nK_{s1}w'_{(x)} + f'_{(x)}u'_{(x)} = 0 \end{array}\right\} \quad (1438)$$

And boundary conditions:

$$\left\{\begin{array}{c} w'_{(0)} = 0 \\ \theta'_{(0)} = 0 \\ \left[K_{s1}+K_{s2}-f_{(0)}\right]u'_{(0)} - (K_{s2}-mK_{s1})\theta_{(0)} - nK_{s1}w_{(0)} = 0 \end{array}\right\} \quad (1439)$$

Integrating the equation once and evaluating at x=0:

$$-\left[K_{s1}+K_{s2}-f_{(x)}\right]u'_{(x)} + (K_{s2}-mK_{s1})\theta_{(x)} + nK_{s1}w_{(x)} = 0 \quad (1440)$$

We have a new system of coupled differential equations:

$$\left\{\begin{array}{c} -K_{b1}w''_{(x)} - nK_{s1}u'_{(x)} - mnK_{s1}\theta_{(x)} + n^2K_{s1}w_{(x)} = 0 \\ -K_{b2}\theta''_{(x)} - (K_{s2}-mK_{s1})u'_{(x)} + (K_{s2}+m^2K_{s1})\theta_{(x)} - mnK_{s1}w_{(x)} = 0 \\ -\left[K_{s1}+K_{s2}-f_{(x)}\right]u'_{(x)} + (K_{s2}-mK_{s1})\theta_{(x)} + nK_{s1}w_{(x)} = 0 \end{array}\right\} \quad (1441)$$

Using the method of coefficients for the solution of the system of equations:

$$\begin{bmatrix} -nK_{s1}D & -K_{b1}D^2 + n^2K_{s1} & -mnK_{s1} \\ -(K_{s2} - mK_{s1})D & -mnK_{s1} & -K_{b2}D^2 + (K_{s2} + m^2K_{s1}) \\ -\left[K_{s1} + K_{s2} - f_{(x)}\right]D & nK_{s1} & (K_{s2} - mK_{s1}) \end{bmatrix} \begin{Bmatrix} u_{(x)} \\ w_{(x)} \\ \theta_{(x)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Which has a solution other than the trivial one if the determinant is equal to zero:

$$\begin{aligned} u_{(x)}''''' - \frac{K_{s1}K_{s2}[n^2K_{b2} + (m+1)^2K_{b1}]}{K_{b1}K_{b2}(K_{s1} + K_{s2})} u_{(x)}'''' \\ - f_{(x)} \left[\frac{1}{(K_{s1} + K_{s2})} u_{(x)}''''' - \frac{K_{b1}(K_{s2} + m^2K_{s1}) + n^2K_{s1}K_{b2}}{K_{b1}K_{b2}(K_{s1} + K_{s2})} u_{(x)}''' \right. \\ \left. + \frac{n^2K_{s1}K_{s2}}{K_{b1}K_{b2}(K_{s1} + K_{s2})} u_{(x)}' \right] = 0 \end{aligned} \quad (1442)$$

A fourth order differential equation is obtained, where the critical load results from the smallest eigenvalue. Normalizing the differential equation by the variable $z=x/H$:

$$\begin{aligned} u_{(z)}''''' - \frac{K_{s1}K_{s2}[n^2K_{b2} + (m+1)^2K_{b1}]}{K_{b1}K_{b2}(K_{s1} + K_{s2})} H^2 u_{(z)}'''' \\ + f_{(z)} \left[-\frac{1}{(K_{s1} + K_{s2})} u_{(z)}''''' + \frac{K_{b1}(K_{s2} + m^2K_{s1}) + n^2K_{s1}K_{b2}}{K_{b1}K_{b2}(K_{s1} + K_{s2})} H^2 u_{(z)}''' \right. \\ \left. - \frac{n^2K_{s1}K_{s2}}{K_{b1}K_{b2}(K_{s1} + K_{s2})} H^4 u_{(z)}' \right] = 0 \end{aligned} \quad (1443)$$

Where:

$$f_{(z)} = q\alpha_{(z)} \quad (1444)$$

The equation can be rewritten as:

$$u_{(z)}''''' - a_0 u_{(z)}''' + q\alpha_{(z)}\left[-a_1 u_{(z)}''''' + a_2 u_{(z)}''' - a_3 u_{(z)}'\right] = 0 \quad (1445)$$

Where:

$$\begin{Bmatrix} a_0 = \dfrac{K_{s1}K_{s2}[n^2K_{b2} + (m+1)^2K_{b1}]}{K_{b1}K_{b2}(K_{s1} + K_{s2})} H^2, a_1 = \dfrac{1}{K_{s1} + K_{s2}} \\ a_2 = \dfrac{K_{b1}(K_{s2} + m^2K_{s1}) + n^2K_{s1}K_{b2}}{K_{b1}K_{b2}(K_{s1} + K_{s2})} H^2, a_3 = \dfrac{n^2K_{s1}K_{s2}}{K_{b1}K_{b2}(K_{s1} + K_{s2})} H^4 \end{Bmatrix} \quad (1446)$$

Expressing the boundary conditions as a function of $u_{(z)}$:

$$\begin{Bmatrix} u_{(1)} = 0 \\ u'_{(1)} = 0 \\ u''_{(0)} = 0 \\ u'''_{(1)} = 0 \\ u''''_{(0)} = 0 \end{Bmatrix} \tag{1447}$$

- **Uniformly Distributed Load**

For beam stability, the governing differential equation is of the form:

$$\left(\frac{d^5}{dz^5} - a_0\frac{d^3}{dz^3}\right)u_{(z)} - q\left[\alpha_{(z)}\left(a_1\frac{d^5}{dz^5} - a_2\frac{d^3}{dz^3} + a_3\frac{d}{dz}\right)\right]u_{(z)} = 0 \tag{1448}$$

Multiplying the equation by $\left(a_1\frac{d^5}{dz^5} - a_2\frac{d^3}{dz^3} + a_3\frac{d}{dz}\right)$, integrating from 0 to 1 and solving:

$$\lambda = \frac{\int_0^1\{a_1 u''''' {}^2_{(z)} + (a_0a_1 + a_2)u'''' {}^2_{(z)} + (a_0a_2 + a_3)u''' {}^2_{(z)} + a_0a_3u'' {}^2_{(z)}\}dz}{\int_0^1 \alpha_{(z)}\left[a_1u'''''_{(z)} - a_2u'''_{(z)} + a_3u'_{(z)}\right]^2 dz} \tag{1449}$$

This Rayleigh ratio represents an approximation of the upper limit of the critical load, and it is exact if and only if the exact equilibrium curve $u_{(z)}$ is used to calculate λ.

For the case of a uniformly distributed load, the function $\alpha_{(z)}$ results in:

$$\alpha_{(z)} = z \rightarrow f_{(z)} = qz \tag{1450}$$

The Rayleigh quotient becomes:

$$\lambda = \frac{\int_0^1\{a_1 u''''' {}^2_{(z)} + (a_0a_1 + a_2)u'''' {}^2_{(z)} + (a_0a_2 + a_3)u''' {}^2_{(z)} + a_0a_3u'' {}^2_{(z)}\}dz}{\int_0^1 z\left[a_1u'''''_{(z)} - a_2u'''_{(z)} + a_3u'_{(z)}\right]^2 dz} \tag{1451}$$

- **Point Load at x=0 (z=0)**

For the case of a point load applied at x=0 (z=0), the function $\alpha_{(z)}$:

$$\alpha_{(z)} = 1 \rightarrow f_{(z)} = q \tag{1452}$$

Substituting into the differential equation:

$$u_{(z)}^{'''''} - a_0 u_{(z)}^{'''} + q\left[-a_1 u_{(z)}^{'''''} + a_2 u_{(z)}^{'''} - a_3 u_{(z)}^{'}\right] = 0 \qquad (1453)$$

The expression for $u_{(z)}$ can be derived as:

$$u_{(z)} = C_0 + C_1 \cosh\left(\sqrt{\xi} z\right) + C_2 \sinh\left(\sqrt{\xi} z\right) + C_3 \cos\left(\sqrt{\beta} z\right) + C_4 \sin\left(\sqrt{\beta} z\right) \qquad (1454)$$

Where:

$$\begin{Bmatrix} \xi = \dfrac{(a_0 - qa_2) + \sqrt{(a_0 - qa_2)^2 + 4qa_3(1 - qa_1)}}{2(1 - qa_1)} \\ \beta = \dfrac{-(a_0 - qa_2) + \sqrt{(a_0 - qa_2)^2 + 4qa_3(1 - qa_1)}}{2(1 - qa_1)} \end{Bmatrix} \qquad (1455)$$

The linear algebraic system resulting from the boundary conditions, written in matrix form, is:

$$\begin{bmatrix} 1 & \cosh\left(\sqrt{\xi}\right) & \sinh\left(\sqrt{\xi}\right) & \cos\left(\sqrt{\beta}\right) & \sin\left(\sqrt{\beta}\right) \\ 0 & \xi^{1/2} \sinh\left(\sqrt{\xi}\right) & \xi^{1/2} \cosh\left(\sqrt{\xi}\right) & -\beta^{1/2} \sin\left(\sqrt{\beta}\right) & \beta^{1/2} \cos\left(\sqrt{\beta}\right) \\ 0 & \xi & 0 & -\beta & 0 \\ 0 & \xi^{3/2} \sinh\left(\sqrt{\xi}\right) & \xi^{3/2} \cosh\left(\sqrt{\xi}\right) & \beta^{3/2} \sin\left(\sqrt{\beta}\right) & -\beta^{3/2} \cos\left(\sqrt{\beta}\right) \\ 0 & \xi^2 & 0 & \beta^2 & 0 \end{bmatrix} \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = 0 \qquad (1456)$$

Which has a different solution than the trivial one if the determinant is equal to zero (the matrix of coefficients is singular), that is, for:

$$Cos\sqrt{\beta} = 0 \rightarrow \sqrt{\beta} = (2n - 1)\frac{\pi}{2} \;/\; n = 1, 2, 3 \ldots \qquad (1457)$$

i.e.,

$$\frac{-(a_0 - qa_2) + \sqrt{(a_0 - qa_2)^2 + 4qa_3(1 - qa_1)}}{2(1 - qa_1)} = (2n - 1)^2 \frac{\pi^2}{4} \qquad (1458)$$

Solving:

$$q_{cr} = \frac{\frac{(2n-1)^4\pi^4}{4} + a_0(2n-1)^2\pi^2}{4a_0 + a_2(2n-1)^2\pi^2 + a_1\frac{(2n-1)^4\pi^4}{4}} \tag{1459}$$

Replacing the coefficients and after some simple manipulations:

$$q_{cr} = \frac{(2n-1)^4\pi^4 K_{b1}K_{b2}(K_{s1}+K_{s2}) + 4H^2(2n-1)^2\pi^2[(m+1)^2K_{b1} + n^2K_{b2}]}{(2n-1)^4\pi^4 K_{b1}K_{b2} + 4H^2(2n-1)^2\pi^2[K_{b1}(K_{s2}+m^2K_{s1}) + n^2K_{s1}K_{b2}] + 16H^4n^2K_{s1}K_{s2}} \tag{1460}$$

It is observed that a formula where the modes interact independently is not possible due to the existing coupling between the bending and shear behaviors produced by the connecting beams. For the case of n=1:

$$q_{cr} = \frac{\pi^4 K_{b1}K_{b2}(K_{s1}+K_{s2}) + 4H^2\pi^2[(m+1)^2K_{b1} + n^2K_{b2}]}{\pi^4 K_{b1}K_{b2} + 4H^2\pi^2[K_{b1}(K_{s2}+m^2K_{s1}) + n^2K_{s1}K_{b2}] + 16H^4n^2K_{s1}K_{s2}} \tag{1461}$$

4.3.11.2 Case 2

- **Calculation of the Transfer Matrix**

According to the coupled differential equations:

$$\begin{Bmatrix} -K_{b1}w''_{(x)} - nK_{s1}u'_{(x)} - mnK_{s1}\theta_{(x)} + n^2K_{s1}w_{(x)} = 0 \\ -K_{b2}\theta''_{(x)} - (K_{s2} - mK_{s1})u'_{(x)} + (K_{s2} + m^2K_{s1})\theta_{(x)} - mnK_{s1}w_{(x)} = 0 \\ -[K_{s1} + K_{s2} - q]u''_{(x)} + (K_{s2} - mK_{s1})\theta'_{(x)} + nK_{s1}w'_{(x)} + f'_{(x)}u'_{(x)} = 0 \end{Bmatrix} \tag{1462}$$

Using the method of coefficients:

$$\begin{bmatrix} -nK_{s1}D & -K_{b1}D^2 + n^2K_{s1} & -mnK_{s1} \\ -(K_{s2} - mK_{s1})D & -mnK_{s1} & -K_{b2}D^2 + (K_{s2} + m^2K_{s1}) \\ -[K_{s1} + K_{s2} - q]D^2 & nK_{s1}D & (K_{s2} - mK_{s1})D \end{bmatrix} \begin{Bmatrix} u_{(x)} \\ w_{(x)} \\ \theta_{(x)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{1463}$$

To avoid trivial solutions, the determinant must be equal to zero, that is:

$$D^2\left\{D^4-\left\{\frac{K_{s1}K_{s2}\left[(m+1)^2K_{b1}+n^2K_{b2}\right]-q\left[K_{b1}\left(K_{s2}+m^2K_{s1}\right)+n^2K_{s1}K_{b2}\right]}{K_{b1}K_{b2}(K_{s1}+K_{s2}-q)}\right\}D^2 - \frac{n^2K_{s1}K_{s2}q}{K_{b1}K_{b2}(K_{s1}+K_{s2}-q)}\right\}=0 \tag{1464}$$

Rewriting:

$$D^2(D^4-r_1D^2-r_2)=0 \tag{1465}$$

Where:

$$\left\{\begin{array}{c} r_1=\frac{K_{s1}K_{s2}\left[(m+1)^2K_{b1}+n^2K_{b2}\right]-q\left[K_{b1}(K_{s2}+m^2K_{s1})+n^2K_{s1}K_{b2}\right]}{K_{b1}K_{b2}(K_{s1}+K_{s2}-q)} \\ r_2=\frac{n^2K_{s1}K_{s2}q}{K_{b1}K_{b2}(K_{s1}+K_{s2}-q)} \end{array}\right\} \tag{1466}$$

The expression for $u_{(z)}$, $\psi_{(z)}$ and $\theta_{(z)}$ is proposed:

$$\left\{\begin{array}{c} u_{(z)}=C_0+C_1z+C_2\cosh\left(\sqrt{\xi}z\right)+C_3\sinh\left(\sqrt{\xi}z\right)+C_4\cos\left(\sqrt{\beta}z\right)+C_5\sin\left(\sqrt{\beta}z\right) \\ w_{(x)}=C_6+C_7z+C_8\cosh\left(\sqrt{\xi}z\right)+C_9\sinh\left(\sqrt{\xi}z\right)+C_{10}\cos\left(\sqrt{\beta}z\right)+C_{11}\sin\left(\sqrt{\beta}z\right) \\ \theta_{(x)}=C_{12}+C_{13}z+C_{14}\cosh\left(\sqrt{\xi}z\right)+C_{15}\sinh\left(\sqrt{\xi}z\right)+C_{16}\cos\left(\sqrt{\beta}z\right)+C_{17}\sin\left(\sqrt{\beta}z\right) \end{array}\right\} \tag{1467}$$

Where:

$$\left\{\begin{array}{c} \xi=\frac{r_1+\sqrt{r_1^2+4r_2}}{2} \\ \beta=\frac{-r_1+\sqrt{r_1^2+4r_2}}{2} \end{array}\right\} \tag{1468}$$

Expressing the coefficients of $\psi_{(z)}$ and $\theta_{(z)}$ as a function of the coefficients of $u_{(z)}$:

$$\left\{\begin{array}{c} u_{(z)}=C_0+C_1z+C_2\cosh\left(\sqrt{\xi}z\right)+C_3\sinh\left(\sqrt{\xi}z\right)+C_4\cos\left(\sqrt{\beta}z\right)+C_5\sin\left(\sqrt{\beta}z\right) \\ w_{(x)}=p_3C_1+\left[\sqrt{\xi}p_1\sinh\left(\sqrt{\xi}z\right)\right]C_2+\left[\sqrt{\xi}p_1\cosh\left(\sqrt{\xi}z\right)\right]C_3-\left[\sqrt{\beta}p_1\sin\left(\sqrt{\beta}z\right)\right]C_4-\left[\sqrt{\beta}p_1\cos\left(\sqrt{\beta}z\right)\right]C_5 \\ \theta_{(x)}=p_4C_1+\left[\sqrt{\xi}p_2\sinh\left(\sqrt{\xi}z\right)\right]C_2+\left[\sqrt{\xi}p_2\cosh\left(\sqrt{\xi}z\right)\right]C_3-\left[\sqrt{\beta}p_2\sin\left(\sqrt{\beta}z\right)\right]C_4-\left[\sqrt{\beta}p_2\cos\left(\sqrt{\beta}z\right)\right]C_5 \end{array}\right\} \tag{1469}$$

Where:

$$\begin{Bmatrix} p_1 = \dfrac{nK_{s1}[(m+1)K_{s2} + \beta K_{b2}]}{(n^2K_{s1} + \beta K_{b1})(K_{s2} + m^2K_{s2} + \beta K_{b2}) - m^2n^2K_{s1}^2} \\ p_2 = \dfrac{(K_{s2} - mK_{s1})[n^2K_{s1}K_{s2} + \beta K_{b1}(K_{s2} - mK_{s1})]}{(n^2K_{s1} + \beta K_{b1})(K_{s2} + m^2K_{s2} + \beta K_{b2}) - m^2n^2K_{s1}^2} \\ p_3 = \left(\dfrac{m+1}{n}\right)\left(1 + m^2\dfrac{K_{s1}}{K_{s2}}\right), p_4 = 1 - m(m-1)\dfrac{K_{s1}}{K_{s2}} \end{Bmatrix} \quad (1470)$$

Internal forces such as bending moment and shear force associated with lateral displacement result in:

$$\begin{Bmatrix} \begin{Bmatrix} M_{1(z)} = K_{b1}w'_{(x)} = [\xi p_1 K_{b1}\cosh(\sqrt{\xi}z)]C_2 + [\xi p_1 K_{b1}\sinh(\sqrt{\xi}z)]C_3 \\ -[\beta p_1 K_{b1}\cos(\sqrt{\beta}z)]C_4 - [\beta p_1 K_{b1}\sin(\sqrt{\beta}z)]C_5 \end{Bmatrix} \\ \begin{Bmatrix} M_{2(z)} = K_{b2}\theta'_{(x)} = [\xi p_2 K_{b2}\cosh(\sqrt{\xi}z)]C_2 + [\xi p_2 K_{b2}\sinh(\sqrt{\xi}z)]C_3 \\ -[\beta p_2 K_{b2}\cos(\sqrt{\beta}z)]C_4 - [\beta p_2 K_{b2}\sin(\sqrt{\beta}z)]C_5 \end{Bmatrix} \\ \begin{Bmatrix} V_{(z)} = (K_{s1} + K_{s2} - q)u'_{(x)} - nK_{s1}w_{(x)} - (K_{s2} - mK_{s1})\theta_{(x)} = R_1C_1 + R_2\sinh(\sqrt{\xi}z)\,C_2 \\ +R_2\cosh(\sqrt{\xi}z)\,C_3 - R_3\sin(\sqrt{\beta}z)\,C_4 + R_4\cos(\sqrt{\beta}z)\,C_5 \end{Bmatrix} \end{Bmatrix} \quad (1471)$$

Where:

$$\begin{Bmatrix} R_1 = (K_{s1} + K_{s2} - q) - nK_{s1}p_3 - (K_{s2} - mK_{s1})p_4 \\ R_2 = (K_{s1} + K_{s2} - q)\sqrt{\xi} - nK_{s1}\sqrt{\xi}p_1 - (K_{s2} - mK_{s1})\sqrt{\xi}p_2 \\ R_3 = -(K_{s1} + K_{s2} - q)\sqrt{\xi} + nK_{s1}\sqrt{\xi}p_1 + (K_{s2} - mK_{s1})\sqrt{\xi}p_2 \\ R_4 = (K_{s1} + K_{s2} - q)\sqrt{\xi} + nK_{s1}\sqrt{\xi}p_1 + (K_{s2} - mK_{s1})\sqrt{\xi}p_2 \end{Bmatrix} \quad (1472)$$

Writing the equations in matrix form:

$$\begin{Bmatrix} u_i(z_i) \\ \psi_i(z_i) \\ \theta_i(z_i) \\ M_{1\mathrm{i}}(z_i) \\ M_{2\mathrm{i}}(z_i) \\ V_i(z_i) \end{Bmatrix} = K_i(z_i)\begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{Bmatrix} \quad (1473)$$

Where:

$$K_i(z_i) = \begin{bmatrix} 1 & z & \cosh(\sqrt{\xi}z) & \sinh(\sqrt{\xi}z) & \cos(\sqrt{\beta}z) & \sin(\sqrt{\beta}z) \\ 0 & p_3 & \sqrt{\xi}p_1 \sinh(\sqrt{\xi}z) & \sqrt{\xi}p_1 \cosh(\sqrt{\xi}z) & -\sqrt{\beta}p_1 \sin(\sqrt{\beta}z) & -\sqrt{\beta}p_1 \cos(\sqrt{\beta}z) \\ 0 & p_4 & \sqrt{\xi}p_2 \sinh(\sqrt{\xi}z) & \sqrt{\xi}p_2 \cosh(\sqrt{\xi}z) & -\sqrt{\beta}p_2 \sin(\sqrt{\beta}z) & -\sqrt{\beta}p_2 \cos(\sqrt{\beta}z) \\ 0 & 0 & \xi p_1 K_{b1} \cosh(\sqrt{\xi}z) & \xi p_1 K_{b1} \sinh(\sqrt{\xi}z) & -\beta p_1 K_{b1} \cos(\sqrt{\beta}z) & -\beta p_1 K_{b1} \sin(\sqrt{\beta}z) \\ 0 & 0 & \xi p_2 K_{b2} \cosh(\sqrt{\xi}z) & \xi p_2 K_{b2} \sinh(\sqrt{\xi}z) & -\beta p_2 K_{b2} \cos(\sqrt{\beta}z) & -\beta p_2 K_{b2} \sin(\sqrt{\beta}z) \\ 0 & R_1 & R_2 \sinh(\sqrt{\xi}z) & R_2 \sinh(\sqrt{\xi}z) & -R_3 \sin(\sqrt{\beta}z) & R_4 \cos(\sqrt{\beta}z) \end{bmatrix}_i$$

(1474)

- **Stability Analysis Under Point Loads Applied at Floor Level**

Applying sequentially from the base to the top of the beam and expressing the equation between the product symbol:

$$\begin{Bmatrix} u_n(0) \\ \psi_n(0) \\ \theta_n(0) \\ M_{1\mathrm{n}}(0) \\ M_{2\mathrm{n}}(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_k(0) \begin{Bmatrix} u_1(h_1) \\ \psi_1(h_1) \\ \theta_1(h_1) \\ M_{11}(h_1) \\ M_{21}(h_1) \\ V_1(h_1) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ \psi_1(h_1) \\ \theta_1(h_1) \\ M_{11}(h_1) \\ M_{21}(h_1) \\ V_1(h_1) \end{Bmatrix}$$

(1475)

Where:

$$\mathrm{t} = \prod_{k=1}^{n} T_k(0)$$

(1476)

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 6x6 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ w_{(1)} = 0 \\ \theta_{(1)} = 0 \\ w'_{(0)} = 0 \\ \theta'_{(0)} = 0 \\ (K_{s1} + K_{s2} - q)u'_{(0)} - nK_{s1}w_{(0)} - (K_{s2} - mK_{s1})\theta_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1(h_1) = 0 \\ \psi_1(h_1) = 0 \\ \theta_1(h_1) = 0 \\ M_{1n\,(0)} = 0 \\ M_{2n\,(0)} = 0 \\ V_{n\,(0)} = 0 \end{Bmatrix}$$

(1477)

Replacing:

$$\begin{Bmatrix} u_n(0) \\ \psi_n(0) \\ \theta_n(0) \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & t_{1,5} & t_{1,6} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} & t_{2,5} & t_{2,6} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} & t_{3,5} & t_{3,6} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,1} & t_{5,2} & t_{5,3} & t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,1} & t_{6,2} & t_{6,3} & t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ M_{11}(h_1) \\ M_{21}(h_1) \\ V_1(h_1) \end{Bmatrix} \qquad (1478)$$

Solving for bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ M_2(h_1) \\ V_1(h_1) \end{Bmatrix} \qquad (1479)$$

Which has a different solution than the trivial if the determinant is equal to zero (the matrix of coefficients is singular). Solving the critical loads of the beam.

4.3.12 Modified Generalized Parallel Coupling of Two Beams of Two Field (GCTB)

4.3.12.1 Case 1

The potential energy of the two-field GCTB model is expressed as follows:

$$V = \frac{1}{2}\int_0^H \left[K_{b1}{w'_{(x)}}^2 + K_{b2}{u''_{(x)}}^2\right] dx + \frac{1}{2}\int_0^H K_{s1}\left[(m+1)u'_{(x)} - nw_{(x)}\right]^2 dx \tag{1480}$$

Where:

$$\left\{K_{b1} = E\left(A_2 + \frac{A_2^2}{A_1}\right), K_{b2} = E(I_1 + I_2), K_{s1} = G_{eq}t_w l_b\right\} \tag{1481}$$

The work done by the external force is expressed as:

$$W = -f_{(x)}dl = -\frac{1}{2}\int_0^H f_{(x)}{u'_{(x)}}^2 dx \tag{1482}$$

Consequently, the total potential energy of the model is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H \left\{K_{b1}{w'_{(x)}}^2 + K_{b2}{u''_{(x)}}^2 + K_{s1}\left[(m+1)u'_{(x)} - nw_{(x)}\right]^2\right\} dx - \frac{1}{2}\int_0^H f(x){u'_{(x)}}^2 dx \tag{1483}$$

Closed-form solutions of the model are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\begin{aligned}\delta\mathcal{U} = \int_0^H &\{K_{b1}w'_{(x)}\delta w'_{(x)} + K_{b2}u''_{(x)}\delta u''_{(x)} + K_{s1}\left[(m+1)u'_{(x)} - nw_{(x)}\right](m+1)\delta u'_{(x)} \\ &- K_{s1}\left[(m+1)u'_{(x)} - nw_{(x)}\right]n\delta w_{(x)} - f_{(x)}u'_{(x)}\delta u'_{(x)}\}dx \\ &- \frac{1}{2}\int_0^H f(x)u'^2_{(x)}\delta f_{(x)}dx\end{aligned} \tag{1484}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$
\begin{aligned}
\delta\mathcal{U} = {} & K_{b1}\left[w'_{(x)}\delta w_{(x)}\right]_0^H + K_{b2}\left[u''_{(x)}\delta u'_{(x)}\right]_0^H \\
& + \left\{\left\{-K_{b2}u'''_{(x)} + \left[(m+1)^2K_{s1} - f_{(x)}\right]u'_{(x)} - n(m+1)K_{s1}w_{(x)}\right\}\delta u_{(x)}\right\}_0^H \\
& + \int_0^H \{-K_{b1}w''_{(x)} - n(m+1)K_{s1}u'_{(x)} + n^2K_{s1}w_{(x)}\}\delta w_{(x)} \\
& + \int_0^H \left\{K_{b2}u''''_{(x)} - \left[(m+1)^2K_{s1} - f_{(x)}\right]u''_{(x)} + n(m+1)K_{s1}w'_{(x)}\right. \\
& \left. + f'_{(x)}u'_{(x)}\right\}\delta u_{(x)} - \frac{1}{2}\int_0^H f'_{(x)}u'^2_{(x)}\delta f_{(x)}dx
\end{aligned} \tag{1485}
$$

Setting the terms equal to zero, the following equations result:

$$
\begin{Bmatrix} -K_{b1}w''_{(x)} - n(m+1)K_{s1}u'_{(x)} + n^2K_{s1}w_{(x)} = 0 \\ K_{b2}u''''_{(x)} - [(m+1)^2K_{s1} - f_{(x)}]u''_{(x)} + n(m+1)K_{s1}w'_{(x)} + f'_{(x)}u'_{(x)} = 0 \end{Bmatrix} \tag{1486}
$$

And boundary conditions:

$$
\begin{Bmatrix} w'_{(0)} = 0 \\ u''_{(0)} = 0 \\ K_{b2}u'''_{(0)} - [(m+1)^2K_{s1} - f_{(0)}]u'_{(0)} + n(m+1)K_{s1}w_{(0)} = 0 \end{Bmatrix} \tag{1487}
$$

Integrating the equation once and evaluating at x=0:

$$
K_{b2}u'''_{(x)} - [(m+1)^2K_{s1} - f_{(x)}]u'_{(x)} + n(m+1)K_{s1}w_{(x)} = 0 \tag{1488}
$$

We have a new system of coupled differential equations:

$$
\begin{Bmatrix} -K_{b1}w''_{(x)} - n(m+1)K_{s1}u'_{(x)} + n^2K_{s1}w_{(x)} = 0 \\ K_{b2}u'''_{(x)} - [(m+1)^2K_{s1} - f_{(x)}]u'_{(x)} + n(m+1)K_{s1}w_{(x)} = 0 \end{Bmatrix} \tag{1489}
$$

Using the method of coefficients for the solution of the system of equations:

$$
\begin{bmatrix} -n(m+1)K_{s1}D & -K_{b1}D^2 + n^2K_{s1} \\ K_{b2}D^3 - [(m+1)^2K_{s1} - f_{(x)}]D & n(m+1)K_{s1} \end{bmatrix} \begin{Bmatrix} u_{(x)} \\ w_{(x)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}
$$

The determinant is equal to zero (the coefficient matrix is singular):

$$\frac{K_{b1}K_{b2}}{K_{s1}}u_{(x)}^{'''''}-[(m+1)^2K_{b1}+n^2K_{b2}]u_{(x)}^{'''}+f_{(x)}\left[\frac{K_{b1}}{K_{s1}}u_{(x)}^{'''}-u_{(x)}^{'}\right]=0 \quad (1490)$$

Reordering:

$$u_{(x)}^{'''''}-\frac{K_{s1}}{K_{b2}}\left[(m+1)^2+n^2\frac{K_{b2}}{K_{b1}}\right]u_{(x)}^{'''}+\frac{f_{(x)}}{K_{b2}}\left[u_{(x)}^{'''}-\frac{K_{s1}}{K_{b1}}u_{(x)}^{'}\right]=0 \quad (1491)$$

A fourth order differential equation is obtained, where the critical load results from the smallest eigenvalue. Normalizing the differential equation by the variable z=x/H:

$$u_{(z)}^{'''''}-\frac{K_{s1}}{K_{b2}}\left[(m+1)^2+n^2\frac{K_{b2}}{K_{b1}}\right]H^2u_{(z)}^{'''}+\frac{f_{(z)}H^2}{K_{b2}}\left[u_{(z)}^{'''}-\frac{K_{s1}H^2}{K_{b1}}u_{(z)}^{'}\right]=0 \quad (1492)$$

Where:

$$f_{(z)}=q\alpha_{(z)} \quad (1493)$$

We define:

$$\left\{\alpha=H\sqrt{\frac{K_{s1}}{K_{b2}}},\kappa=\sqrt{(m+1)^2+n^2\frac{K_{b2}}{K_{b1}}},\lambda=\frac{qH^3}{K_{b2}}\right\} \quad (1494)$$

Rewriting:

$$u_{(z)}^{'''''}-(\alpha\kappa)^2u_{(z)}^{'''}+\lambda\alpha_{(z)}\left\{u_{(z)}^{'''}-\alpha^2\left[\frac{\kappa^2-(m+1)^2}{n^2}\right]u_{(z)}^{'}\right\}=0 \quad (1495)$$

However, the axial extensibility function is of a lesser degree:

$$w_{(z)}^{''''}-(\alpha\kappa)^2w_{(z)}^{''}+\lambda\alpha_{(z)}\left\{w_{(z)}^{''}-\alpha^2\left[\frac{\kappa^2-(m+1)^2}{n^2}\right]w_{(z)}\right\}=0 \quad (1496)$$

Expressing the boundary conditions as a function of $w_{(z)}$:

$$\begin{Bmatrix} w_{(1)} = 0 \\ w'_{(0)} = 0 \\ w''_{(1)} = 0 \\ w'''_{(0)} = 0 \end{Bmatrix} \tag{1497}$$

- **Uniformly Distributed Load**

The stability of the two-field GCTB, the governing differential equation is of the form:

$$\left[\frac{d^4}{dz^4} - (\alpha\kappa)^2 \frac{d^2}{dz^2}\right] w_{(z)} - \lambda \left\{ -\alpha_{(z)} \left\{ \frac{d^2}{dz^2} - \alpha^2 \left[\frac{\kappa^2 - (m+1)^2}{n^2}\right] \right\} \right\} w_{(z)} = 0 \tag{1498}$$

Multiplying the equation by $\left\{ w''_{(z)} - \alpha^2 \left[\frac{\kappa^2-(m+1)^2}{n^2}\right] w_{(z)} \right\}$ and integrating from 0 to 1 :

$$\begin{aligned} \int_0^1 \Bigg\{ & w''_{(z)} w''''_{(z)} - (\alpha\kappa)^2 w''^2_{(z)} - \alpha^2 \left[\frac{\kappa^2 - (m+1)^2}{n^2}\right] w_{(z)} w''''_{(z)} \\ & + \alpha^2 \left[\frac{\kappa^2 - (m+1)^2}{n^2}\right] (\alpha\kappa)^2 w_{(z)} w''_{(z)} \Bigg\} dz \\ & + \lambda \int_0^1 \alpha_{(z)} \left\{ w''_{(z)} - \alpha^2 \left[\frac{\kappa^2 - (m+1)^2}{n^2}\right] w_{(z)} \right\}^2 dz = 0 \end{aligned} \tag{1499}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\begin{aligned} -\int_0^1 \Bigg\{ & w'''^2_{(z)} + \alpha^2 \left[\frac{2\kappa^2 - (m+1)^2}{n^2}\right] w''^2_{(z)} + \alpha^2 \left[\frac{\kappa^2 - (m+1)^2}{n^2}\right] (\alpha\kappa)^2 w'^2_{(z)} \Bigg\} dz \\ & + \lambda \int_0^1 \alpha_{(z)} \left\{ w''_{(z)} - \alpha^2 \left[\frac{\kappa^2 - (m+1)^2}{n^2}\right] w_{(z)} \right\}^2 dz = 0 \end{aligned} \tag{1500}$$

Solving the parameter γ:

$$\lambda = \frac{\int_0^1 \left\{ w'''^2_{(z)} + \alpha^2 \left[\frac{2\kappa^2 - (m+1)^2}{n^2}\right] w''^2_{(z)} + \alpha^2 \left[\frac{\kappa^2 - (m+1)^2}{n^2}\right] (\alpha\kappa)^2 w'^2_{(z)} \right\} dz}{\int_0^1 \alpha_{(z)} \left\{ w''_{(z)} - \alpha^2 \left[\frac{\kappa^2 - (m+1)^2}{n^2}\right] w_{(z)} \right\}^2 dz} \tag{1501}$$

This Rayleigh ratio represents an approximation of the upper limit of the critical load, and it is exact if and only if the exact equilibrium curve $w_{(z)}$ is used to calculate λ.

For the case of a uniformly distributed load, the function $\alpha_{(z)}$ results in:

$$\alpha_{(z)} = z \rightarrow f_{(z)} = qz \tag{1502}$$

The Rayleigh quotient becomes:

$$\lambda = \frac{\int_0^1 \left\{\theta'''^2_{(z)} + \alpha^2 \left[\frac{2\kappa^2 - (m+1)^2}{n^2}\right]\theta''^2_{(z)} + \alpha^2 \left[\frac{\kappa^2 - (m+1)^2}{n^2}\right](\alpha\kappa)^2\theta'^2_{(z)}\right\}dz}{\int_0^1 z\left\{\theta''_{(z)} - \alpha^2 \left[\frac{\kappa^2 - (m+1)^2}{n^2}\right]\theta_{(z)}\right\}^2 dz} \tag{1503}$$

Taking into account the boundary conditions. We consider two simple polynomials of different degrees that satisfy the boundary condition:

$$\phi_1^1 = 1 - \frac{6}{5}z^2 + \frac{1}{5}z^4, \phi_2^1 = 1 - \frac{10}{9}z^2 + \frac{1}{9}z^5 \tag{1504}$$

Taking a linear combination of both terms:

$$\theta_{(z)} = A\phi_1^1 + B\phi_2^1 = A\left(1 - \frac{6}{5}z^2 + \frac{1}{5}z^4\right) + B\left(1 - \frac{10}{9}z^2 + \frac{1}{9}z^5\right) \tag{1505}$$

We expand the integrals and substitute into the Rayleigh quotient:

$$\begin{aligned} U = \int_0^1 &\left\{\theta'''^2_{(z)} + \alpha^2 \left[\frac{2\kappa^2 - (m+1)^2}{n^2}\right]\theta''^2_{(z)} + \alpha^2 \left[\frac{\kappa^2 - (m+1)^2}{n^2}\right](\alpha\kappa)^2\theta'^2_{(z)}\right\}dz \\ &- \lambda \int_0^1 z\left\{\theta''_{(z)} - \alpha^2 \left[\frac{\kappa^2 - (m+1)^2}{n^2}\right]\theta_{(z)}\right\}^2 dz \end{aligned} \tag{1506}$$

Expanding the integrals and joining common terms:

$$U = A^2(a_1 - \lambda a_2) + B^2(b_1 - \lambda b_2) + AB[(ab)_1 - \lambda(ab)_2] \tag{1507}$$

Where:

$$\begin{Bmatrix} a_1 = 7.68 + 3.072\left\{\alpha^2\left[\frac{2\kappa^2-(m+1)^2}{n^2}\right]\right\} + 1.2434\left\{\alpha^2\left[\frac{\kappa^2-(m+1)^2}{n^2}\right](\alpha\kappa)^2\right\} \\ a_2 = 0.96 + 0.1507\left\{\alpha^2\left[\frac{\kappa^2-(m+1)^2}{n^2}\right]\right\}^2 - 1.3166\left\{\alpha^2\left[\frac{\kappa^2-(m+1)^2}{n^2}\right]\right\} \\ b_1 = 8.8889 + 3.1746\left\{\alpha^2\left[\frac{2\kappa^2-(m+1)^2}{n^2}\right]\right\} + 1.2689\left\{\alpha^2\left[\frac{\kappa^2-(m+1)^2}{n^2}\right](\alpha\kappa)^2\right\} \\ b_2 = 1.1111 + 0.1555\left\{\alpha^2\left[\frac{\kappa^2-(m+1)^2}{n^2}\right]\right\}^2 - 1.5089\left\{\alpha^2\left[\frac{\kappa^2-(m+1)^2}{n^2}\right]\right\} \\ (ab)_1 = 16 + 6.2222\left\{\alpha^2\left[\frac{2\kappa^2-(m+1)^2}{n^2}\right]\right\} + 2.5111\left\{\alpha^2\left[\frac{\kappa^2-(m+1)^2}{n^2}\right](\alpha\kappa)^2\right\} \\ (ab)_2 = 2.0571 + 0.3062\left\{\alpha^2\left[\frac{\kappa^2-(m+1)^2}{n^2}\right]\right\}^2 - 3.1030\left\{\alpha^2\left[\frac{\kappa^2-(m+1)^2}{n^2}\right]\right\} \end{Bmatrix} \quad (1508)$$

The condition for the critical load to be the minimum is expressed as:

$$\begin{Bmatrix} \dfrac{\partial U}{\partial A} = 0 \rightarrow 2(a_1 - \lambda a_2)A + [(ab)_1 - \lambda(ab)_2]B \\ \dfrac{\partial U}{\partial B} = 0 \rightarrow [(ab)_1 - \lambda(ab)_2]A + 2(b_1 - \lambda b_2)B \end{Bmatrix} \quad (1509)$$

Expressing in matrix form:

$$\begin{bmatrix} 2(a_1 - \lambda a_2) & [(ab)_1 - \lambda(ab)_2] \\ [(ab)_1 - \lambda(ab)_2] & 2(b_1 - \lambda b_2) \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (1510)$$

For a nontrivial solution (a and b cannot be equal to zero simultaneously), the determinant of the coefficient matrix for a and b must be equal to zero; namely:

$$\begin{vmatrix} 2(a_1 - \lambda a_2) & [(ab)_1 - \lambda(ab)_2] \\ [(ab)_1 - \lambda(ab)_2] & 2(b_1 - \lambda b_2) \end{vmatrix} = 0 \quad (1511)$$

Operating the determinant, we have:

$$[4a_2b_2 - (ab)_2^2]\lambda^2 + [2(ab)_1(ab)_2 - 4(a_1b_2 + a_2b_1)]\lambda + [4a_1b_1 - (ab)_1^2] = 0 \quad (1512)$$

The minimum eigenvalue is obtained from the minimum root of the quadratic equation.

$$\lambda = \frac{qH^3}{K_{b2}} \rightarrow q_{cr}H = \lambda\frac{K_{b2}}{H^2} \quad (1513)$$

To obtain a new polynomial of higher degree and that takes into account the eigenvalue calculated in the previous iteration, we will integrate the resulting differential equation of the two-field GCTB beam model four times:

The first polynomial to be considered will be the one with the highest degree of the previous iteration:

$$\phi_1^2 = 1 - \frac{10}{9}z^2 + \frac{1}{9}z^5 \tag{1514}$$

To obtain a new polynomial of higher degree and that takes into account the eigenvalue calculated in the previous iteration, we will integrate the resulting differential equation of the two-field GCTB beam model four times:

$$\theta_{(z)} = (\alpha\kappa)^2 \iint \theta_{(z)} dz - \lambda \iiiint \alpha_{(z)} \theta''_{(z)} dz + \lambda\alpha^2[\kappa^2 - (m+1)^2] \iiiint \alpha_{(z)} \theta_{(z)} dz + C_3 z^3 + C_2 z^2 + C_1 z + C_0 \tag{1515}$$

For the case of a uniform load:

$$\theta_{(z)} = (\alpha\kappa)^2 \iint \theta_{(z)} dz - \lambda \iiiint z \theta''_{(z)} dz + \lambda\alpha^2[\kappa^2 - (m+1)^2] \iiiint z \theta_{(z)} dz + C_3 z^3 + C_2 z^2 + C_1 z + C_0 \tag{1516}$$

When evaluating the boundary conditions, the constants C_0, C_1, C_2 and C_3 are determined and the new polynomial to be used in the second iteration is determined.

Taking a linear combination of both terms:

$$\theta_{(z)} = A\phi_1^2 + B\phi_2^2 = A\phi_2^1 + B\phi_2^2 \tag{1517}$$

A closer approximation to the exact value can be achieved by repeating the two iteration steps, resulting in polynomials of higher and higher degree.

- **Point Load at x=0 (z=0)**

For the case of a point load applied at x=H (z=1), the function $\alpha_{(z)}$:

$$\alpha_{(z)} = 1 \rightarrow f_{(z)} = q \tag{1518}$$

Substituting into the differential equation:

$$\theta''''_{(z)} - (\alpha\kappa)^2\theta''_{(z)} + \frac{\lambda}{H}\left\{\theta''_{(z)} - \alpha^2\left[\frac{\kappa^2 - (m+1)^2}{n^2}\right]\theta_{(z)}\right\} = 0 \tag{1519}$$

The expression for $\theta_{(z)}$ can be derived as:

$$\theta_{(z)} = C_1\cosh\left(\sqrt{\xi}z\right) + C_2\sinh\left(\sqrt{\xi}z\right) + C_3\cos\left(\sqrt{\beta}z\right) + C_4\sin\left(\sqrt{\beta}z\right) \tag{1520}$$

Where:

$$\left\{\begin{matrix} \xi = \dfrac{-\left[\dfrac{\lambda}{H} - (\alpha\kappa)^2\right] + \sqrt{\left[\dfrac{\lambda}{H} - (\alpha\kappa)^2\right]^2 + 4\dfrac{\lambda}{H}\alpha^2\left[\dfrac{\kappa^2 - (m+1)^2}{n^2}\right]}}{2} \\ \beta = \dfrac{\left[\dfrac{\lambda}{H} - (\alpha\kappa)^2\right] + \sqrt{\left[\dfrac{\lambda}{H} - (\alpha\kappa)^2\right]^2 + 4\dfrac{\lambda}{H}\alpha^2\left[\dfrac{\kappa^2 - (m+1)^2}{n^2}\right]}}{2} \end{matrix}\right\} \tag{1521}$$

The linear algebraic system resulting from the boundary conditions, written in matrix form, is:

$$\begin{bmatrix} \cosh\sqrt{\xi} & \sinh\sqrt{\xi} & \cos\sqrt{\beta} & \sin\sqrt{\beta} \\ 0 & \xi^{1/2} & 0 & \beta^{1/2} \\ \xi\cosh\sqrt{\xi} & \xi\sinh\sqrt{\xi} & -\beta\cos\sqrt{\beta} & -\beta\sin\sqrt{\beta} \\ 0 & \xi^{3/2} & 0 & -\beta^{3/2} \end{bmatrix}\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = 0 \tag{1522}$$

Which has a solution different from the trivial one if the determinant is equal to zero (the matrix of coefficients is singular), that is, for:

$$Cos\sqrt{\beta} = 0 \rightarrow \sqrt{\beta} = (2n-1)\frac{\pi}{2}\ /\ n = 1, 2, 3\ ... \tag{1523}$$

i.e.,

$$\frac{\left[\dfrac{\lambda}{H} - (\alpha\kappa)^2\right] + \sqrt{\left[\dfrac{\lambda}{H} - (\alpha\kappa)^2\right]^2 + 4\dfrac{\lambda}{H}\alpha^2\left[\dfrac{\kappa^2 - (m+1)^2}{n^2}\right]}}{2} = (2n-1)^2\frac{\pi^2}{4} \tag{1524}$$

After some simple manipulations:

$$\frac{\lambda}{H} = (2n-1)^2 \frac{\pi^2}{4} + \frac{1}{\frac{4\left[\frac{\kappa^2-(m+1)^2}{n^2}\right]}{(2n-1)^2\pi^2} + \frac{1}{\alpha^2}} \tag{1525}$$

Replacing by its characteristic rigidities:

$$q_{cr} = (2n-1)^2 \frac{\pi^2 K_{b2}}{4H^2} + \frac{1}{\frac{4H^2}{(2n-1)^2\pi^2 K_{b1}} + \frac{1}{K_{s1}}} \tag{1526}$$

Sorting properly:

$$q_{cr} = (2n-1)^2 \frac{\pi^2 K_{b2}}{4H^2} + \left\{\left[(2n-1)^2 \frac{\pi^2 K_{b1}}{4H^2}\right]^{-1} + {K_{s1}}^{-1}\right\}^{-1} \tag{1527}$$

For the case when $n = 1$, we have:

$$q_{cr} = \frac{\pi^2 K_{b2}}{4H^2} + \left[\left(\frac{\pi^2 K_{b1}}{4H^2}\right)^{-1} + {K_{s1}}^{-1}\right]^{-1} \tag{1528}$$

i.e.,

$$q_{cr} = q_{cr,flexiòn\,local} + \left[{q_{cr,flexiòn\,global}}^{-1} + {q_{cr,corte}}^{-1}\right]^{-1} \tag{1529}$$

Since the resulting critical load is independent of some approximation function, it can be considered exact and identical to the one that would be obtained by applying Föppl's theorem.

4.3.12.2 Case 2

- **Calculation of the Transfer Matrix**

According to the coupled differential equations:

$$\begin{Bmatrix} -K_{b1}w''_{(x)} - n(m+1)K_{s1}u'_{(x)} + n^2K_{s1}w_{(x)} = 0 \\ K_{b2}u''''_{(x)} - [(m+1)^2K_{s1} - q]u''_{(x)} + n(m+1)K_{s1}w'_{(x)} = 0 \end{Bmatrix} \tag{1530}$$

Using the method of coefficients:

$$\begin{bmatrix} -n(m+1)K_{s1}D & -K_{b1}D^2+n^2K_{s1} \\ K_{b2}D^4-[(m+1)^2K_{s1}-q]D^2 & n(m+1)K_{s1}D \end{bmatrix}\begin{Bmatrix} u_{(x)} \\ w_{(x)} \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1531}$$

To avoid trivial solutions, the determinant must be equal to zero, that is:

$$D^2\left\{D^4-\left\{\frac{K_{s1}}{K_{b2}}\left[(m+1)^2+n^2\frac{K_{b2}}{K_{b1}}\right]-\frac{q}{K_{b2}}\right\}D^2-\frac{K_{s1}q}{K_{b1}K_{b2}}\right\}=0 \tag{1532}$$

Rewriting:

$$D^2\left\{D^4-[(\alpha^*\kappa)^2-\lambda]D^2-\left\{\alpha^{*2}\left[\frac{\kappa^2-(m+1)^2}{n^2}\right]\lambda\right\}\right\}=0 \tag{1533}$$

Where:

$$\left\{\alpha^*=\sqrt{\frac{K_{s1}}{K_{b2}}},\kappa=\sqrt{(m+1)^2+n^2\frac{K_{b2}}{K_{b1}}},\lambda=\frac{q}{K_{b2}}\right\} \tag{1534}$$

The expression for $u_{(z)}$ and $w_{(z)}$ is proposed:

$$\begin{Bmatrix} u_{(z)}=C_0+C_1z+C_2\cosh\left(\sqrt{\xi}z\right)+C_3\sinh\left(\sqrt{\xi}z\right)+C_4\cos\left(\sqrt{\beta}z\right)+C_5\sin\left(\sqrt{\beta}z\right) \\ w_{(z)}=C_6+C_7z+C_8\cosh\left(\sqrt{\xi}z\right)+C_9\sinh\left(\sqrt{\xi}z\right)+C_{10}\cos\left(\sqrt{\beta}z\right)+C_{11}\sin\left(\sqrt{\beta}z\right) \end{Bmatrix} \tag{1535}$$

Where:

$$\begin{Bmatrix} \xi=\dfrac{\left[(\alpha^*\kappa)^2-\lambda\right]+\sqrt{[(\alpha^*\kappa)^2-\lambda]^2+4\alpha^{*2}\left[\dfrac{\kappa^2-(m+1)^2}{n^2}\right]\lambda}}{2} \\ \beta=\dfrac{-\left[(\alpha^*\kappa)^2-\lambda\right]+\sqrt{[(\alpha^*\kappa)^2-\lambda]^2+4\alpha^{*2}\left[\dfrac{\kappa^2-(m+1)^2}{n^2}\right]\lambda}}{2} \end{Bmatrix} \tag{1536}$$

Expressing the coefficients of $\psi_{(z)}$ and $\theta_{(z)}$ as a function of the coefficients of $u_{(z)}$:

$$\begin{Bmatrix} u_{(z)}=C_0+C_1z+C_2\cosh\left(\sqrt{\xi}z\right)+C_3\sinh\left(\sqrt{\xi}z\right)+C_4\cos\left(\sqrt{\beta}z\right)+C_5\sin\left(\sqrt{\beta}z\right) \\ w_{(z)}=p_3C_1+p_1\sinh\left(\sqrt{\xi}z\right)C_2+p_1\cosh\left(\sqrt{\xi}z\right)C_3-p_2\sin\left(\sqrt{\beta}z\right)C_4+p_2\cos\left(\sqrt{\beta}z\right)C_5 \end{Bmatrix} \tag{1537}$$

Where:

$$\left\{p_1 = \frac{n(m+1)K_{s1}\sqrt{\xi}}{n^2K_{s1} - \xi K_{b1}}, p_2 = \frac{n(m+1)K_{s1}\sqrt{\beta}}{n^2K_{s1} + \beta K_{b1}}, p_3 = \left(\frac{m+1}{n}\right)\right\} \tag{1538}$$

Internal forces such as bending moment and shear force associated with lateral displacement result in:

$$\left\{\begin{array}{c} \left\{\begin{array}{c} M_{1(z)} = K_{b1}w'_{(x)} = p_1\sqrt{\xi}K_{b1}\cosh\left(\sqrt{\xi}z\right)C_2 + p_1\sqrt{\xi}K_{b1}\sinh\left(\sqrt{\xi}z\right)C_3 \\ -p_2\sqrt{\beta}K_{b1}\cos\left(\sqrt{\beta}z\right)C_4 - p_2\sqrt{\beta}K_{b1}\sin\left(\sqrt{\beta}z\right)C_5 \end{array}\right\} \\ \left\{\begin{array}{c} M_{2(z)} = K_{b2}u''_{(x)} = \xi K_{b2}\cosh\left(\sqrt{\xi}z\right)C_2 + \xi K_{b2}\sinh\left(\sqrt{\xi}z\right)C_3 \\ -\beta K_{b2}\cos\left(\sqrt{\beta}z\right)C_4 - \beta K_{b2}\sin\left(\sqrt{\beta}z\right)C_5 \end{array}\right\} \\ \left\{\begin{array}{c} V_{(z)} = -K_{b2}u'''_{(x)} + [(m+1)^2K_{s1} - q]u'_{(x)} - n(m+1)K_{s1}w_{(x)} = p_4C_1 + p_5\sinh\left(\sqrt{\xi}z\right)C_2 \\ p_5\cosh\left(\sqrt{\xi}z\right)C_3 + p_6\sin\left(\sqrt{\beta}z\right)C_4 + p_7\cos\left(\sqrt{\beta}z\right)C_5 \end{array}\right\} \end{array}\right\} \tag{1539}$$

Where:

$$\left\{\begin{array}{c} p_4 = (m+1)K_{s1}[(m+1) - np_3] - q \\ p_5 = -K_{b2}\xi^{\frac{3}{2}} + [(m+1)^2K_{s1} - q]\xi^{\frac{1}{2}} - n(m+1)K_{s1}p_1 \\ p_6 = -K_{b2}\beta^{\frac{3}{2}} + [(m+1)^2K_{s1} - q]\beta^{\frac{1}{2}} + n(m+1)K_{s1}p_2 \\ p_7 = K_{b2}\beta^{\frac{3}{2}} + [(m+1)^2K_{s1} - q]\beta^{\frac{1}{2}} - n(m+1)K_{s1}p_2 \end{array}\right\} \tag{1540}$$

Writing the equations in matrix form:

$$\left\{\begin{array}{c} u_i(z_i) \\ u'_i(z_i) \\ w_i(z_i) \\ M_{1\mathrm{i}}(z_i) \\ M_{2\mathrm{i}}(z_i) \\ V_i(z_i) \end{array}\right\} = K_i(z_i)\left\{\begin{array}{c} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{array}\right\} \tag{1541}$$

Where:

$$K_i(z_i) = \begin{bmatrix} 1 & z & \cosh\left(\sqrt{\xi}z\right) & \sinh\left(\sqrt{\xi}z\right) & \cos\left(\sqrt{\beta}z\right) & \sin\left(\sqrt{\beta}z\right) \\ 0 & 1 & \sqrt{\xi}\sinh\left(\sqrt{\xi}z\right) & \sqrt{\xi}\cosh\left(\sqrt{\xi}z\right) & -\sqrt{\beta}\sin\left(\sqrt{\beta}z\right) & \sqrt{\beta}\cos\left(\sqrt{\beta}z\right) \\ 0 & p_3 & p_1\sinh\left(\sqrt{\xi}z\right) & p_1\cosh\left(\sqrt{\xi}z\right) & -p_2\sin\left(\sqrt{\beta}z\right) & p_2\cos\left(\sqrt{\beta}z\right) \\ 0 & 0 & p_1\sqrt{\xi}K_{b1}\cosh\left(\sqrt{\xi}z\right) & p_1\sqrt{\xi}K_{b1}\sinh\left(\sqrt{\xi}z\right) & -p_2\sqrt{\beta}K_{b1}\cos\left(\sqrt{\beta}z\right) & -p_2\sqrt{\beta}K_{b1}\sin\left(\sqrt{\beta}z\right) \\ 0 & 0 & \xi K_{b2}\cosh\left(\sqrt{\xi}z\right) & \xi K_{b2}\sinh\left(\sqrt{\xi}z\right) & -\beta K_{b2}\cos\left(\sqrt{\beta}z\right) & -\beta K_{b2}\sin\left(\sqrt{\beta}z\right) \\ 0 & p_4 & p_5\sinh\left(\sqrt{\xi}z\right) & p_5\cosh\left(\sqrt{\xi}z\right) & p_6\sin\left(\sqrt{\beta}z\right) & +p_7\cos\left(\sqrt{\beta}z\right) \end{bmatrix}_i$$

(1542)

- **Stability Analysis Under Point Loads Applied at Floor Level**

Applying sequentially from the base to the top of the beam and expressing the equation between the product symbol:

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ w_n(0) \\ M_{1\mathrm{n}}(0) \\ M_{2\mathrm{n}}(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_k(0) \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ w_1(h_1) \\ M_{11}(h_1) \\ M_{21}(h_1) \\ V_1(h_1) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ u_1'(h_1) \\ w_1(h_1) \\ M_{11}(h_1) \\ M_{21}(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1543}$$

Where:

$$\mathrm{t} = \prod_{k=1}^{n} T_k(0) \tag{1544}$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 6x6 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ u'_{(1)} = 0 \\ w_{(1)} = 0 \\ w'_{(0)} = 0 \\ u''_{(0)} = 0 \\ -K_{b2}u'''_{(0)} + [(m+1)^2 K_{s1} - q]u'_{(0)} - n(m+1)K_{s1}w_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1(h_1) = 0 \\ u_1'(h_1) = 0 \\ w_1(h_1) = 0 \\ M_{1\mathrm{n}}(0) = 0 \\ M_{2\mathrm{n}}(0) = 0 \\ V_n(0) = 0 \end{Bmatrix} \tag{1545}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ u_n'(0) \\ w_n(0) \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & t_{1,5} & t_{1,6} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} & t_{2,5} & t_{2,6} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} & t_{3,5} & t_{3,6} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,1} & t_{5,2} & t_{5,3} & t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,1} & t_{6,2} & t_{6,3} & t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ M_1(h_1) \\ M_2(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1546}$$

Solving for bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{4,4} & t_{4,5} & t_{4,6} \\ t_{5,4} & t_{5,5} & t_{5,6} \\ t_{6,4} & t_{6,5} & t_{6,6} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ M_2(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1547}$$

Which has a different solution than the trivial if the determinant is equal to zero (the matrix of coefficients is singular). Solving the critical loads of the beam.

4.3.13 Generalized Parallel Coupling of Two Beams of a Field (GCTB)

4.3.13.1 Case 1

The potential energy of the GCTB model of a field is:

$$V = \frac{1}{2}\int_0^H \left[K_b {u''_{(x)}}^2 + (m+1)^2 K_s u'^2_{(x)}\right] dx \tag{1548}$$

Where:

$$\{K_b = E(I_1 + I_2), K_s = G_{eq} t_w l_b\} \tag{1549}$$

The work done by the external force is expressed as:

$$W = -f_{(x)} dl = -\frac{1}{2}\int_0^H f_{(x)} {u'_{(x)}}^2 \, dx \tag{1550}$$

Consequently, the total potential energy of the model is expressed as:

$$\mathcal{U} = \frac{1}{2}\int_0^H \left[K_b {u''_{(x)}}^2 + (m+1)^2 K_s u'^2_{(x)}\right] dx - \frac{1}{2}\int_0^H f(x) {u'_{(x)}}^2 \, dx \tag{1551}$$

Closed-form solutions of the model are achieved by solving the differential system that arises from the stationarity of the equation. Stationarity due to equilibrium implies:

$$\delta\mathcal{U} = \int_0^H \left[K_b u''_{(x)} \delta u''_{(x)} + (m+1)^2 K_s u'_{(x)} \delta u'_{(x)} - f_{(x)} u'_{(x)} \delta u'_{(x)}\right] dx - \frac{1}{2}\int_0^H f(x) u'^2_{(x)} \delta f_{(x)} dx \tag{1552}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\begin{aligned}\delta\mathcal{U} = {} & \left[K_b u''_{(x)} \delta u'_{(x)}\right]_0^H + \left\{\left\{-K_b u'''_{(x)} + \left[(m+1)^2 K_s - f_{(x)}\right] u'_{(x)}\right\} \delta u_{(x)}\right\}_0^H \\ & + \int_0^H \left[K_b u''''_{(x)} - (m+1)^2 K_s u''_{(x)} + f'_{(x)} u'_{(x)} + f_{(x)} u''_{(x)}\right] \delta u_{(x)} dx \\ & - \frac{1}{2}\int_0^H f(x) u'^2_{(x)} \delta f_{(x)} dx\end{aligned} \tag{1553}$$

Setting the terms equal to zero, the following equation results:

$$K_b u''''_{(x)} - (m+1)^2 K_s u''_{(x)} + f'_{(x)} u'_{(x)} + f_{(x)} u''_{(x)} = 0 \tag{1554}$$

And boundary conditions:

$$\begin{Bmatrix} u''_{(0)} = 0 \\ -K_b u'''_{(0)} + \left[(m+1)^2 K_s - f_{(0)}\right] u'_{(0)} = 0 \end{Bmatrix} \tag{1555}$$

Integrating the equation once and evaluating at $x = 0$:

$$K_b u'''_{(x)} - \left[(m+1)^2 K_s - f_{(x)}\right] u'_{(x)} = 0 \tag{1556}$$

A third order differential equation is obtained, where the critical load results from the smallest eigenvalue. Normalizing the differential equation by the variable $z = x/H$:

$$u'''_{(z)} - \frac{(m+1)^2 K_s - f_{(z)}}{K_b} H^2 u'_{(z)} = 0 \tag{1557}$$

The equation can be rewritten as:

$$u'''_{(z)} - \alpha^2 u'_{(z)} + \lambda \alpha_{(z)} u'_{(z)} = 0 \tag{1558}$$

Where:

$$\begin{Bmatrix} \alpha = H\sqrt{\dfrac{(m+1)^2 K_s}{K_b}}, \lambda = \dfrac{qH^3}{K_{b2}} \end{Bmatrix} \tag{1559}$$

- **Uniformly Distributed Load**

The stability of the beam GCTB of a field, the governing differential equation is of the form:

$$\left(\frac{d^3}{dz^3} - \alpha^2 \frac{d}{dz}\right) u_{(z)} - \lambda \left[-\alpha_{(z)} \frac{d}{dz}\right] u_{(z)} = 0 \tag{1560}$$

Multiplying the equation by $u'_{(z)}$ and integrating from 0 to 1:

$$\int_0^1 \left[u'_{(z)} u'''_{(z)} - \alpha^2 u'^2_{(z)}\right] dz + \lambda \int_0^1 \alpha_{(z)} \left[u'_{(z)}\right]^2 dz = 0 \tag{1561}$$

After integrating by parts and replacing it in the equation, we order the common terms:

$$\int_0^1 \left[-u''^2_{(z)} - \alpha^2 u'^2_{(z)}\right] dz + \lambda \int_0^1 \alpha_{(z)} \left[u'_{(z)}\right]^2 dz = 0 \tag{1562}$$

Clearing the parameter λ:

$$\lambda = \frac{\int_0^1 \left[u''^2_{(z)} + \alpha^2 u'^2_{(z)}\right] dz}{\int_0^1 \alpha_{(z)} u'^2_{(z)} dz} \tag{1563}$$

Where λ is the Rayleigh quotient. For the case of a uniformly distributed load, the function $\alpha_{(z)}$ results in:

$$\alpha_{(z)} = z \rightarrow f_{(z)} = qz \tag{1564}$$

The Rayleigh quotient becomes:

$$\lambda = \frac{\int_0^1 \left[u''^2_{(z)} + \alpha^2 u'^2_{(z)}\right] dz}{\int_0^1 z u'^2_{(z)} dz} \tag{1565}$$

Taking into account the boundary conditions. We consider two simple polynomials of different degrees that satisfy the boundary condition:

$$\phi_1^1 = 1 - \frac{4}{3} z + \frac{1}{3} z^4, \phi_2^1 = 1 - \frac{5}{4} z + \frac{1}{4} z^5 \tag{1566}$$

Taking a linear combination of both terms:

$$u_{(z)} = A\phi_1^1 + B\phi_2^1 = A\left(1 - \frac{4}{3} z + \frac{1}{3} z^4\right) + B\left(1 - \frac{5}{4} z + \frac{1}{4} z^5\right) \tag{1567}$$

We expand the integrals and substitute into the Rayleigh quotient:

$$\mathcal{U} = \int_0^1 \left[u''^2_{(z)} + \alpha^2 u'^2_{(z)}\right] dz - \lambda \int_0^1 z \left[u'_{(z)}\right]^2 dz \tag{1568}$$

Expanding the integrals and grouping common terms:

$$\mathcal{U} = A^2[(3.2 + 1.1429\alpha^2) - 0.4\lambda] + B^2[(3.5714 + 1.1111\alpha^2) - 0.4167\lambda] + AB[(6.6667 + 2.25\alpha^2) - 0.8148\lambda] \tag{1569}$$

The condition for the critical load to be the minimum is expressed as:

$$\begin{Bmatrix} \dfrac{\partial \mathcal{U}}{\partial A} = 0 \to [(6.4 + 2.2858\alpha^2) - 0.8\lambda]A + [(6.6667 + 2.25\alpha^2) - 0.8148\lambda]B = 0 \\ \dfrac{\partial \mathcal{U}}{\partial B} = 0 \to [(6.6667 + 2.25\alpha^2) - 0.8148\lambda]A + [(7.1428 + 2.2222\alpha^2) - 0.8334\lambda]B = 0 \end{Bmatrix} \tag{1570}$$

Expressing in matrix form:

$$\begin{bmatrix} (6.4 + 2.2858\alpha^2) - 0.8\lambda & (6.6667 + 2.25\alpha^2) - 0.8148\lambda \\ (6.6667 + 2.25\alpha^2) - 0.8148\lambda & (7.1428 + 2.2222\alpha^2) - 0.8334\lambda \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1571}$$

For a nontrivial solution (a and b cannot be equal to zero simultaneously), the determinant of the coefficient matrix for a and b must be equal to zero. Operating the determinant:

$$\lambda^2 - (66.8571 + 5.7857\alpha^2)\lambda + (6.1473\alpha^4 + 200.0205\alpha^2 + 462.4561) = 0 \tag{1572}$$

The minimum eigenvalue is obtained from the minimum root of the quadratic equation.

$$\lambda_1 = (33.4286 + 2.8929\alpha^2) - \sqrt{2.2213\alpha^4 - 6.6123\alpha^2 + 655.0133}$$
$$q_{cr}H = \lambda_1 \frac{K_b}{H^2} \to q_{cr}H = \lambda_1 \frac{K_b}{H^2} \tag{1573}$$

Which is the first approximation to the value of the critical load of the beam.

- 2nd Iteration:

The first polynomial to be considered will be the one with the highest degree of the previous iteration:

$$\phi_1^2 = 1 - \frac{5}{4}z + \frac{1}{4}z^5 \tag{1574}$$

To obtain a new polynomial of higher degree and that takes into account the eigenvalue calculated in the previous iteration, we will integrate the differential equation resulting from the beam model three times:

$$u_{(z)} = \iint_0^z \alpha^2 u_{(z)} dz - \lambda \iiint_0^z \alpha_{(z)} u'_{(z)} dzdz + C_2 z^2 + C_1 z + C_0 \tag{1575}$$

For the case of a uniform load:

$$u_{(z)} = \iint_0^z \alpha^2 u_{(z)} dz - \lambda \iiint_0^z z u'_{(z)} dzdz + C_2 z^2 + C_1 z + C_0 \tag{1576}$$

When evaluating the boundary conditions, the constants C_0, C_1, C_2 and C_3 are determined and the new polynomial ϕ_2^2 to be used in the second iteration is determined.

Taking a linear combination of both terms:

$$u_{(z)} = A\phi_1^2 + B\phi_2^2 \tag{1577}$$

Solving similarly to iteration 1, the new eigenvalue λ_2 is obtained. A closer approximation to the exact value can be achieved by repeating the two iteration steps, resulting in polynomials of higher and higher degree. Numerically it is observed that with a third iteration the approximation can be considered exact.

- **Point load at x=0 (z=0)**

For the case of a point load applied at x=0 (z=0), the function $\alpha_{(z)}$:

$$\alpha_{(z)} = 1 \rightarrow f_{(z)} = q \tag{1578}$$

Substituting into the differential equation:

$$u'''_{(z)} + (\lambda/H - \alpha^2) u'_{(z)} = 0 \tag{1579}$$

The expression for $u_{(z)}$ can be derived as:

$$u_{(z)} = C_0 + C_1 Cos\left(\sqrt{\lambda/H - \alpha^2} z\right) + C_2 Sen\left(\sqrt{\lambda/H - \alpha^2} z\right) \tag{1580}$$

The linear algebraic system resulting from the boundary conditions, written in matrix form, is:

$$\begin{bmatrix} 1 & \cos\left(\sqrt{\lambda/H-\alpha^2}\right) & \sin\left(\sqrt{\lambda/H-\alpha^2}\right) \\ 0 & -\sin\left(\sqrt{\lambda/H-\alpha^2}\right) & \cos\left(\sqrt{\lambda/H-\alpha^2}\right) \\ 0 & \cos\left(\sqrt{\lambda/H-\alpha^2}\right) & 0 \end{bmatrix} \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \end{Bmatrix} = 0 \tag{1581}$$

Which has a different solution than the trivial one $(C_0 = C_1 = C_2 = 0)$ if the determinant is equal to zero (the matrix of coefficients is singular), that is:

$$Cos\left(\sqrt{\lambda/H-\alpha^2}\right) = 0 \rightarrow \sqrt{\lambda/H-\alpha^2} = (2n-1)\frac{\pi}{2} \; / \; n = 1, 2, 3 \; \dots \tag{1582}$$

Solving, it is found that the critical load is:

$$q_{cr} = (m+1)^2 K_s + (2n-1)^2 \frac{\pi^2}{4} \frac{K_b}{H^2} \tag{1583}$$

For the case when $n = 1$, we have:

$$q_{cr} = (m+1)^2 K_s + \frac{\pi^2 K_b}{4H^2} = q_{cr,flexión} + q_{cr,corte} \tag{1584}$$

Since the resulting critical load is independent of some approximation function, it can be considered exact and identical to the one that would be obtained by applying Föppl's theorem.

4.3.13.2 Case 2

- **Calculation of the Transfer Matrix**

According to fourth degree differential equations:

$$K_b u''''_{(x)} + [q - (m+1)^2 K_s] u''_{(x)} = 0 \tag{1585}$$

Using the method of coefficients:

$$D^2(D^2 + \xi^2) = 0$$

The expression for $u_{(z)}$ and $u'_{(z)}$ is proposed:

$$\begin{Bmatrix} u_{(z)} = C_0 + C_1 z + C_2 \cos\left(\sqrt{\xi} z\right) + C_3 \sin\left(\sqrt{\xi} z\right) \\ u'_{(z)} = C_1 - C_2\sqrt{\xi} \sin\left(\sqrt{\xi} z\right) + C_3\sqrt{\xi} \cos\left(\sqrt{\xi} z\right) \end{Bmatrix} \tag{1586}$$

Where:

$$\left\{ \xi = \frac{q}{K_b} - \alpha^{*2}, \alpha^* = \sqrt{\frac{(m+1)^2 K_s}{K_b}} \right\} \tag{1587}$$

Internal forces such as bending moment and shear force associated with lateral displacement result in:

$$\begin{Bmatrix} M_{(z)} = K_b u''_{(x)} = -\left[\xi K_b \cos\left(\sqrt{\xi} z\right)\right] C_2 - \left[\xi K_b \sin\left(\sqrt{\xi} z\right)\right] C_3 \\ V_{(z)} = K_b u'''_{(x)} + [q - (m+1)^2 K_s] u'_{(x)} = [q - (m+1)^2 K_s] C_1 \end{Bmatrix} \tag{1588}$$

Writing the equations in matrix form:

$$\begin{Bmatrix} u_i(z_i) \\ u'_i(z_i) \\ M_i(z_i) \\ V_i(z_i) \end{Bmatrix} = K_i(z_i) \begin{Bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{Bmatrix} \tag{1589}$$

Where:

$$K_i(z_i) = \begin{bmatrix} 1 & z_i & \cos\left(\sqrt{\xi} z\right) & \sin\left(\sqrt{\xi} z\right) \\ 0 & 1 & -\sqrt{\xi} \sin\left(\sqrt{\xi} z\right) & \sqrt{\xi} \cos\left(\sqrt{\xi} z\right) \\ 0 & 0 & -\xi K_b \cos\left(\sqrt{\xi} z\right) & -\xi K_b \sin\left(\sqrt{\xi} z\right) \\ 0 & q - (m+1)^2 K_s & 0 & 0 \end{bmatrix}_i \tag{1590}$$

- **Stability Analysis Under Point Loads Applied at Floor Level**

Applying sequentially from the base to the top of the beam and expressing the equation between the product symbol:

$$\begin{Bmatrix} u_n(0) \\ u'_n(0) \\ M_n(0) \\ V_n(0) \end{Bmatrix} = \prod_{k=1}^{n} T_k(0) \begin{Bmatrix} u_1(h_1) \\ u'_1(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} = \mathrm{t} \begin{Bmatrix} u_1(h_1) \\ u'_1(h_1) \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1591}$$

Where:

$$\mathrm{t} = \prod_{k=1}^{n} T_k(0) \tag{1592}$$

This equation expresses the relationship between the forces and displacements of the top and bottom of the beam. An important point to note is that the size of the transfer matrix is 4x4 and remains constant across all floors.

According to the boundary conditions defined in case 1:

$$\begin{Bmatrix} u_{(1)} = 0 \\ u'_{(1)} = 0 \\ K_b u''_{(0)} = 0 \\ K_b u'''_{(0)} + [q - (m+1)^2 K_s] u'_{(0)} = 0 \end{Bmatrix} \rightarrow \begin{Bmatrix} u_1(h_1) = 0 \\ u'_1(h_1) = 0 \\ M_n(0) = 0 \\ V_n(0) = 0 \end{Bmatrix} \tag{1593}$$

Replacing:

$$\begin{Bmatrix} u_n(0) \\ \theta_n(0) \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\ t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} \\ t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} \\ t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1594}$$

Solving for bending moment and shear force at the base of the model:

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} t_{3,3} & t_{3,4} \\ t_{4,3} & t_{4,4} \end{bmatrix} \begin{Bmatrix} M_1(h_1) \\ V_1(h_1) \end{Bmatrix} \tag{1595}$$

Which has a different solution than the trivial if the determinant is equal to zero (the coefficient matrix is singular). Solving the critical loads of the beam.

4.4 EQUIVALENT REPLACEMENT BEAM OF TALL BUILDING

Figures 97 and 98 show how the building is modeled as a system of structural elements joined by inextensible fixed bars that represent the rigid diaphragm. Two options are presented for modeling the building: a sandwich equivalent replacement beam and a generalized sandwich equivalent replacement beam.

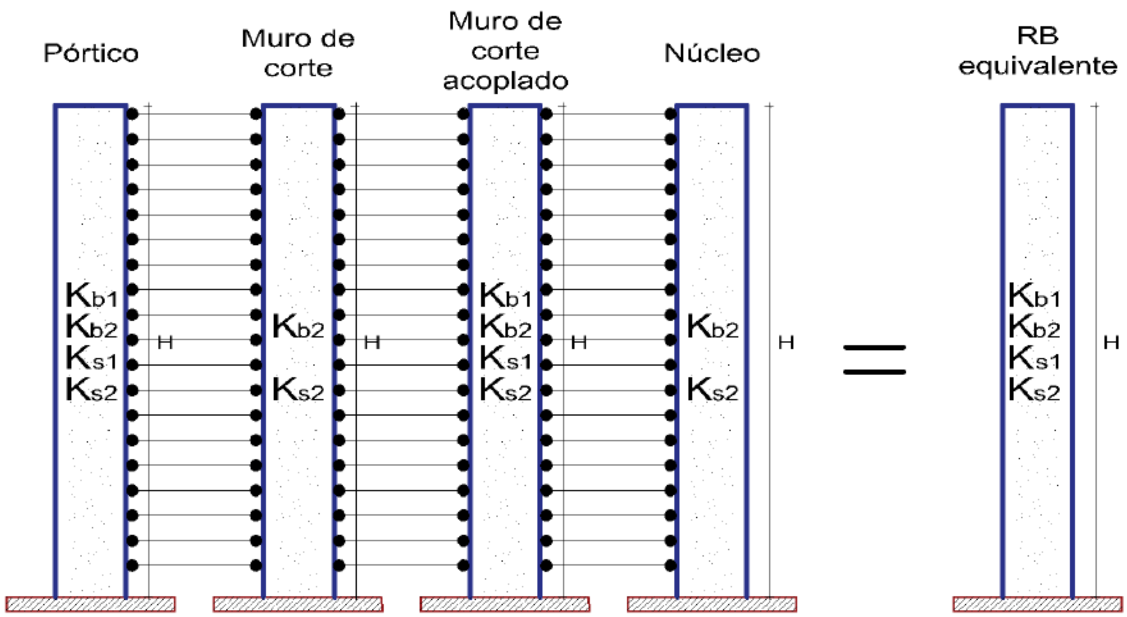


Figure 97. Structural elements and the equivalent sandwich beam replacement beam.

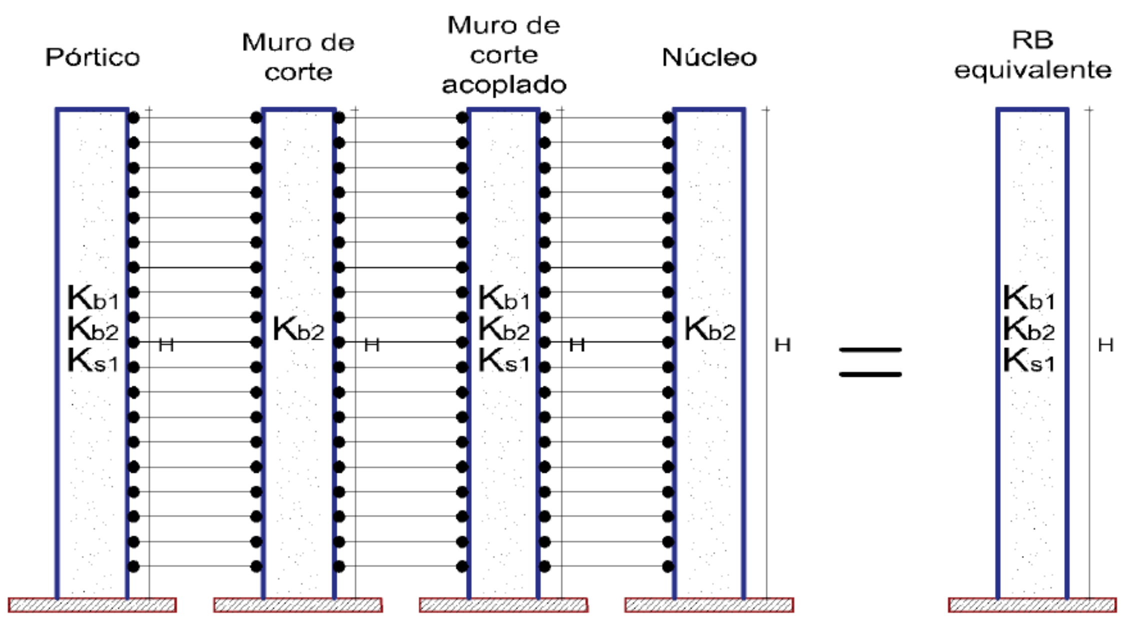


Figure 98. Structural elements and the generalized sandwich girder equivalent replacement beam.

As defined in previous chapters, a structural system can be outlined by an appropriate replacement beam. However, it is more complex to map the entire tall building using a single suitable replacement beam connecting all the lateral load-resisting structural systems. Six strategies are presented to solve this problem:

4.4.1.1 Strategy 1

In this strategy, each structural system is considered as a "Sandwich Beam" replacement beam and the equivalent stiffness properties of the overall replacement beam of the building are obtained by directly adding the stiffness properties of each structural system. The following relationships are suggested:

$$K_{b1} = \sum_{k=1}^{n} K_{b1k}\, , K_{b2} = \sum_{k=1}^{n} K_{b2k}\, , K_{s1} = \sum_{k=1}^{n} K_{s1k} \qquad (1596)$$

4.4.1.2 Strategy 2

In this strategy, each structural system is considered as a "Generalized Sandwich Beam" type replacement beam and the equivalent stiffness properties of the overall replacement beam of the building are obtained by directly adding the stiffness properties of each structural system. The following relationships are suggested:

$$K_{b1} = \sum_{k=1}^{n} K_{b1k}\, , K_{b2} = \sum_{k=1}^{n} K_{b2k}\, , K_{s1} = \sum_{k=1}^{n} K_{s1k}\, , K_{s1} = \sum_{k=1}^{n} K_{s2k} \qquad (1597)$$

4.4.1.3 Strategy 3

Potzta (2002) considered each structural system as a "sandwich beam" type replacement beam and proposed the global replacement beam of the building based on an energy formulation. The proposed replacement beam considers n resistant structural elements, where the kth element has stiffnesses K_{b1k}, K_{s1k} and K_{b2k} and the stiffnesses of the sandwich beam that replaces the building structure are indicated by K_{b1}, K_{s1} and K_{b2}.

Following Potzta (2002), the equivalent stiffnesses of the replacement beam are derived from the strain energy balance of the equivalent sandwich beam:

$$V = \frac{1}{2}\int_0^H \left\{K_{b1}{\theta'_{(x)}}^2 + K_{s1}\left[\theta_{(x)} - u'_{(x)}\right]^2 + K_{b2}{u''_{(x)}}^2\right\}dx \tag{1598}$$

From the energetic derivation of the equations of motion, the following property is derived:

$$K_{s1}\left[\theta_{(x)} - u'_{(x)}\right] - K_{b1}\theta''_{(x)} = 0 \tag{1599}$$

From the sum of the strain energies of each structural system:

$$\frac{1}{2}\int_0^H \left\{K_{b1}{\theta'_{(x)}}^2 + K_{s1}\left[\theta_{(x)} - u'_{(x)}\right]^2 + K_{b2}{u''_{(x)}}^2\right\}dx$$
$$= \frac{1}{2}\int \sum_{k=1}^{n}\left[K_{b1k}{\theta'_{(x)}}^2 + K_{s1k}\left[\theta_{(x)} - u'_{(x)}\right]^2 + K_{b2k}{u''_{(x)}}^2\right]dx \tag{1600}$$

Applying a sinusoidal displacement:

$$u_{(x)} = u_0 \sin\left(\frac{\pi}{H}x\right)$$
$$\theta_{(x)} = \theta_0 \cos\left(\frac{\pi}{H}x\right) \tag{1601}$$

From the property derived from the energetic derivation:

$$\theta_0 = \frac{\left(\frac{\pi}{H}\right)}{1 + \left(\frac{\pi}{H}\right)^2 \frac{K_{b1}}{K_{s1}}} u_0 \tag{1602}$$

Replacing and integrating in the equivalence of strain energies, we obtain:

$$\frac{4}{u_0^2\left(\frac{\pi}{H}\right)^4}V = K_{b2} + \frac{K_{b1}}{1 + \left(\frac{\pi}{H}\right)^2 \frac{K_{b1}}{K_{s1}}} = \sum_{k=1}^{n}\left[K_{b2k} + \frac{K_{b1k}}{1 + \left(\frac{\pi}{H}\right)^2 \frac{K_{b1k}}{K_{s1k}}}\right] \tag{1603}$$

Applying Taylor series with respect to $\left(\frac{1}{H}\right)^2$ approximately at $\left(\frac{1}{H_0}\right)^2$:

$$K_{b2}+\sum_{i=0}^{\infty}\frac{K_{b1}\left(-\pi^2\frac{K_{b1}}{K_{s1}}\right)^i}{\left[1+\left(\frac{\pi}{H_0}\right)^2\frac{K_{b1}}{K_{s1}}\right]^{i+1}}\left(\frac{1}{l^2}-\frac{1}{l_0^2}\right)^i=\sum_{k=1}^{n}\left\{K_{b2k}+\sum_{i=0}^{\infty}\frac{K_{b1k}\left(-\pi^2\frac{K_{b1k}}{K_{s1k}}\right)^i}{\left[1+\left(\frac{\pi}{H_0}\right)^2\frac{K_{b1k}}{K_{s1k}}\right]^{i+1}}\left(\frac{1}{l^2}-\frac{1}{l_0^2}\right)^i\right\} \tag{1604}$$

Considering the first three terms of the series; is obtained:

$$\begin{gathered}
K_{b2}+\frac{K_{b1}}{1+\left(\frac{1}{H_0}\right)^2\pi^2\frac{K_{b1}}{K_{s1}}}=\sum_{k=1}^{n}\left[K_{b2k}+\frac{K_{b1k}}{1+\left(\frac{1}{H_0}\right)^2\pi^2\frac{K_{b1k}}{K_{s1k}}}\right]\\
\frac{K_{b1}}{\left[1+\left(\frac{1}{H_0}\right)^2\pi^2\frac{K_{b1}}{K_{s1}}\right]^2}\left(\pi^2\frac{K_{b1}}{K_{s1}}\right)=\sum_{k=1}^{n}\left\{\frac{K_{b1k}}{\left[1+\left(\frac{1}{H_0}\right)^2\pi^2\frac{K_{b1k}}{K_{s1k}}\right]^2}\left(\pi^2\frac{K_{b1k}}{K_{s1k}}\right)\right\}\\
\frac{K_{b1}}{\left[1+\left(\frac{1}{H_0}\right)^2\pi^2\frac{K_{b1}}{K_{s1}}\right]^3}\left(\pi^2\frac{K_{b1}}{K_{s1}}\right)^2=\sum_{k=1}^{n}\left\{\frac{K_{b1k}}{\left[1+\left(\frac{1}{H_0}\right)^2\pi^2\frac{K_{b1k}}{K_{s1k}}\right]^3}\left(\pi^2\frac{K_{b1k}}{K_{s1k}}\right)^2\right\}
\end{gathered} \tag{1605}$$

Simultaneously solving the equations, we obtain:

$$K_{b1}=\frac{1}{\frac{C}{B^2}-\frac{1}{l_0^2}\frac{C^2}{B^3}},K_{b2}=A-\frac{B^2}{C},K_{s1}=\pi^2\frac{B^3}{C^2} \tag{1606}$$

Where:

$$\begin{gathered}
A=\sum_{k=1}^{n}\left[\frac{K_{b1k}}{1+\left(\frac{1}{H_0}\right)^2\pi^2\frac{K_{b1k}}{K_{s1k}}}\right]\\
B=\sum_{k=1}^{n}\left\{\frac{K_{b0k}}{\left[1+\left(\frac{1}{H_0}\right)^2\pi^2\frac{K_{b1k}}{K_{s1k}}\right]^2}.\pi^2\frac{K_{b1k}}{K_{s1k}}\right\}\\
C=\sum_{k=1}^{n}\left\{\frac{K_{b0k}}{\left[1+\left(\frac{1}{H_0}\right)^2\pi^2\frac{K_{b1k}}{K_{s1k}}\right]^3}.\left(\pi^2\frac{K_{b1k}}{K_{s1k}}\right)^2\right\}
\end{gathered} \tag{1607}$$

4.4.1.4 Strategy 4

Considering each structural element as a "generalized sandwich beam" type replacement beam, the global replacement beam of the building is proposed based on an energy formulation. The proposed replacement beam considers n resistant structural elements, where the kth element has stiffnesses K_{b1k}, K_{s1k}, K_{b2k} and K_{s2k} and the stiffnesses of the sandwich beam that replaces the building structure are indicated by K_{b1}, K_{s1}, K_{b2} and K_{s2}.

The equivalent stiffnesses of the replacement beam are derived from the strain energy balance of the equivalent generalized sandwich beam:

$$V = \frac{1}{2}\int_0^H \left\{K_{b1}\psi'^2 + K_{s1}\left[u'_{(x)} - \psi_{(x)}\right]^2 + K_{b2}{\theta'_{(x)}}^2 + K_{s2}\left[u'_{(x)} - \theta_{(x)}\right]^2\right\} dx \tag{1608}$$

From the energetic derivation of the equations of motion, the following property is derived:

$$\begin{Bmatrix} K_{b1}\psi''_{(x)} + K_{s1}\left[u'_{(x)} - \psi_{(x)}\right] = 0 \\ K_{b2}\theta''_{(x)} + K_{s2}\left[u'_{(x)} - \theta_{(x)}\right] = 0 \end{Bmatrix} \tag{1609}$$

From the sum of the strain energies of each structural system:

$$\frac{1}{2}\int_0^H \left\{K_{b1}\psi'^2 + K_{s1}\left[u'_{(x)} - \psi_{(x)}\right]^2 + K_{b2}{\theta'_{(x)}}^2 + K_{s2}\left[u'_{(x)} - \theta_{(x)}\right]^2\right\} dx$$
$$= \frac{1}{2}\int \sum_{k=1}^{n} \left\{K_{b1k}\psi'^2 + K_{s1k}\left[u'_{(x)} - \psi_{(x)}\right]^2 + K_{b2k}{\theta'_{(x)}}^2 + K_{s2k}\left[u'_{(x)} - \theta_{(x)}\right]^2\right\} dx \tag{1610}$$

Applying a sinusoidal displacement:

$$\begin{Bmatrix} u_{(x)} = u_0 \sin\left(\frac{\pi}{H}x\right) \\ \varphi_{(x)} = \varphi_0 \cos\left(\frac{\pi}{H}x\right) \\ \theta_{(x)} = \theta_0 \cos\left(\frac{\pi}{H}x\right) \end{Bmatrix} \tag{1611}$$

From the property derived from the energetic derivation:

$$\begin{cases} \varphi_0 = \dfrac{\left(\frac{\pi}{H}\right)}{1+\left(\frac{\pi}{H}\right)^2 \frac{K_{b1}}{K_{s1}}} u_0 \\ \theta_0 = \dfrac{\left(\frac{\pi}{H}\right)}{1+\left(\frac{\pi}{H}\right)^2 \frac{K_{b2}}{K_{s2}}} u_0 \end{cases} \tag{1612}$$

Replacing and integrating in the equivalence of strain energies, we obtain:

$$\frac{4}{u_0^2\left(\frac{\pi}{H}\right)^4} V = \frac{K_{b1}}{1+\left(\frac{\pi}{H}\right)^2 \frac{K_{b1}}{K_{s1}}} + \frac{K_{b2}}{1+\left(\frac{\pi}{H}\right)^2 \frac{K_{b2}}{K_{s2}}} = \sum_{k=1}^{n} \left[\frac{K_{b1k}}{1+\left(\frac{\pi}{H}\right)^2 \frac{K_{b1k}}{K_{s1k}}} + \frac{K_{b2k}}{1+\left(\frac{\pi}{H}\right)^2 \frac{K_{b2k}}{K_{s2k}}} \right] \tag{1613}$$

Applying Taylor series with respect to $\left(\frac{1}{H}\right)^2$ approximately at $\left(\frac{1}{H_0}\right)^2$:

$$\sum_{i=0}^{\infty} \frac{K_{b1}\left(-\pi^2 \frac{K_{b1}}{K_{s1}}\right)^i}{\left[1+\left(\frac{\pi}{H_0}\right)^2 \frac{K_{b1}}{K_{s1}}\right]^{i+1}} \left(\frac{1}{l^2} - \frac{1}{l_0^2}\right)^i + \sum_{i=0}^{\infty} \frac{K_{b2}\left(-\pi^2 \frac{K_{b2}}{K_{s2}}\right)^i}{\left[1+\left(\frac{\pi}{H_0}\right)^2 \frac{K_{b2}}{K_{s2}}\right]^{i+1}} \left(\frac{1}{l^2} - \frac{1}{l_0^2}\right)^i$$
$$= \sum_{k=1}^{n} \left\{ \sum_{i=0}^{\infty} \frac{K_{b1k}\left(-\pi^2 \frac{K_{b1k}}{K_{s1k}}\right)^i}{\left[1+\left(\frac{\pi}{H_0}\right)^2 \frac{K_{b1k}}{K_{s1k}}\right]^{i+1}} \left(\frac{1}{l^2} - \frac{1}{l_0^2}\right)^i + \sum_{i=0}^{\infty} \frac{K_{b2k}\left(-\pi^2 \frac{K_{b2k}}{K_{s2k}}\right)^i}{\left[1+\left(\frac{\pi}{H_0}\right)^2 \frac{K_{b2k}}{K_{s2k}}\right]^{i+1}} \left(\frac{1}{l^2} - \frac{1}{l_0^2}\right)^i \right\} \tag{1614}$$

Considering the first four terms of the series; is obtained:

$$\left\{ \frac{K_{b1}}{1+\left(\frac{1}{H_0}\right)^2 \pi^2 \frac{K_{b1}}{K_{s1}}} + \frac{K_{b2}}{1+\left(\frac{1}{H_0}\right)^2 \pi^2 \frac{K_{b2}}{K_{s2}}} = \sum_{k=1}^{n} \left[\frac{K_{b1k}}{1+\left(\frac{1}{H_0}\right)^2 \pi^2 \frac{K_{b1k}}{K_{s1k}}} + \frac{K_{b2k}}{1+\left(\frac{1}{H_0}\right)^2 \pi^2 \frac{K_{b2k}}{K_{s2k}}} \right] \right\}$$

$$\left\{\frac{K_{b1}}{\left[1+\left(\frac{1}{H_0}\right)^2\pi^2\frac{K_{b1}}{K_{s1}}\right]^2}\left(\pi^2\frac{K_{b1}}{K_{s1}}\right)+\frac{K_{b2}}{\left[1+\left(\frac{1}{H_0}\right)^2\pi^2\frac{K_{b2}}{K_{s2}}\right]^2}\left(\pi^2\frac{K_{b2}}{K_{s2}}\right)\right.$$

$$\left.=\sum_{k=1}^{n}\left\{\frac{K_{b1k}}{\left[1+\left(\frac{1}{H_0}\right)^2\pi^2\frac{K_{b1k}}{K_{s1k}}\right]^2}\left(\pi^2\frac{K_{b1k}}{K_{s1k}}\right)+\frac{K_{b2k}}{\left[1+\left(\frac{1}{H_0}\right)^2\pi^2\frac{K_{b2k}}{K_{s2k}}\right]^2}\left(\pi^2\frac{K_{b2k}}{K_{s2k}}\right)\right\}\right\}$$

$$\left\{\frac{K_{b1}}{\left[1+\left(\frac{1}{H_0}\right)^2\pi^2\frac{K_{b1}}{K_{s1}}\right]^3}\left(\pi^2\frac{K_{b1}}{K_{s1}}\right)^2+\frac{K_{b2}}{\left[1+\left(\frac{1}{H_0}\right)^2\pi^2\frac{K_{b2}}{K_{s2}}\right]^3}\left(\pi^2\frac{K_{b2}}{K_{s2}}\right)^2\right.$$

$$\left.=\sum_{k=1}^{n}\left\{\frac{K_{b1k}}{\left[1+\left(\frac{1}{H_0}\right)^2\pi^2\frac{K_{b1k}}{K_{s1k}}\right]^3}\left(\pi^2\frac{K_{b1k}}{K_{s1k}}\right)^2+\frac{K_{b2k}}{\left[1+\left(\frac{1}{H_0}\right)^2\pi^2\frac{K_{b2k}}{K_{s2k}}\right]^3}\left(\pi^2\frac{K_{b2k}}{K_{s2k}}\right)^2\right\}\right\}$$

$$\left\{\frac{K_{b1}}{\left[1+\left(\frac{1}{H_0}\right)^2\pi^2\frac{K_{b1}}{K_{s1}}\right]^4}\left(\pi^2\frac{K_{b1}}{K_{s1}}\right)^3+\frac{K_{b2}}{\left[1+\left(\frac{1}{H_0}\right)^2\pi^2\frac{K_{b2}}{K_{s2}}\right]^4}\left(\pi^2\frac{K_{b2}}{K_{s2}}\right)^3\right.$$

$$\left.=\sum_{k=1}^{n}\left\{\frac{K_{b1k}}{\left[1+\left(\frac{1}{H_0}\right)^2\pi^2\frac{K_{b1k}}{K_{s1k}}\right]^4}\left(\pi^2\frac{K_{b1k}}{K_{s1k}}\right)^3+\frac{K_{b2k}}{\left[1+\left(\frac{1}{H_0}\right)^2\pi^2\frac{K_{b2k}}{K_{s2k}}\right]^4}\left(\pi^2\frac{K_{b2k}}{K_{s2k}}\right)^3\right\}\right\}$$

(1615)

i.e.,

$$
\left\{
\begin{array}{c}
\dfrac{K_{b1}}{1+\left(\dfrac{1}{H_0}\right)^2 \pi^2 \dfrac{K_{b1}}{K_{s1}}} + \dfrac{K_{b2}}{1+\left(\dfrac{1}{H_0}\right)^2 \pi^2 \dfrac{K_{b2}}{K_{s2}}} = A \\
\dfrac{K_{b1}}{\left[1+\left(\dfrac{1}{H_0}\right)^2 \pi^2 \dfrac{K_{b1}}{K_{s1}}\right]^2}\left(\pi^2 \dfrac{K_{b1}}{K_{s1}}\right) + \dfrac{K_{b2}}{\left[1+\left(\dfrac{1}{H_0}\right)^2 \pi^2 \dfrac{K_{b2}}{K_{s2}}\right]^2}\left(\pi^2 \dfrac{K_{b2}}{K_{s2}}\right) = B \\
\dfrac{K_{b1}}{\left[1+\left(\dfrac{1}{H_0}\right)^2 \pi^2 \dfrac{K_{b1}}{K_{s1}}\right]^3}\left(\pi^2 \dfrac{K_{b1}}{K_{s1}}\right)^2 + \dfrac{K_{b2}}{\left[1+\left(\dfrac{1}{H_0}\right)^2 \pi^2 \dfrac{K_{b2}}{K_{s2}}\right]^3}\left(\pi^2 \dfrac{K_{b2}}{K_{s2}}\right)^2 = C \\
\dfrac{K_{b1}}{\left[1+\left(\dfrac{1}{H_0}\right)^2 \pi^2 \dfrac{K_{b1}}{K_{s1}}\right]^4}\left(\pi^2 \dfrac{K_{b1}}{K_{s1}}\right)^3 + \dfrac{K_{b2}}{\left[1+\left(\dfrac{1}{H_0}\right)^2 \pi^2 \dfrac{K_{b2}}{K_{s2}}\right]^4}\left(\pi^2 \dfrac{K_{b2}}{K_{s2}}\right)^3 = D
\end{array}
\right\} \tag{1616}
$$

Where:

$$
\left\{
\begin{array}{c}
A = \displaystyle\sum_{k=1}^{n}\left[\dfrac{K_{b1k}}{1+\left(\dfrac{1}{H_0}\right)^2 \pi^2 \dfrac{K_{b1k}}{K_{s1k}}} + \dfrac{K_{b2k}}{1+\left(\dfrac{1}{H_0}\right)^2 \pi^2 \dfrac{K_{b2k}}{K_{s2k}}}\right] \\
B = \displaystyle\sum_{k=1}^{n}\left\{\dfrac{K_{b1k}}{\left[1+\left(\dfrac{1}{H_0}\right)^2 \pi^2 \dfrac{K_{b1k}}{K_{s1k}}\right]^2}.\pi^2 \dfrac{K_{b1k}}{K_{s1k}} + \dfrac{K_{b2k}}{\left[1+\left(\dfrac{1}{H_0}\right)^2 \pi^2 \dfrac{K_{b2k}}{K_{s2k}}\right]^2}.\pi^2 \dfrac{K_{b2k}}{K_{s2k}}\right\} \\
C = \displaystyle\sum_{k=1}^{n}\left\{\dfrac{K_{b1k}}{\left[1+\left(\dfrac{1}{H_0}\right)^2 \pi^2 \dfrac{K_{b1k}}{K_{s1k}}\right]^3}.\left(\pi^2 \dfrac{K_{b1k}}{K_{s1k}}\right)^2 + \dfrac{K_{b2k}}{\left[1+\left(\dfrac{1}{H_0}\right)^2 \pi^2 \dfrac{K_{b2k}}{K_{s2k}}\right]^2}.\pi^2 \dfrac{K_{b2k}}{K_{s2k}}\right\} \\
D = \displaystyle\sum_{k=1}^{n}\left\{\dfrac{K_{b1k}}{\left[1+\left(\dfrac{1}{H_0}\right)^2 \pi^2 \dfrac{K_{b1k}}{K_{s1k}}\right]^4}.\left(\pi^2 \dfrac{K_{b1k}}{K_{s1k}}\right)^3 + \dfrac{K_{b2k}}{\left[1+\left(\dfrac{1}{H_0}\right)^2 \pi^2 \dfrac{K_{b2k}}{K_{s2k}}\right]^4}.\left(\pi^2 \dfrac{K_{b2k}}{K_{s2k}}\right)^3\right\}
\end{array}
\right\} \tag{1617}
$$

The equivalent characteristic stiffnesses can be obtained after numerically solving the system of equations.

4.4.1.5 Strategy 5

Following the methodology proposed by Zalka (2020), each structural system is modeled differently as proposed in this research project.

The external load is distributed according to the stiffness of each structural element:

$$q_i(z) = p_i . f(z) \tag{1618}$$

Where $f(z)$ is the total external load acting on the building and q_i is the external load allocator to the individual structural elements. Since the field is linear elastic, this load distributor can be used in both forces and moments. The value of the load sharer is defined as:

$$p_i = \frac{S_i}{\sum_{i=1}^{n} S_i} \tag{1619}$$

Where S_i is the lateral stiffness of the i-th structural element. The lateral stiffness of a structural element is defined as:

$$S_i = \frac{1}{y_i(H)} \tag{1620}$$

Where $y_i(H)$ is the maximum upper lateral displacement of the i-th structural element.

Whichever structural element is chosen to evaluate the lateral displacement of the building, the result will be the same. Therefore, it seems practical to choose a shear wall as the ith structural element, since its equation is simpler. It is important to clarify that, however, to calculate the external load spreader in the structural element, it is necessary to determine the maximum deflection of each structural element of the bracing system, and, therefore, it is necessary to solve the lateral displacement of all the structural elements present in the building.

The method is simple and easy to apply. The drawback is that it is not possible to take into account the direct interaction between the structural elements and, therefore, this complex interaction is ignored.

4.4.1.6 Strategy 6

The proposed replacement beam considers n load-resistant lateral subsystems, where the kth “Sandwich Beam” type elements have stiffnesses K_{b1k}, K_{b2k} and K_{s1k}; and the j-th “Bending Beam (EBB)” type element has only local bending stiffness $K_{b1,j}$.

The equivalent stiffnesses of the replacement beam are derived from the strain energy balance of the SWB and EBB beam, respectively:

$$V = \frac{1}{2}\int \sum_{k=1}^{n1} \left\{K_{b1k}\theta'^{\,2}_{(x)} + K_{s1k}\left[\theta_{(x)} - u'_{(x)}\right]^2 + K_{b2k}u''^{\,2}_{(x)}\right\} dx + \frac{1}{2}\int_0^H \sum_{j=1}^{n2}\left[K_{b2j}u''^{\,2}_{(x)}\right] dx \tag{1621}$$

Sorting properly:

$$V = \frac{1}{2}\int \sum_{k=1}^{n1} \left\{K_{b1k}\theta'^{\,2}_{(x)} + K_{s1k}\left[\theta_{(x)} - u'_{(x)}\right]^2\right\} dx + \frac{1}{2}\int_0^H \left(\sum_{k=1}^{n1} K_{b2k} + \sum_{j=1}^{n2} K_{b2}\right) u''^{\,2}_{(x)} dx \tag{1622}$$

In terms of energy, including the local bending strain energy of the EBB beam in the sum of the local bending strain energies of the SWB beams only increases the local bending stiffness of the members. Another important aspect is that by including the local bending stiffness of the EBB beam in the SWB beam, it is ensured that the interaction between both beams (direct interaction) is automatically taken into account.

Following this concept, Zalka (2020) distributed the total local bending stiffness of the shear walls and/or cores to the coupled frames and/or shear walls based on their relative stiffness.

$$S_i = \frac{1}{y_i(H)} \tag{1623}$$

Where $y_i(H)$ is the maximum upper lateral displacement of the i-th frame and/or coupled shear wall. The value of the moment dealer is defined as:

$$p_i = \frac{S_i}{\sum_{i=1}^{n} S_i} \tag{1624}$$

The resulting system consists of a series of frames and/or shear walls coupled with a new local bending stiffness (product of the distribution of the local bending stiffness of the shear walls and/or cores); then the procedure described in strategy 5 is followed.

4.5 STATIC STRUCTURAL ANALYSIS OF THE TALL BUILDING

4.5.1 Lateral Displacement of the Building

When the building is doubly symmetrical in plan, the lateral displacements are calculated directly from the analysis of the replacement beam with its equivalent characteristic stiffnesses calculated according to the appropriate strategy.

4.5.2 Torsional Displacement of the Building

There is an analogy known as the "Vlasov analogy" for thin-walled structures subjected to bending and torsion. According to this analogy, deflections, bending moments, and shear forces correspond to rotation, strain moments, and torsional moments, respectively.

In order to calculate the rotation of the building we will only refer to the analogy regarding deflections and rotations. We establish the stiffnesses corresponding to the deflections and rotations:

$$\begin{Bmatrix} K_{b1}^{*} = t^2 . K_{b1} \\ K_{b2}^{*} = t^2 . K_{b2} \\ K_{s1}^{*} = t^2 . K_{s1} \\ K_{s2}^{*} = t^2 . K_{s2} \end{Bmatrix} \qquad (1625)$$

Where t is the distance from the shear center of the building to the shear center of each bracing element.

It is concluded that the pure deflection analysis of a tall building can be used to perform the torsional analysis if the stiffnesses of the bracing elements meet their equivalent torsional stiffness.

A drawback arises, there is no exact methodology for calculating the shear center of a building consisting of different structural elements. For the purposes of this research project, the Zalka (2020) approach will be followed:

$$\left\{ \bar{X}_0 = \frac{\sum_{i=1}^{n} S_{y,i} . \bar{X}_i}{\sum_{i=1}^{n} S_{y,i}} , \bar{Y}_0 = \frac{\sum_{i=1}^{n} S_{x,i} . \bar{Y}_i}{\sum_{i=1}^{n} S_{x,i}} \right\} \qquad (1626)$$

Where $\bar{X}_i$, $\bar{Y}_i$ are the perpendicular distances of the i-th shear centers to the origin of the coordinate system, n is the number of structural elements and S_i is the stiffness of the i-th bracing element calculated as the inverse of the maximum deflection due to a unit load.

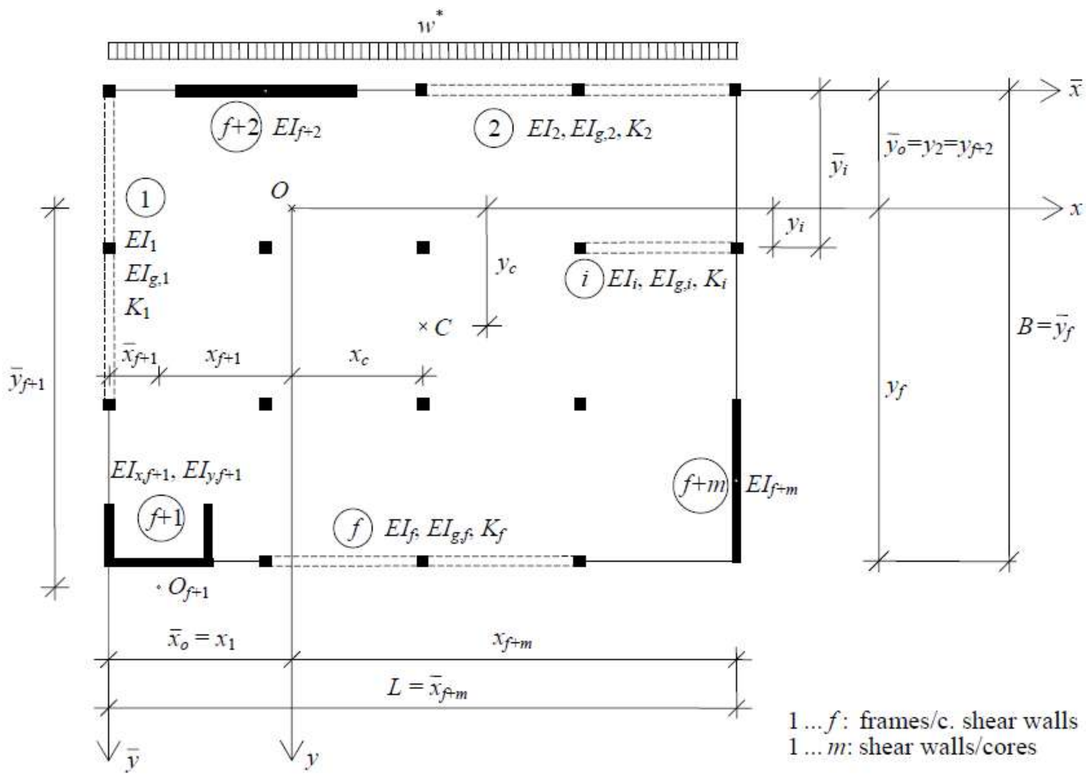


Figure 99. Structural elements with their respective torsion arm (Zalka, 2020)

The total torsional moment of the building is:

$$m = w.X_c = w\left(\frac{L}{2} - \bar{X}_0\right) \tag{1627}$$

The shared torsional moment in the i-th unit of bracing is:

$$m = q_{w,i}.m \tag{1628}$$

Where the torsional parameter $q_{w,i}$ fulfills the same sharing function as the translational parameter q_i used in the previous section.

$$q_{w,i} = \frac{S_{w,i}}{\sum_{i=1}^{f+m} S_{w,i}} \tag{1629}$$

The governing torsional stiffness of the i-th reinforcement unit is defined as:

$$S_{w,i} = S_i . t_i^2 = \frac{t_i^2}{y_{max}} = \frac{t_i^2}{y_H} \tag{1630}$$

Whichever structural element is chosen to evaluate the rotation of the building, the result will be the same. Therefore, it seems practical to choose a shear wall, since its equation is simpler. It is important to clarify that, however, to calculate the external load spreader in the structural element, it is necessary to determine the maximum deflection of each structural element of the bracing system, and, therefore, it is necessary to solve the lateral displacement of all the structural elements present in the building.

4.5.3 Coupled Lateral-Torsional Displacement of the Building

A well-known concept among structural engineers is that “there is no building without torsion”. Although it is true that this concept is relative because mathematically it can be achieved that the building is doubly symmetrical, the term "accidental eccentricity" that is provided in most of the seismic codes of the world, makes this definition absolutely true.

When a tall building is subjected to a horizontal lateral load, the building responds in a complex way by developing two phenomena: pure translational displacement and rotation around the center of stiffness of the building.

The fact that we are under a linear elastic behavior of the building means that, by means of the superposition principle, both phenomena can be separated, which allows the building to be analyzed under a purely lateral displacement and under a pure rotation around the center of rigidity.

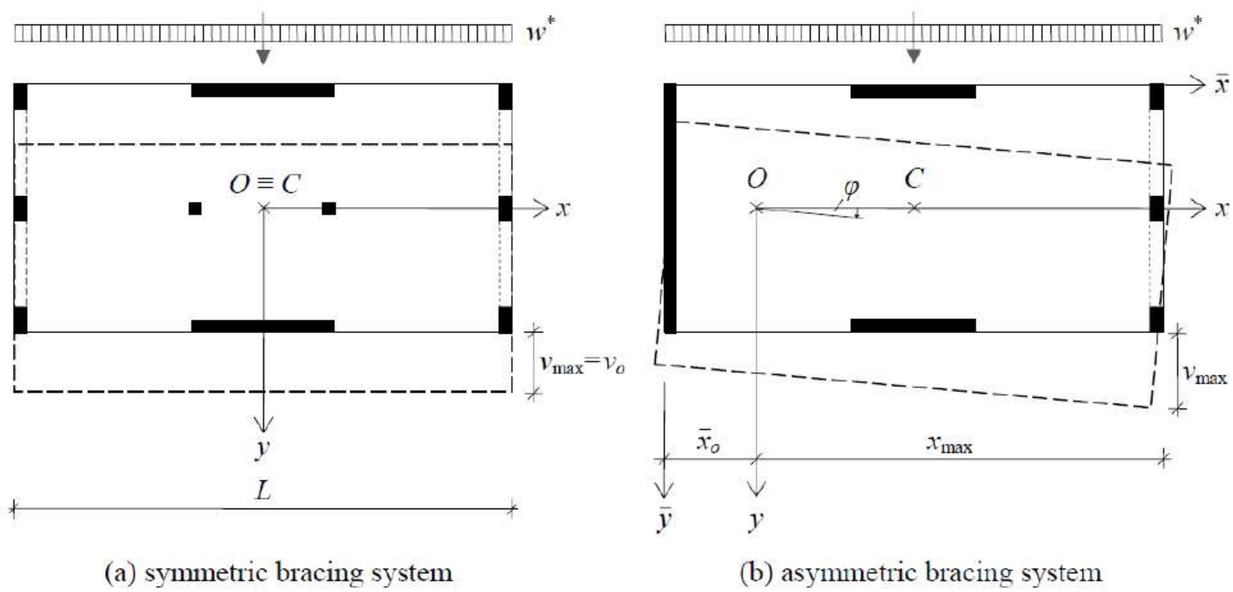


Figure 100. Typical bracing system arrangements (Zalka, 2020).

Where point "O" locates the center of mass and point "C" locates the center of stiffness.

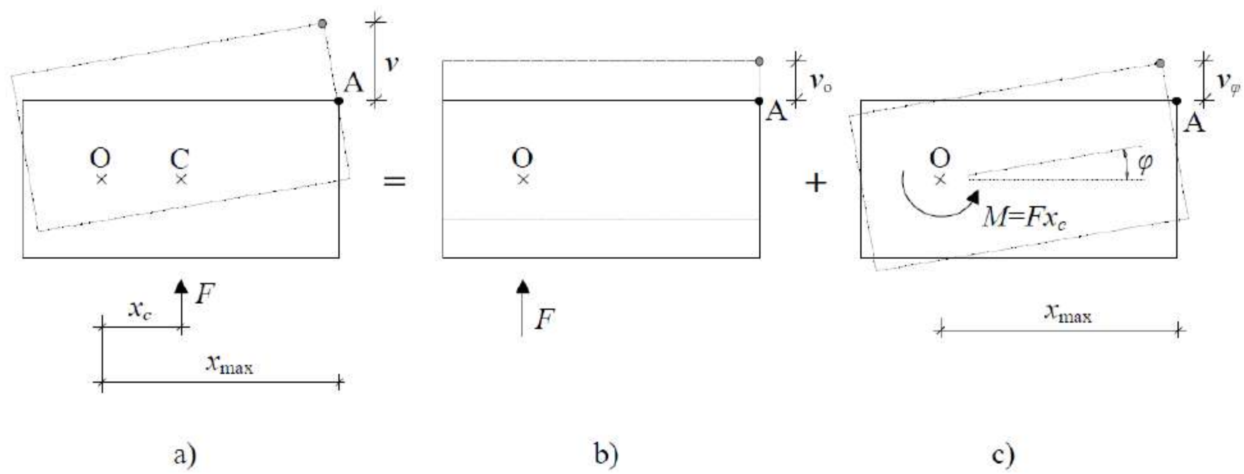


Figure 101. Total displacement of an asymmetrical building. a) v =maximum displacement, b) v_0=displacement due to an applied force at its center of rigidity, and c) v_{φ}=displacement due to the torsional moment at its center of rigidity (Zalka, 2020)

The behavior of the building is then analyzed by transferring the horizontal lateral load located in the center of mass towards the center of rigidity, producing a torsional moment due to the transfer of the horizontal lateral force $(M = F.X_c)$. The horizontal load develops only lateral displacements while the torsional moment develops only rotation in the building (around the center of stiffness).

The maximum displacement of the building is developed at the farthest point from the center of rigidity and using the angle of rotation, we have:

$$v = v_0 + v_\varphi \tag{1631}$$

The maximum displacement is:

$$v_{max} = v_{(H)} = v_{0(H)} + x_{max} . \varphi_{(H)} \tag{1632}$$

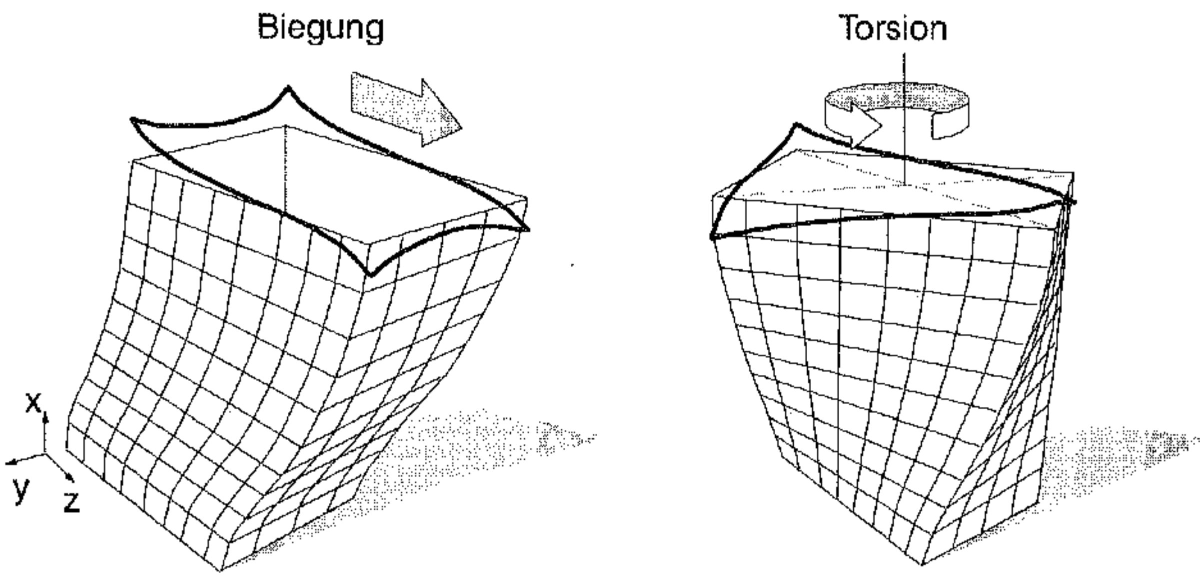


Figure 102. Lateral and torsional displacement of a building (Schmidts, 1998).

4.6 DYNAMIC STRUCTURAL ANALYSIS OF THE TALL BUILDING

4.6.1 Lateral Period of the Building

When the building is doubly symmetric in plan, the lateral periods are calculated directly from the analysis of the replacement beam with its equivalent characteristic stiffnesses calculated according to the appropriate strategy.

4.6.2 Torsional Period of the Building

The Vlasov analogy is used for the calculation of the characteristic stiffnesses and the approach of Zalka (2020) is adopted to calculate the shear center:

$$\left\{\bar{X}_0 = \frac{\sum_{i=1}^{n} f_{y,i}^2 . \bar{X}_i}{\sum_{i=1}^{n} f_{y,i}^2}, \bar{Y}_0 = \frac{\sum_{i=1}^{n} f_{x,i}^2 . \bar{Y}_i}{\sum_{i=1}^{n} f_{x,i}^2}\right\} \tag{1633}$$

Where f_i is the frequency of the i-th bracing element. It should be noted that the distributed mass correction factor must be applied to the characteristic stiffnesses and the distributed mass corresponds to the torsional mass and is equal to the lateral mass multiplied by the radius of gyration squared.

4.6.3 Lateral-Torsional Coupled Period of the Building

Once the lateral and torsional decoupled frequencies are known, to calculate the coupled frequencies, Zalka (2020) proposes the following equation:

$$(f^2)^3 + a_2(f^2)^2 + a_1(f^2) + a_0 = 0 \tag{1634}$$

Where:

$$\left\{\begin{array}{c} a_0 = \dfrac{f_x^2 f_y^2 f_\varphi^2}{1 - t_x^2 - t_y^2}, a_1 = \dfrac{f_x^2 f_y^2 + f_\varphi^2 f_x^2 + f_\varphi^2 f_y^2}{1 - t_x^2 - t_y^2} \\ a_2 = \dfrac{f_x^2 t_x^2 + f_y^2 t_y^2 - f_x^2 - f_y^2 - f_\varphi^2}{1 - t_x^2 - y_y^2}, t_x = \dfrac{x_c}{i_p}, t_y = \dfrac{y_c}{i_p} \end{array}\right\} \tag{1635}$$

The smallest root of the cube equation yields the torsional lateral coupled frequency of the building.

4.7 STABILITY ANALYSIS OF THE TALL BUILDING

4.7.1 Lateral Critical Load of the Building

When the building is doubly symmetrical in plan, the critical lateral loads are calculated directly from the analysis of the replacement beam with its equivalent characteristic stiffnesses calculated according to the appropriate strategy.

4.7.2 Critical Torsional Load of the Building

The Vlasov analogy is used for the calculation of the characteristic stiffnesses and the approach of Zalka (2020) is adopted to calculate the shear center:

$$\left\{\bar{X}_0 = \frac{\sum_{i=1}^{n} N_{y,i}.\bar{X}_i}{\sum_{i=1}^{n} N_{y,i}}, \bar{Y}_0 = \frac{\sum_{i=1}^{n} N_{x,i}.\bar{Y}_i}{\sum_{i=1}^{n} N_{x,i}}\right\} \tag{1636}$$

Where N_i is the frequency of the i-th bracing element. It should be noted that the distributed mass correction factor must be applied to the characteristic stiffnesses and the distributed mass corresponds to the torsional mass and is equal to the lateral mass multiplied by the radius of gyration squared.

4.7.3 Lateral-Torsional Coupled Critical Load of the Building

Once the critical lateral and torsional uncoupled loads are known, to calculate the critical coupled loads, Zalka (2020) proposes the following equation:

$$(N)^3 + b_2(N)^2 + b_1(N) + b_0 = 0 \tag{1637}$$

Where:

$$\left\{\begin{matrix} b_0 = \dfrac{N_{cr,x}N_{cr,y}N_{cr,\varphi}}{1 - t_x^2 - t_y^2}, b_1 = \dfrac{N_{cr,x}N_{cr,y} + N_{cr,\varphi}N_{cr,x} + N_{cr,\varphi}N_{cr,y}}{1 - t_x^2 - t_y^2} \\ b_2 = \dfrac{N_{cr,x}t_x^2 + N_{cr,y}t_y^2 - N_{cr,x} - N_{cr,y} - N_{cr,\varphi}}{1 - t_x^2 - y_y^2}, t_x = \dfrac{x_c}{i_p}, t_y = \dfrac{y_c}{i_p} \end{matrix}\right\} \tag{1638}$$

The smallest root of the cube equation yields the critical torsional lateral coupled load of the building.

4.8 NUMERICAL APPLICATIONS

In order to verify the efficiency of the replacement beam models developed in this research project, in this section the precision and reliability analysis of the proposed methodology will be developed. For the comparison, the finite element programs SAP 2000 and ETABS 2016 will be used, which will be considered exact. A positive difference means an overestimation and a negative answer means an underestimation of the approximate answer compared to the answer considered exact.

4.8.1 Shear Wall

A total of 90 shear walls will be analyzed, consisting of ten W1-W10 walls whose height will be varied from five to eighty stories (5, 10, 15, 20, 25, 30, 40, 6, 80 stories). Modulus of elasticity is $E = 25x10^6$ kN/m2, Poisson's ratio is ν =0.20, shear modulus is $G = 14.42x10^6$ kN/m2, height of floor is $h = 3.00$ m, the base is $b = 0.25$ m and the uniformly distributed wind load is $w = 5.00$ kN/m.

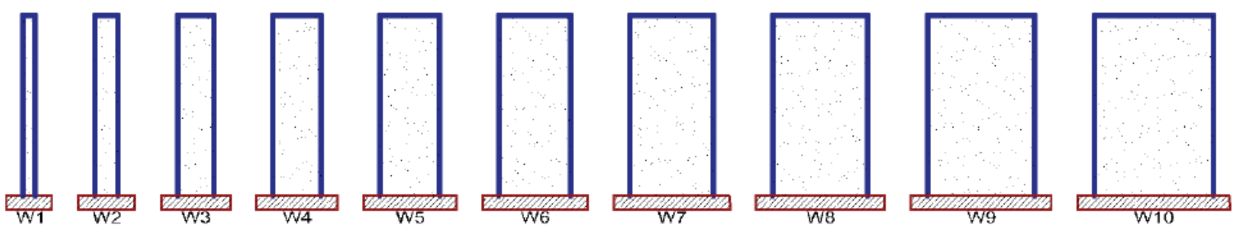


Figure 103. Shear walls W1-W10 for precision analysis.

An accuracy analysis of the Timoshenko beam as a replacement beam for shear walls was performed. The summary of the analyzes is given in Table 10 where the term "difference range" refers to the difference between the approximate solution and the FEM solution.

Tabla.10 *Accuracy of Timoshenko beam (TB) for maximum deflection analysis of W1-W10 shear walls.*

Shear wall	Difference Range (%)	Average absolute difference (%)	Maximum difference (%)
continuous solution	-0.035% - 0.030%	0.005%	0.035%

Figure 104 shows how the difference range varies as the height of the structural element increases.

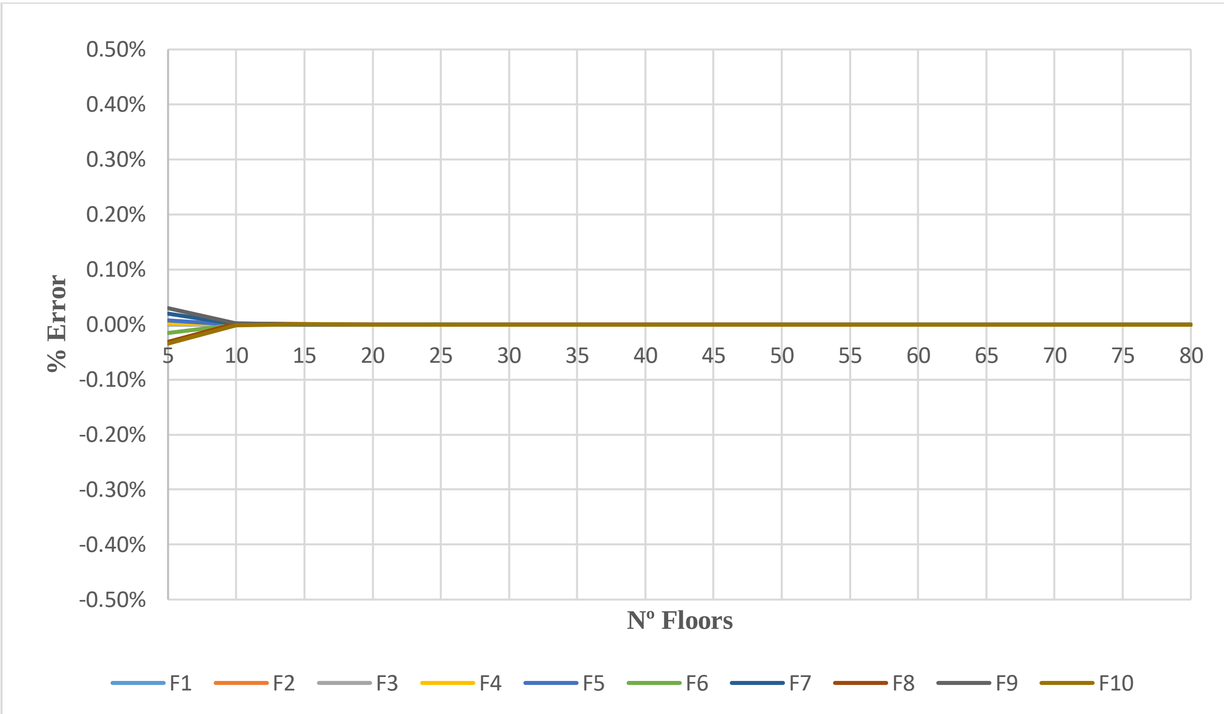


Figure 104. Accuracy of the Timoshenko beam (TB) as a replacement beam for shear walls.

An analysis of table 10 and figure 105 show the excellent accuracy of the Timoshenko beam as a replacement beam for shear walls. The results show a range of difference between -0.035% and 0.030%, an average absolute difference of 0.005% and a maximum difference of 0.035%.

In order to analyze the influence of the shear effect, shear walls were also idealized as bending beams (EBB). The summary can be seen in table 11.

Tabla.11 *Bending Beam Accuracy (EBB) for Maximum Deflection Analysis of W1-W10 Shear Walls.*

shear walls	Difference Range (%)	Average absolute difference (%)	Maximum difference (%)
continuous solution	-29.9% - 0.0%	29.9%	29.9%

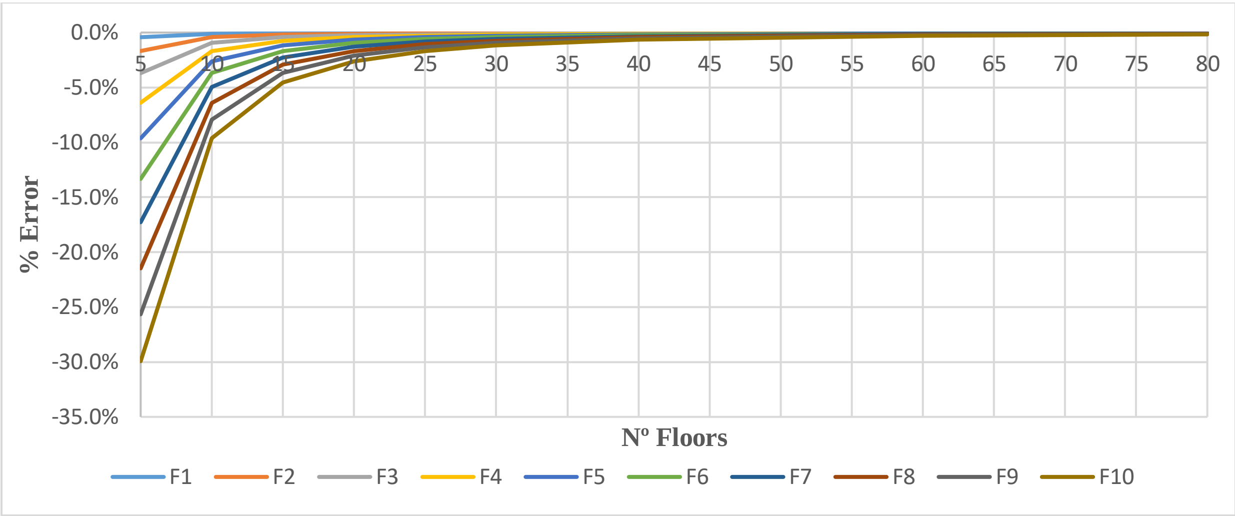


Figure 105. Precision Bending Beam (EBB) as replacement beam for shear walls.

As expected, large errors are expected for less slender walls because the shear effect is important and cannot be neglected. In all the cases analyzed, it is observed that the bending beam model (EBB) underestimates the results. The results show a difference range between -29.9% and 0%, an average absolute difference of 29.9% and a maximum difference of 29.9%.

It is concluded that using the replacement beam type "Tymoshenko Beam (TB)" to analyze shear walls is appropriate and shows excellent precision, being considered exact.

4.8.2 Frame

A total of 540 frames will be analyzed. The analysis consists of 60 frames F1-F60, whose height will vary from five to eighty stories (5, 10, 15, 20, 25, 30, 40, 6, 80 stories) and the number of bays in one and four. Modulus of elasticity is $E = 25x10^6$ kN/m2, Poisson's ratio is ν =0.20, shear modulus is $G = 14.42x10^6$ kN/m2, height of floor is $h = 3.00$ m, the centerline length between columns is $L = 6.00$ m and the uniformly distributed wind load is $w = 5.00$ kN/m. Table 11 shows the sections of the beam and column structural elements.

Tabla.12 *Column and beam section for frames F1-F60*

Frame	Beam		Column		Frame	Beam		Column	
	b (m)	h (m)	b (m)	h (m)		b (m)	h (m)	b (m)	h (m)
F1	0.4	0.4	0.4	0.4	F31	0.4	0.4	0.4	1.6
F2	0.4	0.6	0.4	0.4	F32	0.4	0.6	0.4	1.6
F3	0.4	0.8	0.4	0.4	F33	0.4	0.8	0.4	1.6
F4	0.4	1.0	0.4	0.4	F34	0.4	1.0	0.4	1.6
F5	0.4	1.2	0.4	0.4	F35	0.4	1.2	0.4	1.6
F6	0.4	0.4	0.4	0.6	F36	0.4	0.4	0.4	1.8
F7	0.4	0.6	0.4	0.6	F37	0.4	0.6	0.4	1.8
F8	0.4	0.8	0.4	0.6	F38	0.4	0.8	0.4	1.8
F9	0.4	1.0	0.4	0.6	F39	0.4	1.0	0.4	1.8
F10	0.4	1.2	0.4	0.6	F40	0.4	1.2	0.4	1.8
F11	0.4	0.4	0.4	0.8	F41	0.4	0.4	0.4	2.0
F12	0.4	0.6	0.4	0.8	F42	0.4	0.6	0.4	2.0
F13	0.4	0.8	0.4	0.8	F43	0.4	0.8	0.4	2.0
F14	0.4	1.0	0.4	0.8	F44	0.4	1.0	0.4	2.0
F15	0.4	1.2	0.4	0.8	F45	0.4	1.2	0.4	2.0
F16	0.4	0.4	0.4	1.0	F46	0.4	0.4	0.4	0.4
F17	0.4	0.6	0.4	1.0	F47	0.4	0.6	0.4	0.4
F18	0.4	0.8	0.4	1.0	F48	0.4	0.8	0.4	0.4
F19	0.4	1.0	0.4	1.0	F49	0.4	0.4	0.4	0.8
F20	0.4	1.2	0.4	1.0	F50	0.4	0.6	0.4	0.8
F21	0.4	0.4	0.4	1.2	F51	0.4	0.8	0.4	0.8
F22	0.4	0.6	0.4	1.2	F52	0.4	0.4	0.4	1.2
F23	0.4	0.8	0.4	1.2	F53	0.4	0.6	0.4	1.2
F24	0.4	1.0	0.4	1.2	F54	0.4	0.8	0.4	1.2
F25	0.4	1.2	0.4	1.2	F55	0.4	0.4	0.4	1.6
F26	0.4	0.4	0.4	1.4	F56	0.4	0.6	0.4	1.6
F27	0.4	0.6	0.4	1.4	F57	0.4	0.8	0.4	1.6
F28	0.4	0.8	0.4	1.4	F58	0.4	0.4	0.4	2.0
F29	0.4	1.0	0.4	1.4	F59	0.4	0.6	0.4	2.0
F30	0.4	1.2	0.4	1.4	F60	0.4	0.8	0.4	2.0

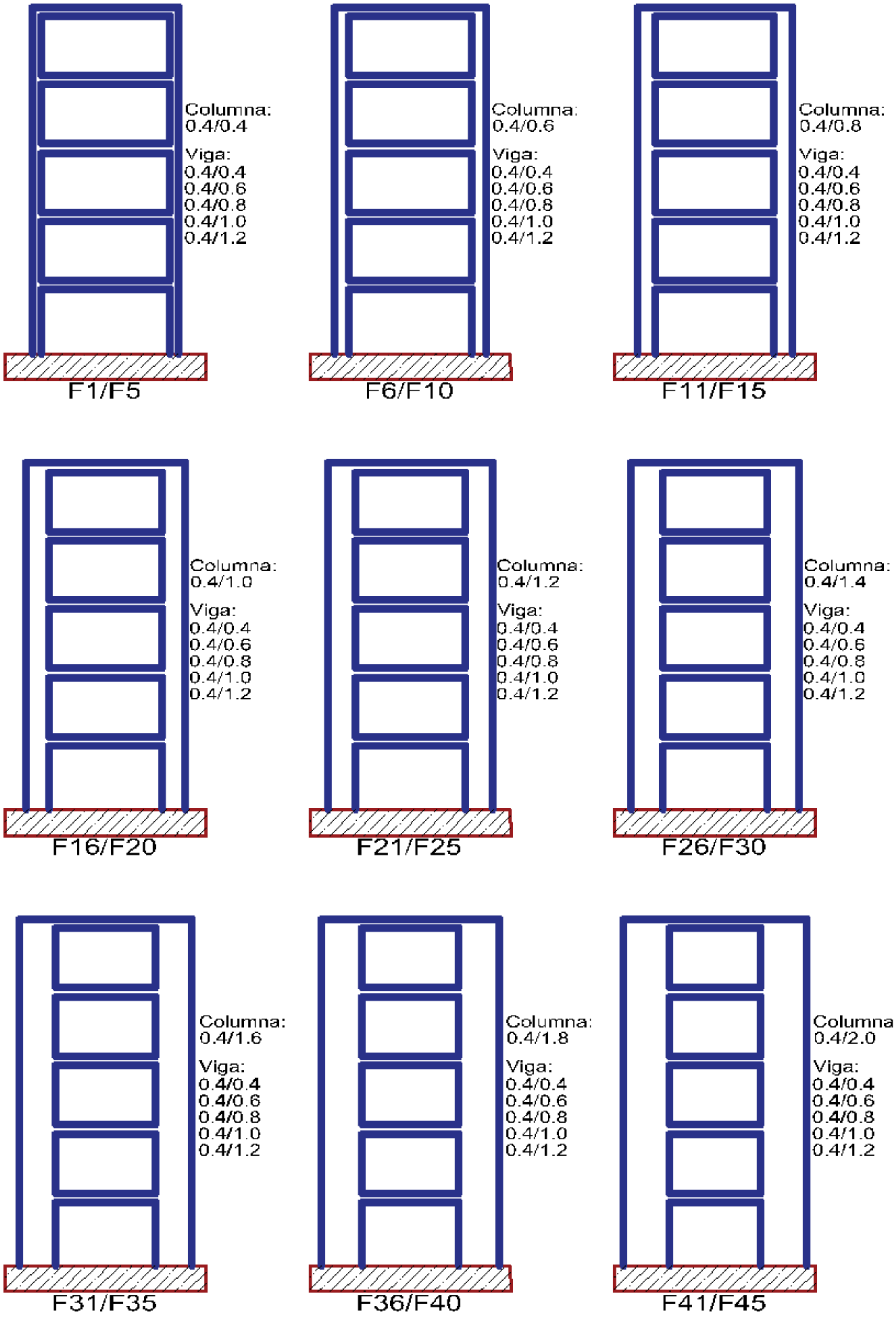


Figure 106. Frames F1-F45 of a section with cross section (base/cant) in meters for precision analysis.

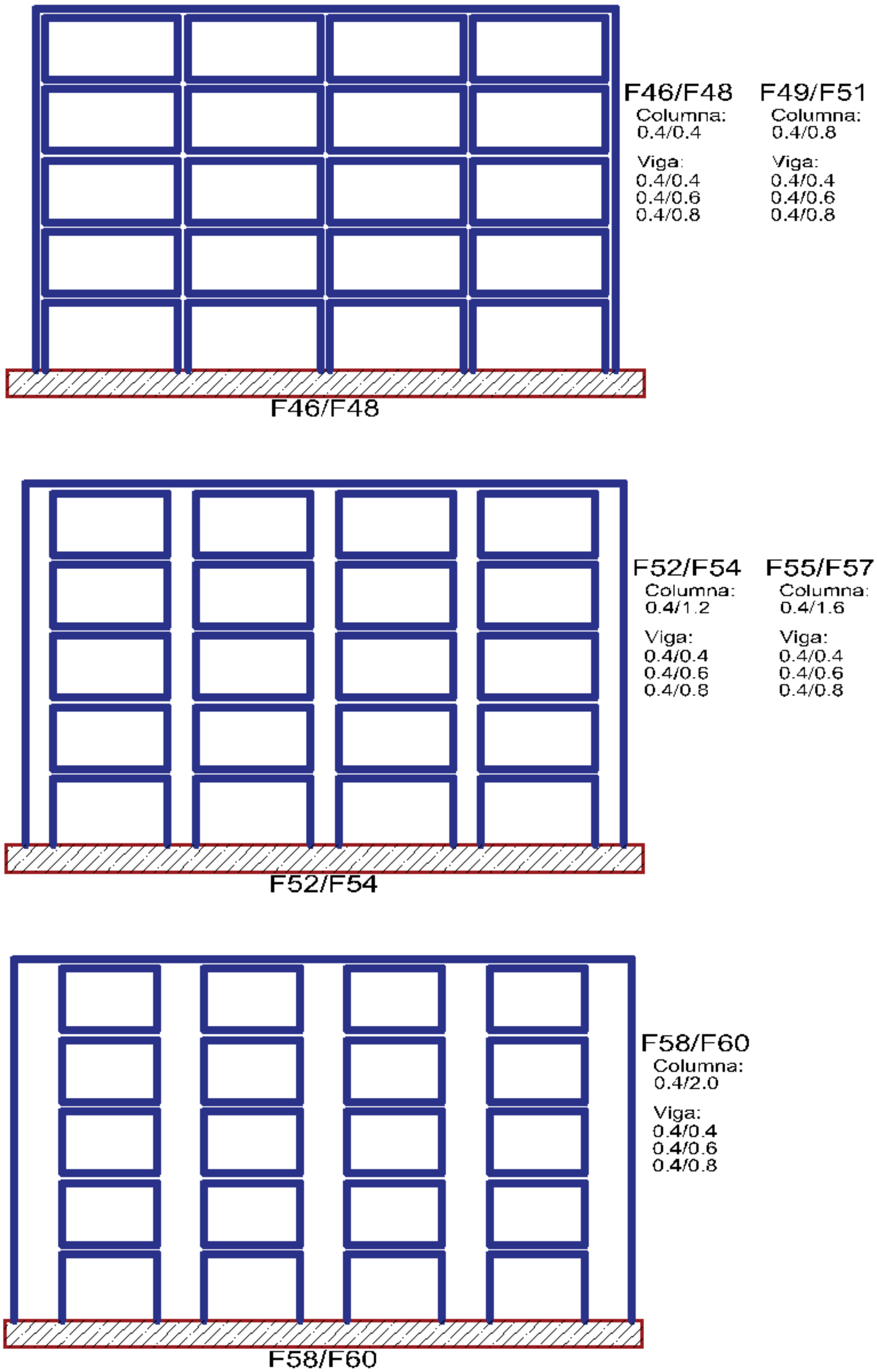


Figure 107. Frames F1-F45 with four sections with cross section (base/superelevation) in meters for precision analysis.

- **Precision Analysis of the Sandwich Beam (SWB) as a Replacement Beam**

Tabla.13 *Accuracy of the sandwich beam (SWB) for the analysis of maximum deflection of frames F1-F27 of a span for N≥5 floors*

Frames N≥5	Difference Range (%)	Average Absolute Difference (%)	Maximum Difference (%)
continuous solution	-1.50% - 5.33%	1.37%	5.33%

Tabla.14 *Accuracy of the sandwich beam (SWB) for the maximum deflection analysis of frames F1-F27 of a span N≥10 stories*

Frames N≥10	Difference Range (%)	Average Absolute Difference (%)	Maximum Difference (%)
continuous solution	-0.49% - 1.79%	0.43%	1.79%

Tabla.15 *Accuracy of the sandwich beam (SWB) for the maximum deflection analysis of frames F1-F27 of a span N≥15 floors*

Frames N≥15	Difference Range (%)	Average Absolute Difference (%)	Maximum Difference (%)
continuous solution	-0.49% - 0.53%	0.17%	0.53%

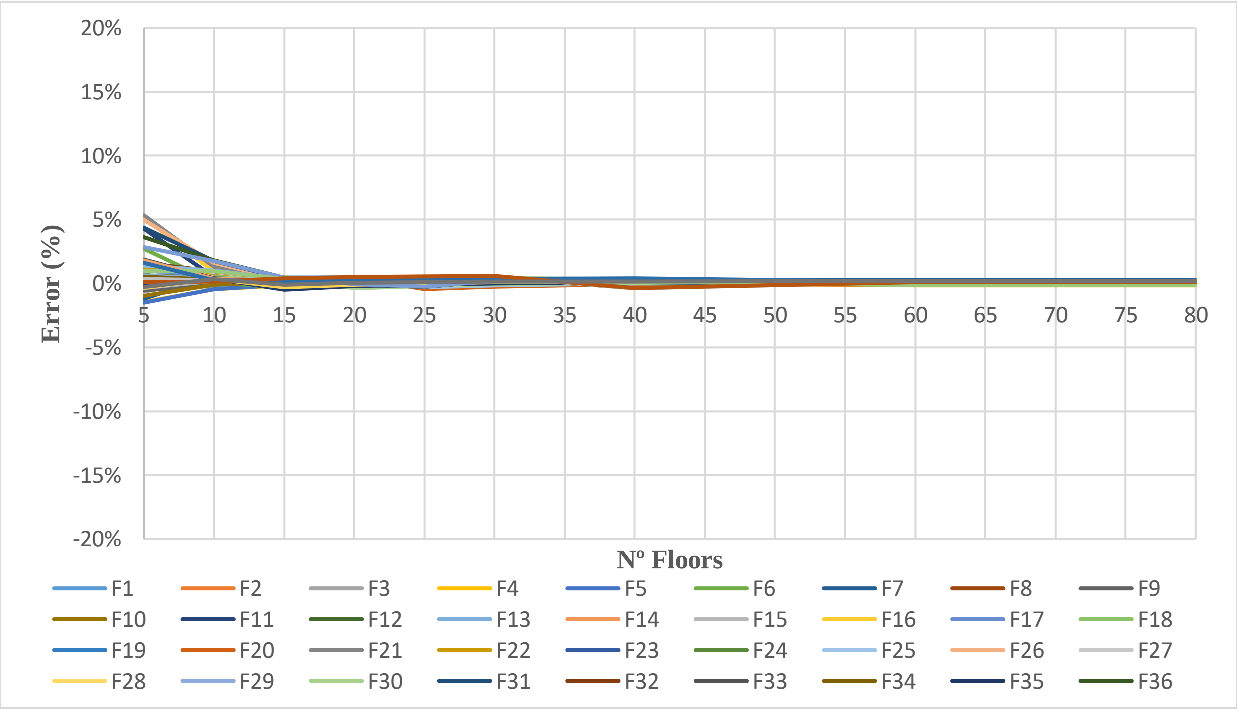


Figure 108. Precision sandwich beam (SWB) as a replacement beam for frames F1-F45 of a span.

Tabla.16 *Accuracy of the sandwich beam (SWB) for the maximum deflection analysis of the four-span frames F1-F27 for N≥5 floors.*

Frames N≥5	Difference Range (%)	Average Absolute Difference (%)	Maximum Difference (%)
continuous solution	-5.61% - 2.95%	2.09%	5.61%

Tabla.17 *Accuracy of the sandwich beam (SWB) for the maximum deflection analysis of the four-span frames F1-F27 for N≥10 stories.*

Frames N≥10	Difference Range (%)	Average Absolute Difference (%)	Maximum Difference (%)
continuous solution	-5.61% - 0.44%	1.89%	5.61%

Tabla.18 *Accuracy of the sandwich beam (SWB) for the maximum deflection analysis of the four-span frames F1-F27 for N≥15 floors.*

Frames N≥15	Difference Range (%)	Average Absolute Difference (%)	Maximum Difference (%)
continuous solution	-5.61% - 0.14%	1.83%	5.61%

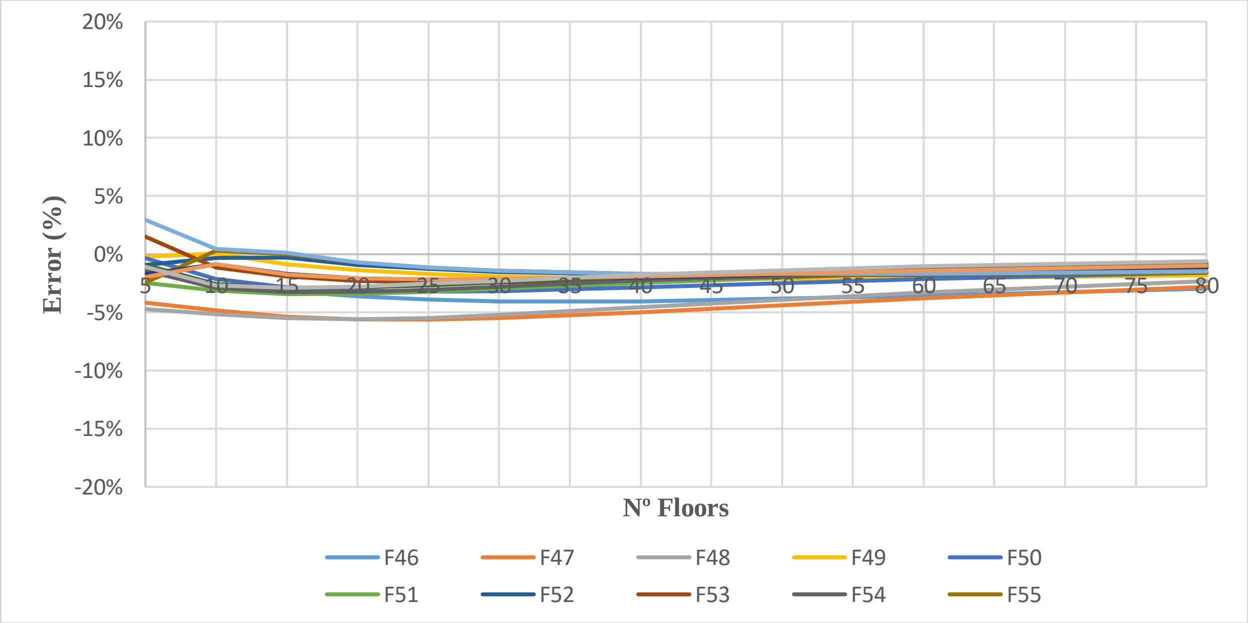


Figure 109. Precision sandwich beam (SWB) as replacement beam for frames F1-F45 four spans.

It is concluded that the use of the sandwich beam type replacement beam (SWB) to analyze the frames is appropriate and shows excellent precision within the engineering criteria; furthermore, it is observed that the error decreases drastically as the height of the building increases.

4.8.3 Coupled Shear Wall

A total of 324 coupled shear walls will be analyzed. The analysis consists of 36 coupled shear walls CSW1-CSW36, whose height will vary from five to eighty stories (5, 10, 15, 20, 25, 30, 40, 6, 80 stories) and the number of sections in one. Modulus of elasticity is $E = 25x10^6$ kN/m2, Poisson's ratio is ν =0.20, shear modulus is $G = 14.42x10^6$ kN/m2, height of floor is $h = 3.00$ m, the free length between shear walls is $L = 6.00$ m and the uniformly distributed wind load is $w = 5.00$ kN/m. Table 18 shows the sections of the beam structural elements and shear walls.

Tabla.19 *Wall and beam section for coupled shear walls CSW1-CSW36.*

CSW	Viga		Muro		CSW	Viga		Muro	
	b (m)	h (m)	b (m)	h (m)		b (m)	h (m)	b (m)	h (m)
CSW1	0.4	0.4	0.4	3.0	CSW19	0.4	0.4	0.4	6.0
CSW2	0.4	0.6	0.4	3.0	CSW20	0.4	0.6	0.4	6.0
CSW3	0.4	0.8	0.4	3.0	CSW21	0.4	0.8	0.4	6.0
CSW4	0.4	1.0	0.4	3.0	CSW22	0.4	1.0	0.4	6.0
CSW5	0.4	1.2	0.4	3.0	CSW23	0.4	1.2	0.4	6.0
CSW6	0.4	1.5	0.4	3.0	CSW24	0.4	1.5	0.4	.0
CSW7	0.4	0.4	0.4	4.0	CSW25	0.4	0.4	0.4	7.0
CSW8	0.4	0.6	0.4	4.0	CSW26	0.4	0.6	0.4	7.0
CSW9	0.4	0.8	0.4	4.0	CSW27	0.4	0.8	0.4	7.0
CSW10	0.4	1.0	0.4	4.0	CSW28	0.4	1.0	0.4	7.0
CSW11	0.4	1.2	0.4	4.0	CSW29	0.4	1.2	0.4	7.0
CSW12	0.4	1.5	0.4	4.0	CSW30	0.4	1.5	0.4	7.0
CSW13	0.4	0.4	0.4	5.0	CSW31	0.4	0.4	0.4	8.0
CSW14	0.4	0.6	0.4	5.0	CSW32	0.4	0.6	0.4	8.0
CSW15	0.4	0.8	0.4	5.0	CSW33	0.4	0.8	0.4	8.0
CSW16	0.4	1.0	0.4	5.0	CSW34	0.4	1.0	0.4	8.0
CSW17	0.4	1.2	0.4	5.0	CSW35	0.4	1.2	0.4	8.0
CSW18	0.4	1.5	0.4	5.0	CSW36	0.4	1.5	0.4	8.0

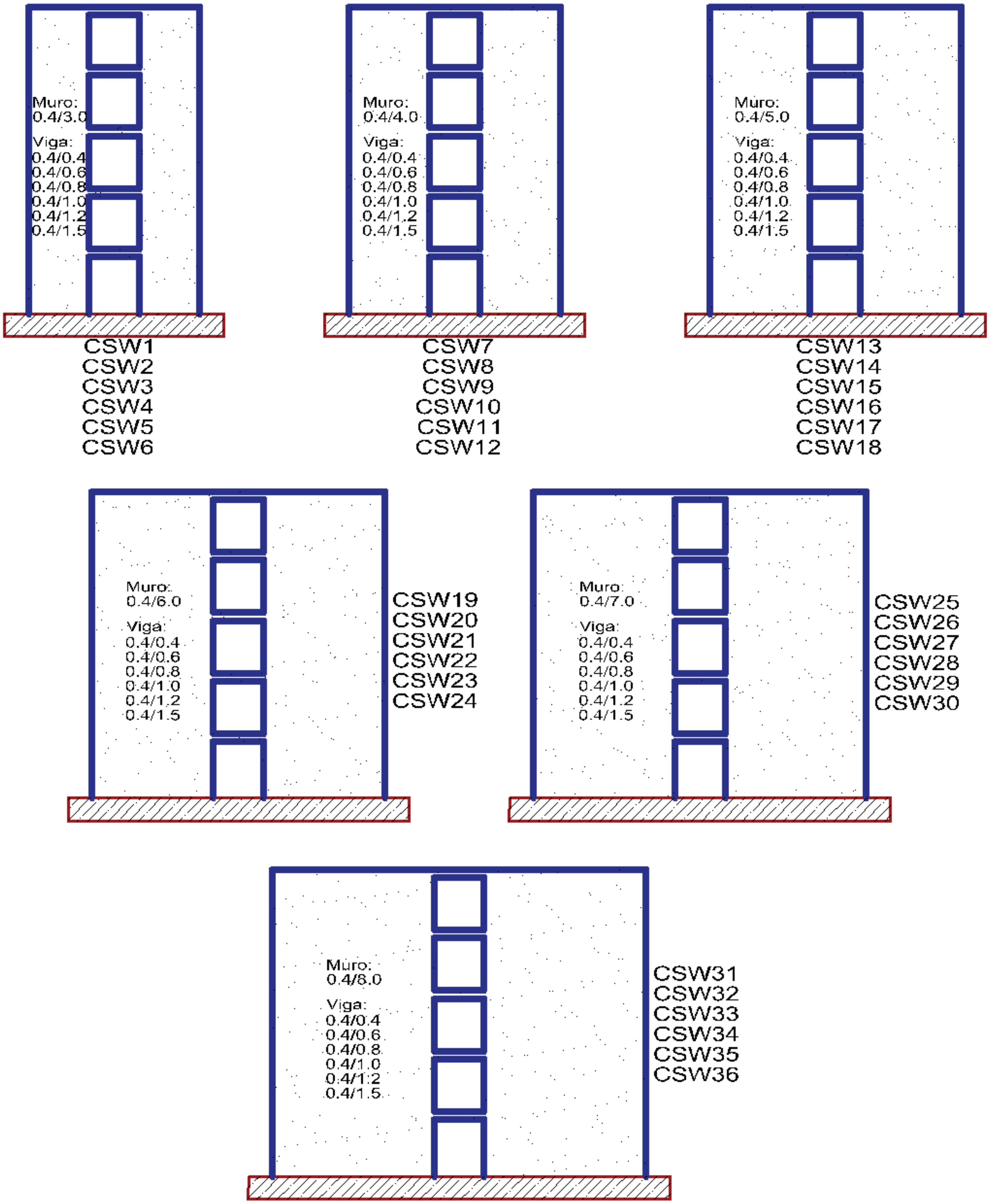


Figure 110. Coupled shear walls CSW 1-36 of a section with cross section (base/superelevation) in meters for precision analysis.

Precision Analysis of Sandwich Beam (SWB) as Replacement Beam

- The local shear stiffness of shear walls is not taken into account.

Tables 20 and 21 show the results of the structural analysis of coupled shear walls modeled as sandwich beams without taking into account the local shear stiffness of the shear walls.

Tabla.20 *Accuracy of the sandwich beam (SWB) for the maximum deflection analysis of the CSW1-CSW36 coupled shear walls of a span N≥10 stories.*

CSW N≥10	Difference Range (%)	Average Absolute Difference (%)	Maximum Difference (%)
continuous solution	-23.39% - 1.21%	10.22%	23.39%

Tabla.21 *Accuracy of the sandwich beam (SWB) for the maximum deflection analysis of the CSW1-CSW36 coupled shear walls of a span N≥15 stories.*

CSW N≥15	Difference Range (%)	Average Absolute Difference (%)	Maximum Difference (%)
continuous solution	-13.34% - 1.21%	5.83%	13.34%

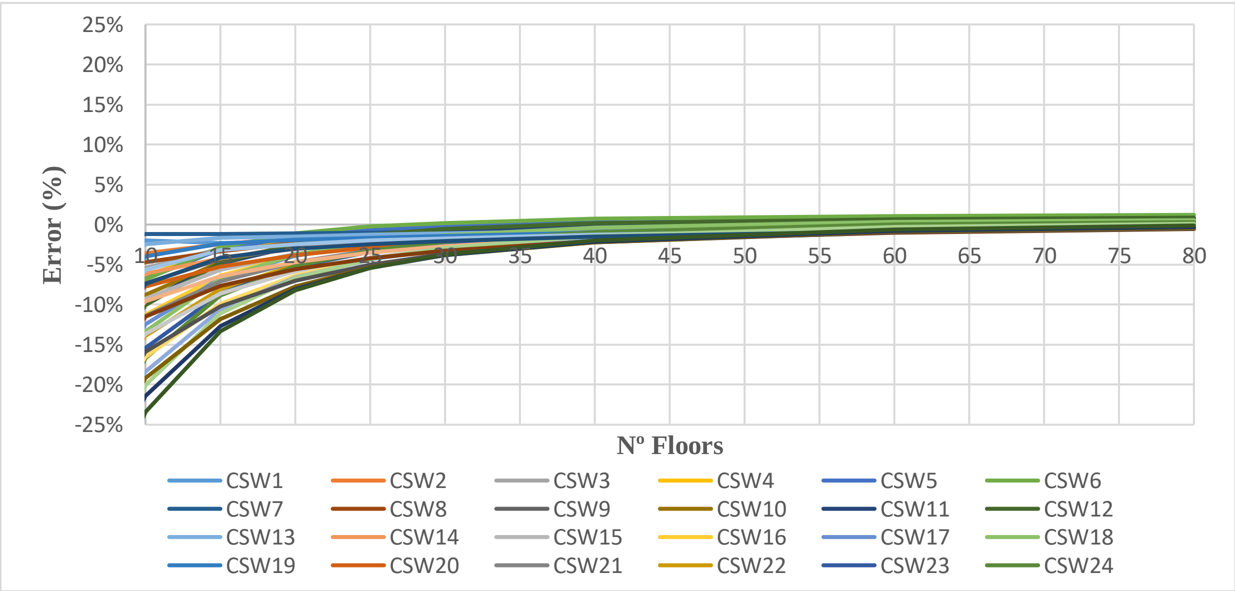


Figure 111. Precision Sandwich Beam (SWB) as Replacement Beam for Single Span CSW1-CSW36 Coupled Shear Walls.

It is observed that not including the local shear stiffness of the shear walls in the first floors leads to errors that are not negligible.

- The local shear stiffness of the shear walls is taken into account.

Tables 22 and 23 show the results of the structural analysis of the coupled shear walls modeled as sandwich beams taking into account the local shear stiffness of the shear walls.

Tabla.22 *Accuracy of the sandwich beam (SWB) for the maximum deflection analysis of the CSW1-CSW36 coupled shear walls of a span N≥10 stories*

CSW N≥10	Difference Range (%)	Average Absolute Difference (%)	Maximum Difference (%)
continuous solution	-5.42% - 7.43%	2.19%	7.43%

Tabla.23 *Accuracy of the sandwich beam (SWB) for the maximum deflection analysis of the CSW1-CSW36 coupled shear walls of a span N≥15 stories*

CSW N≥15	Difference Range (%)	Average Absolute Difference (%)	Maximum Difference (%)
continuous solution	-2.26% - 7.43%	1.56%	7.43%

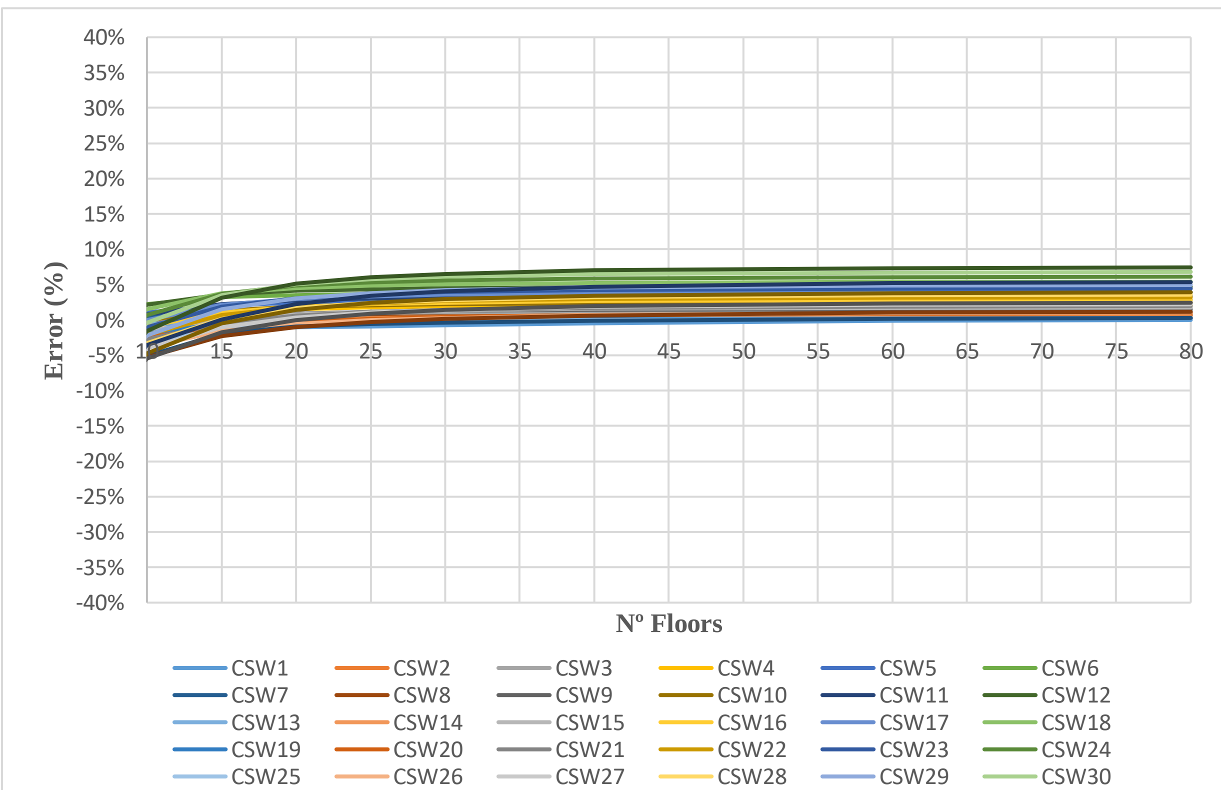


Figure 112. Precision Sandwich Beam (SWB) as Replacement Beam for Single Span CSW1-CSW36 Coupled Shear Walls.

It is observed that including the local shear stiffness of the shear walls in the first floors drastically reduces the errors.

- The local shear stiffness of the shear walls is taken into account only on the first floors.

Tables 24 and 25 show the results of the structural analysis of the coupled shear walls modeled as sandwich beams, taking into account the local shear stiffness of the shear walls on the first floors and neglecting the local shear stiffness of the walls on the lower floors.

Tabla.24 *Accuracy of the sandwich beam (SWB) for the maximum deflection analysis of the coupled shear walls CSW1-CSW36 of a section N≥10 stories.*

CSW N≥10	Difference Range (%)	Average Absolute Difference (%)	Maximum Difference (%)
continuous solution	-5.44% - 5.11%	1.13%	5.44%

Tabla.25 *Accuracy of the sandwich beam (SWB) for the maximum deflection analysis of the coupled shear walls CSW1-CSW36 of a section N≥15 floors.*

CSW N≥15	Difference Range (%)	Average Absolute Difference (%)	Maximum Difference (%)
continuous solution	-5.44% - 5.11%	0.56%	5.44%

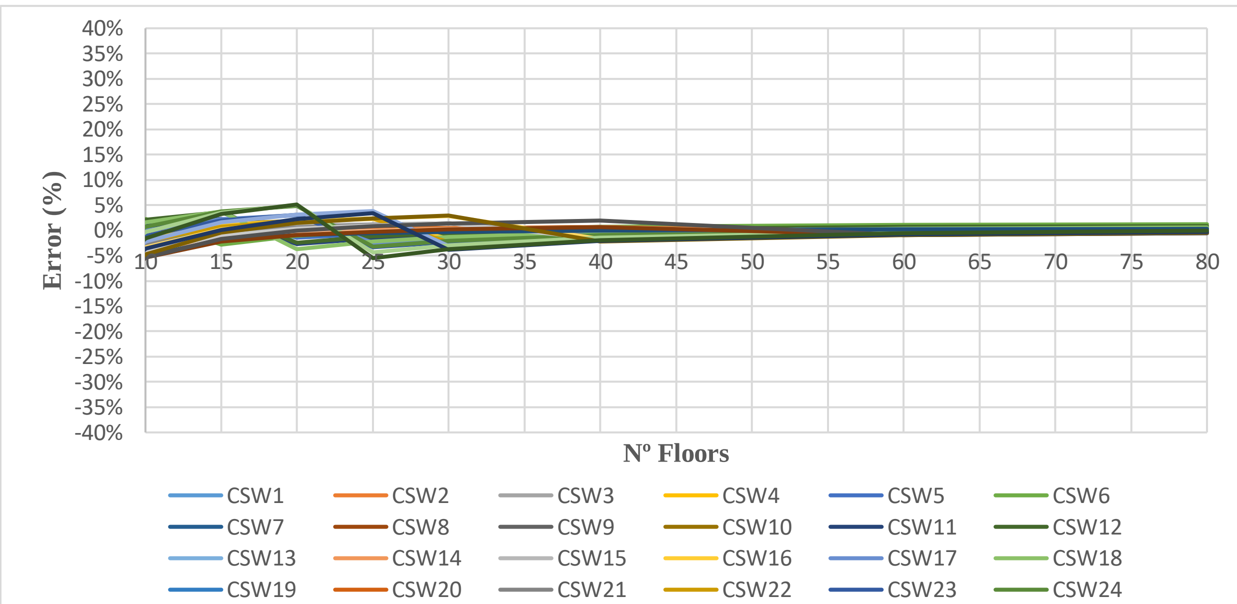


Figure 113. Precision Sandwich Beam (SWB) as Replacement Beam for Single Span CSW1-CSW36 Coupled Shear Walls

The sandwich beam model that included the local shear of the walls in the first floors and ignored the local shear of the walls in the upper floors has shown excellent accuracy with a maximum error of 5% and therefore it is concluded that using sandwich beam (SWB) replacement beam for testing coupled shear walls is appropriate and shows excellent accuracy within engineering criteria.

4.8.4 Reinforced Concrete Frame

A total of 18 frames will be analyzed. The analysis consists of two frames whose height will vary from five to eighty stories (5, 10, 15, 20, 25, 30, 40, 6, 80 stories) and the number of bays between two and three. The mechanical properties are summarized in table 26.

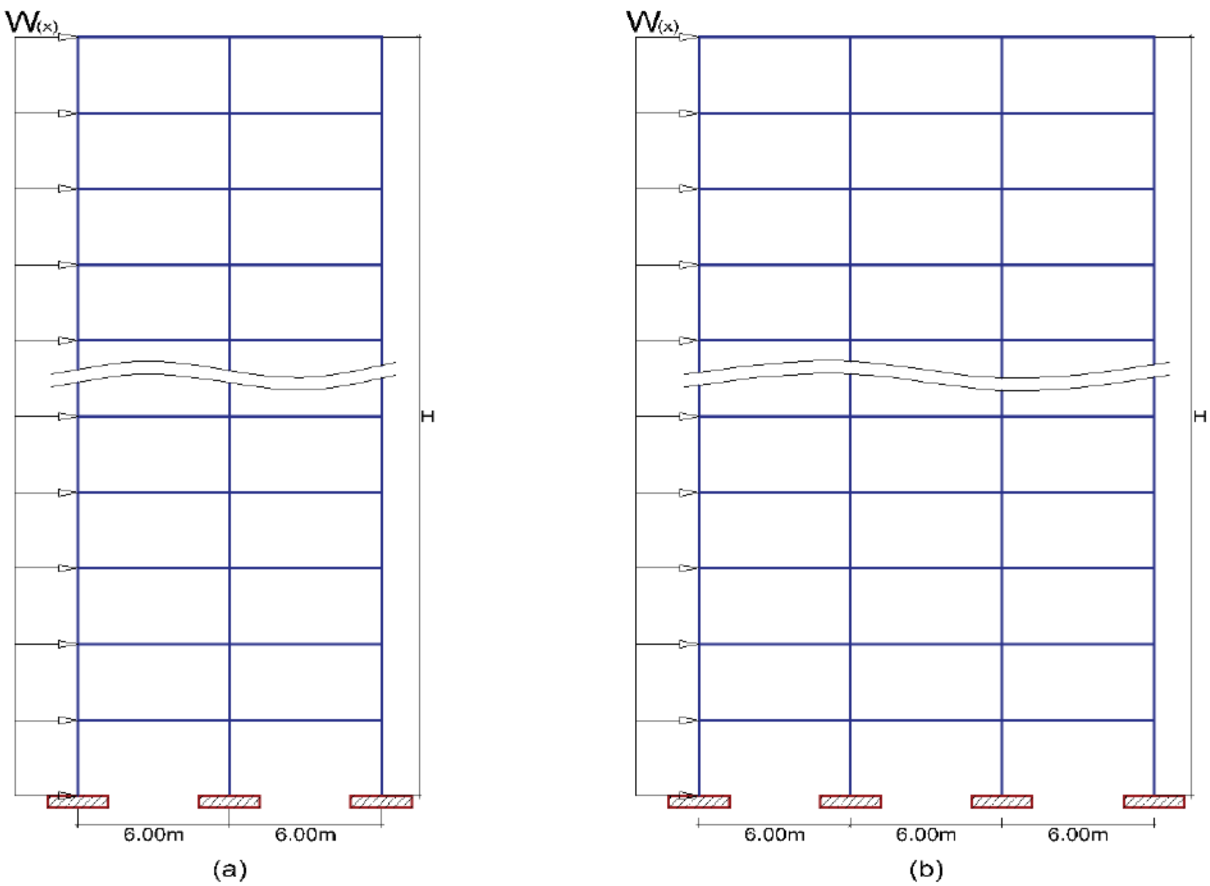


Figure 114. Concrete frame: (a) one span (b) two span.

Tabla.26 *Structural properties and geometries of frames.*

Floor height	3	m
Length to axes of columns	6	m
Beam depth	0.7	m
Beam width	0.4	m
Column depth	0.4	m
Column width	0.4	m
Modulus of elasticity	25000000	kN/m2
Mass density per unit length	50	kN/m
Poisson's ratio	0.2	-
Uniformly distributed load	5	kN/m

4.8.4.1 Static Analysis

Figures 115 and 126 demonstrate the accuracy of the approximate solution. In general, it was found that the continuous solution leads to average estimates of 1.41% for the two-span frame and 2.10% for the three-span frame. It is important to mention that the error tends to stabilize and reduce as the height of the frame increases. In addition, the almost exact trend of the analysis can be observed with the axis length of the beams and considering the rigid zone at the nodes.

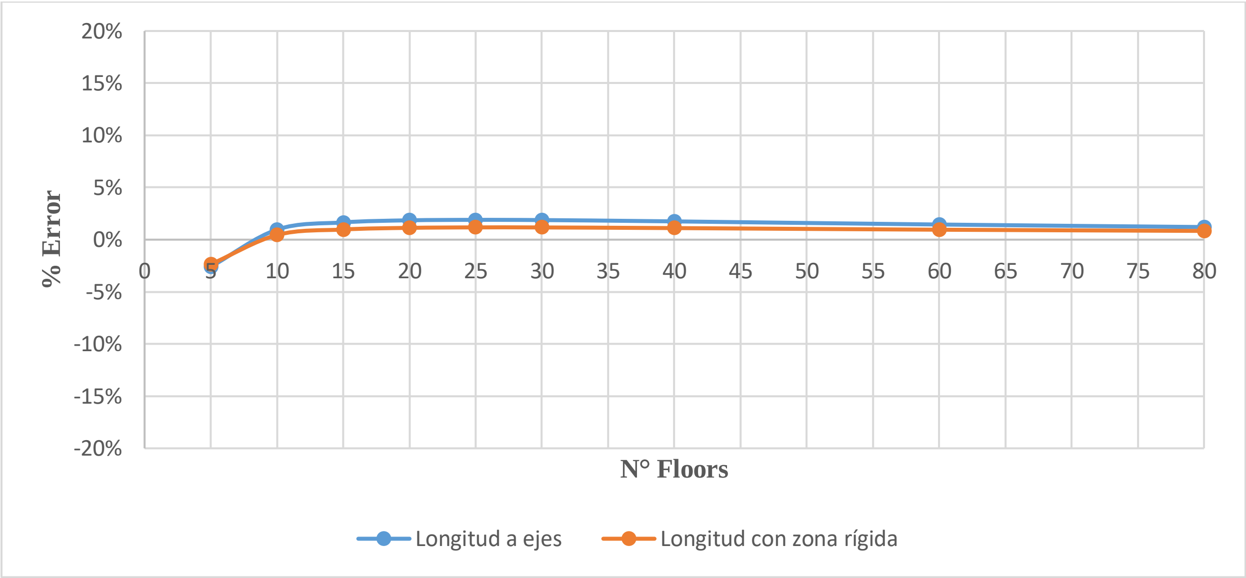


Figure 115. Accuracy of the maximum displacement of the two-span frame.

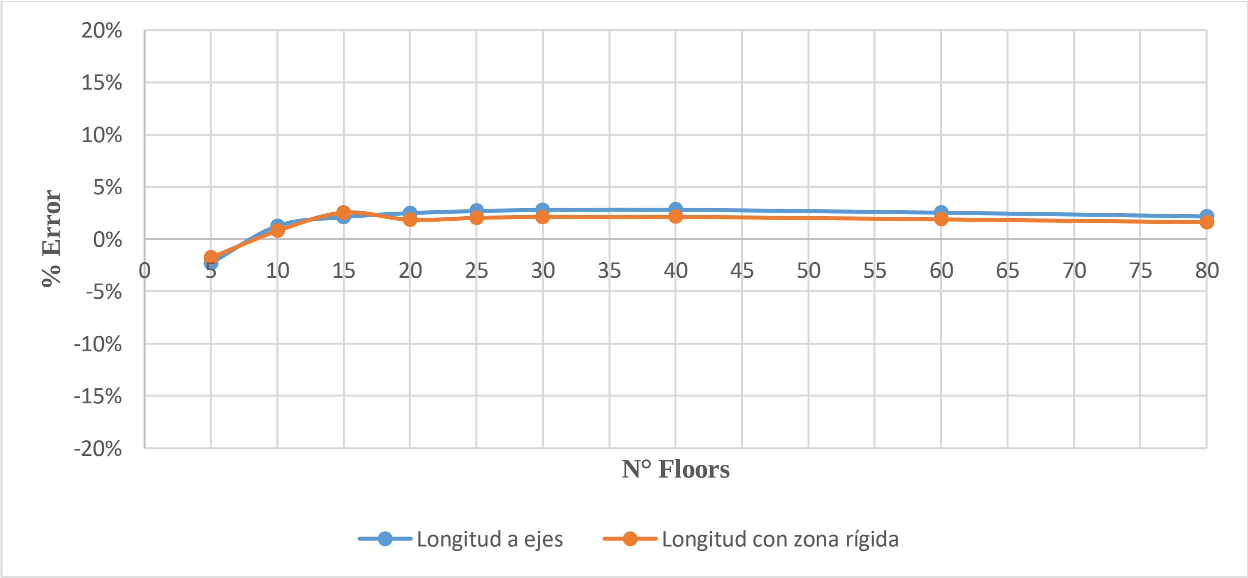


Figure 116. Accuracy of the maximum displacement of the three-span frame.

4.8.4.2 Dynamic Analysis

The results are similar to the static case. Figures 117 and 118 demonstrate the accuracy of the approximate solution. In general, the continuous solution was found to lead to average estimates of 1.45% for the two-span frame and 1.47% for the three-span frame. It is important to mention that the error tends to stabilize and reduce as the height of the frame increases. In addition, the almost exact trend of the analysis can be observed with the axis length of the beams and considering the rigid zone at the nodes.

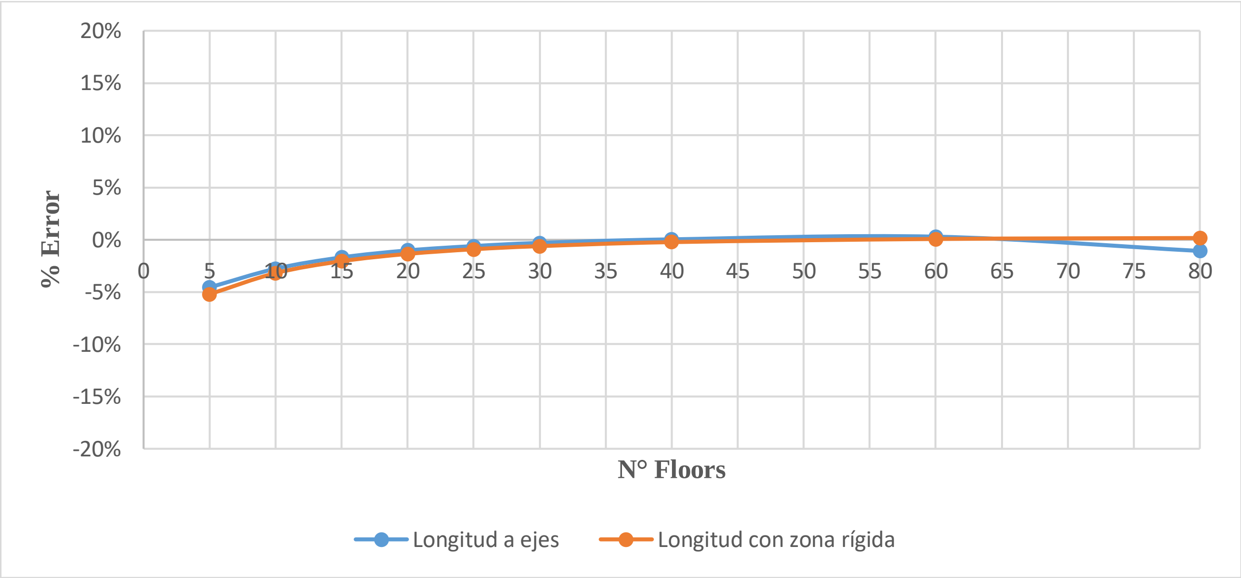


Figure 117. Precision of the fundamental period of the two-span frame.

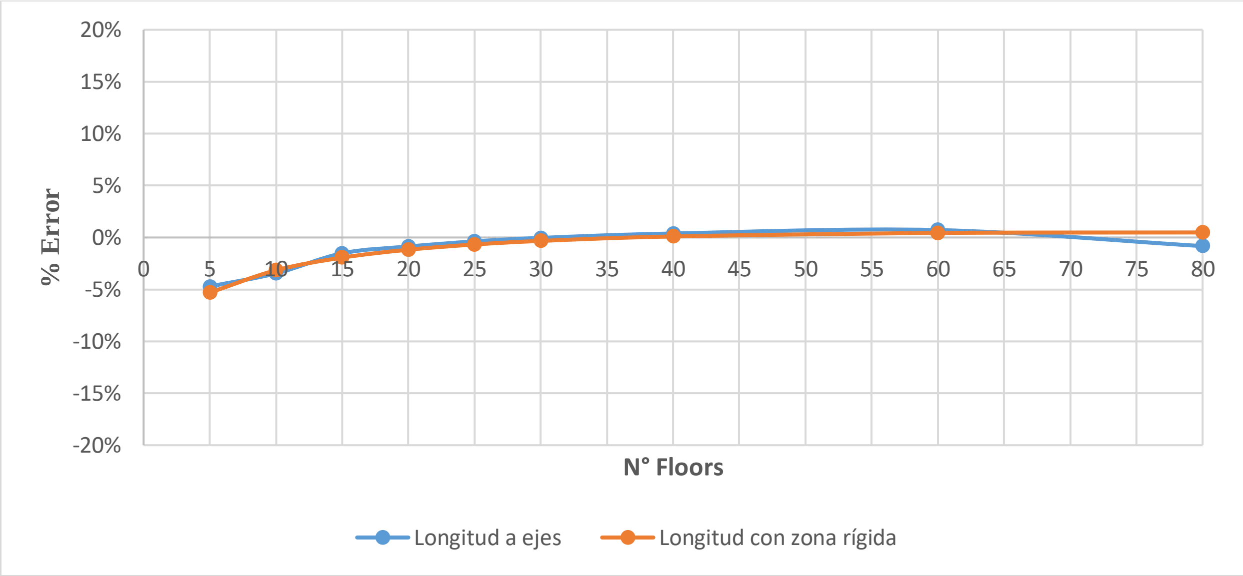


Figure 118. Precision of the fundamental period of the three-span frame.

4.8.4.3 Stability Analysis

Figures 119 and 120 demonstrate the accuracy of the approximate solution. In general, the continuous solution was found to lead to average estimates of 1.62% for the two-span frame and 1.01% for the three-span frame. It is important to mention that only four iterations were used for the calculation of the critical load, a higher number of iterations would lead to a smaller error. In addition, it was found that as the value of the parameter α grows, a greater number of iterations are needed for a value closer to the exact one.

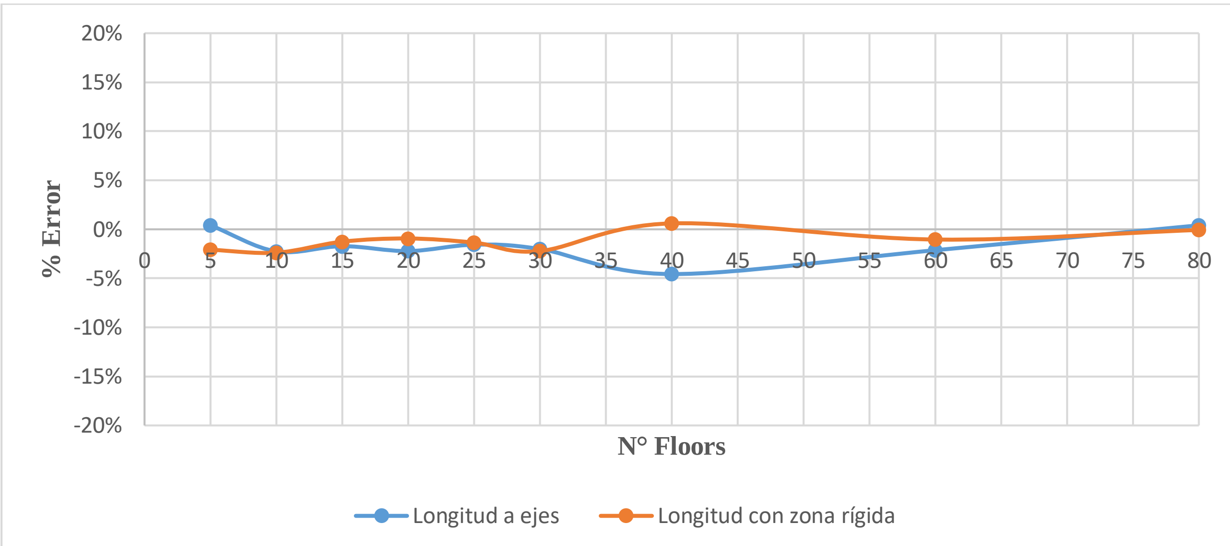


Figure 119. Accuracy of the critical load of the two-span frame.

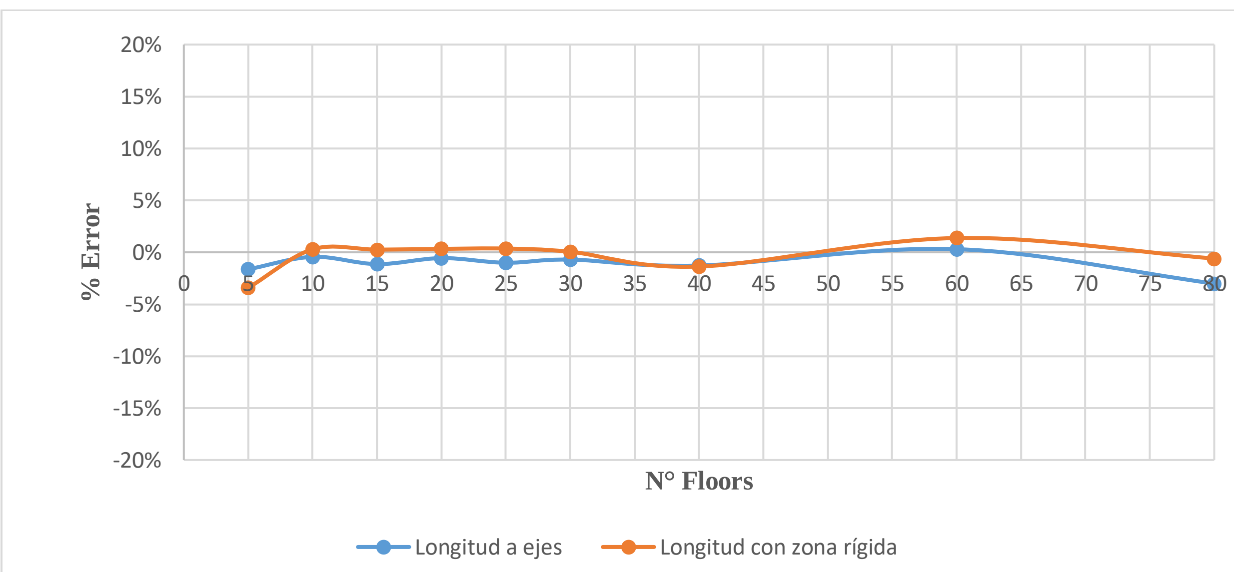


Figure 120. Accuracy of the critical load of the three-span frame.

4.8.5 Coupled Shear Wall

A total of 9 coupled shear walls will be analyzed. The analysis consists of a coupled shear wall whose height will be varied from five to eighty stories (5, 10, 15, 20, 25, 30, 40, 6, 80 stories). The mechanical properties are summarized in table 27.

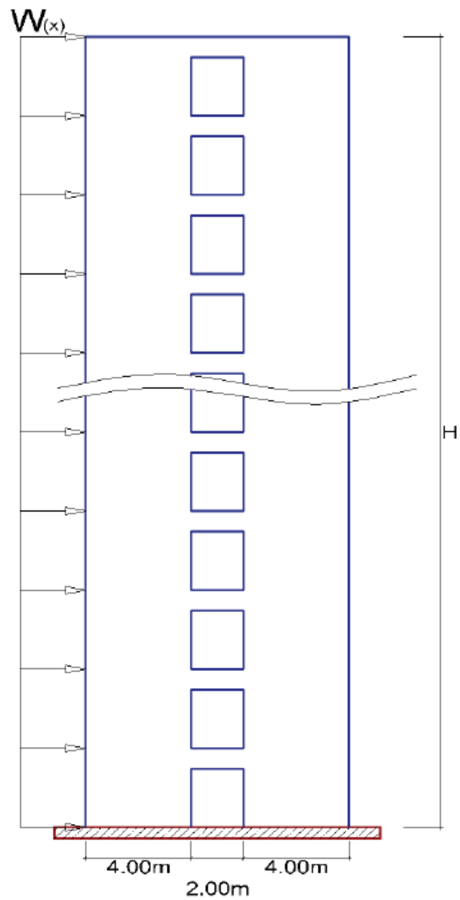


Figure 121. Coupled shear Wall.

Tabla.27 *Structural properties and geometries of the coupled shear Wall.*

Floor height	3	m
Length to axes of shear walls	6	m
Beam depth	0.8	m
Beam width	0.4	m
Shear wall depth	4.0	m
Shear wall width	0.4	m
Modulus of elasticity	25000000	kN/m2
Mass density per unit length	50	kN/m
Poisson's ratio	0.2	-
Uniformly distributed load	5	kN/m

4.8.5.1 Static Analysis

Figure 122 demonstrates the accuracy of the approximate solution. In general, it was found that the continuous solution leads to average estimates of -2.20% for the axis analysis of the beams and 1.80% for the case of considering the rigid zone at the nodes. It is important to mention that for both cases the error tends to stabilize and reduce as the height of the shear wall increases.

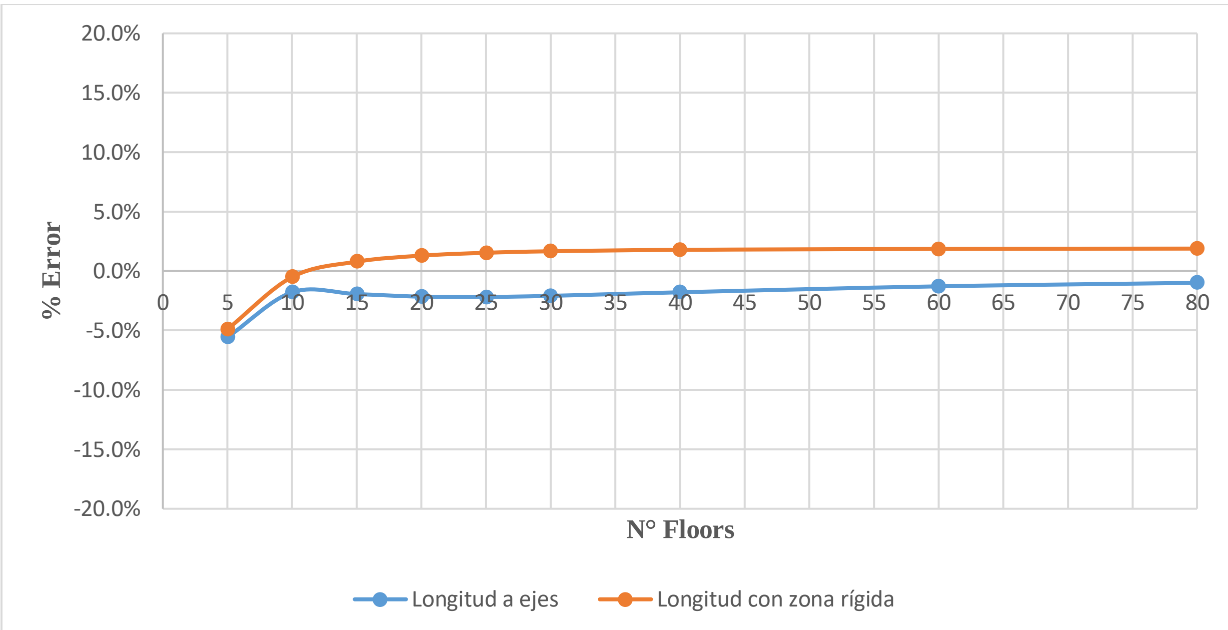


Figure 122. Accuracy of the maximum displacement of the coupled shear Wall.

4.8.5.2 Dynamic Analysis

The results are similar to the static case. Figure 123 demonstrates the accuracy of the approximate solution. In general, an average error of -1.01% was found for the case of axis length and 0.95% for the case of considering rigid zones in the nodes. It is important to mention that for both cases the error tends to stabilize and reduce as the height of the frame increases.

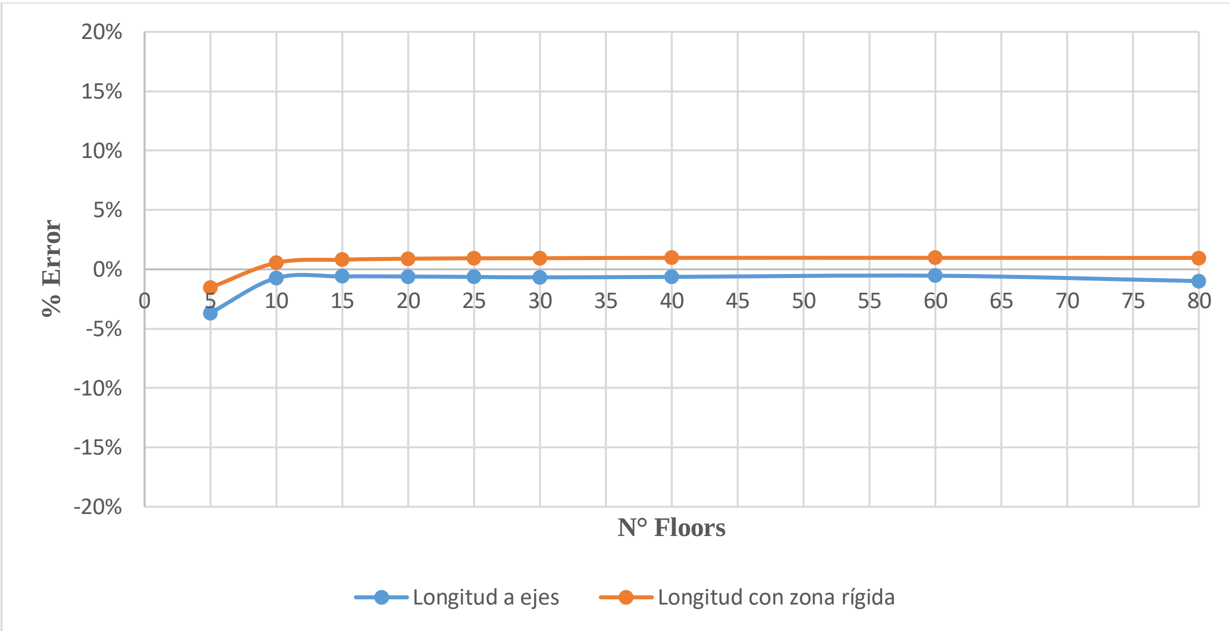


Figure 123. Accuracy of the fundamental period of the coupled shear Wall.

4.8.5.3 Stability Analysis

Figure 124 demonstrates the accuracy of the approximate solution. Only four iterations were used and an average error of 2.54% was found for the case of axis length and 2.84% for the case of considering rigid zones in the nodes.

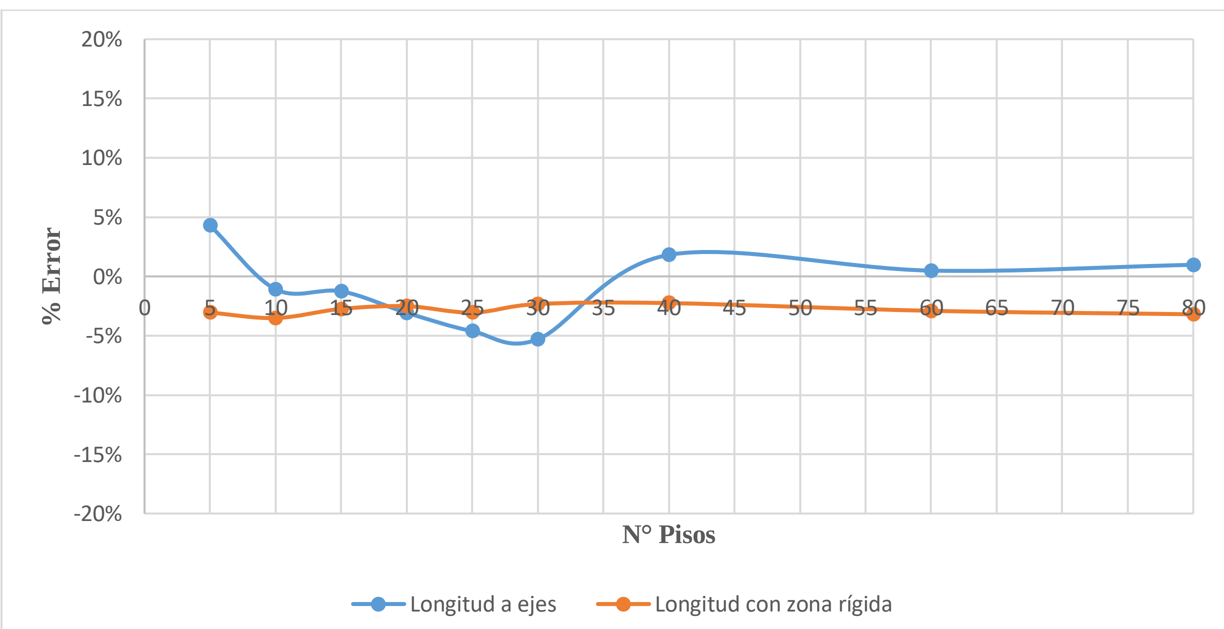


Figure 124. Precisión de la carga crítica del muro de corte acoplado

4.8.6 Frame Building

A total of 9 frame buildings will be analyzed. The analysis consists of a three-story frame building whose height will be varied from five to eighty stories (5, 10, 15, 20, 25, 30, 40, 6, 80 stories). The mechanical properties are summarized in table 28.

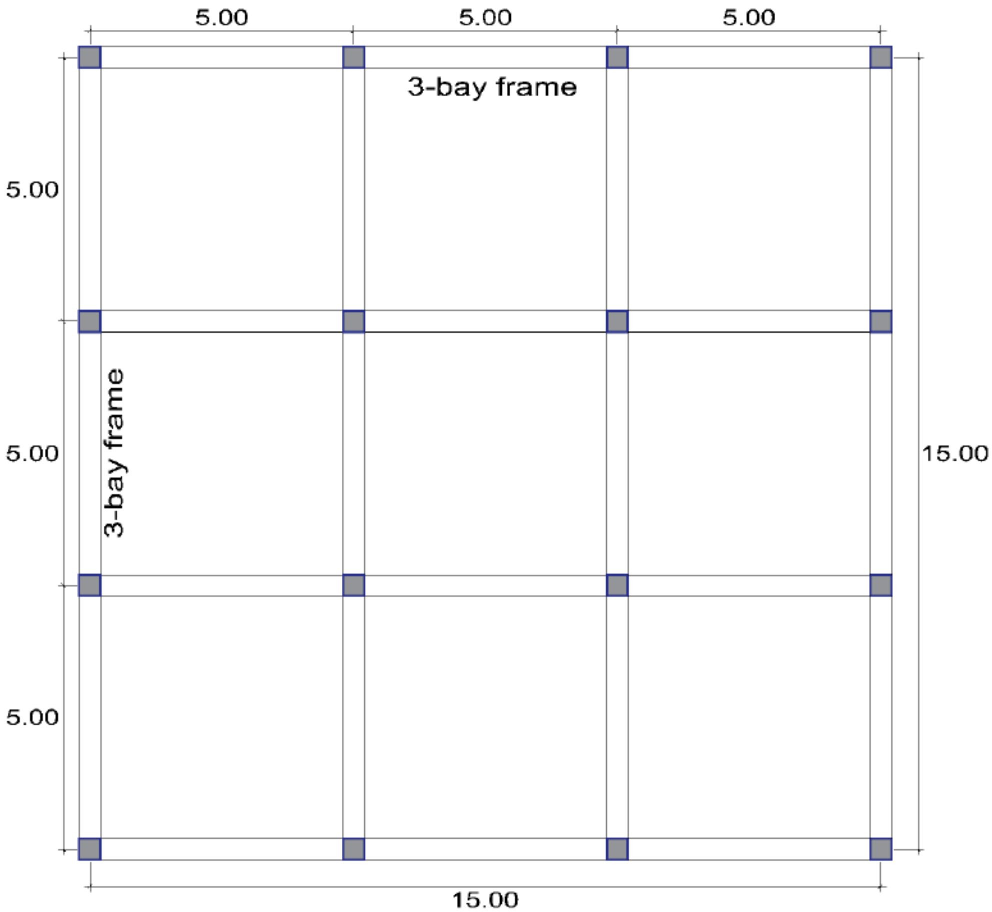


Figure 125. Frame Building.

Tabla.28 *Structural properties and geometries of the frame Building.*

Floor height	3	m
Length to axes of columns	5	m
Beam depth	0.6	m
Beam width	0.4	m
Column depth	0.4	m
Column width	0.4	m
Modulus of elasticity	25000000	kN/m2
Mass density per unit length	10	kN/m2
Poisson's ratio	0.2	-
Uniformly distributed load	5	kN/m2

4.8.6.1 Static Analysis

Figure 126 demonstrates the accuracy of the approximate solution. In general, it was found that the continuous solution leads to average estimates of -3.71% for the axis analysis of the beams and -3.88% for the case of considering the rigid zone at the nodes. It is important to mention that for both cases the error tends to stabilize and reduce as the height of the frame increases.

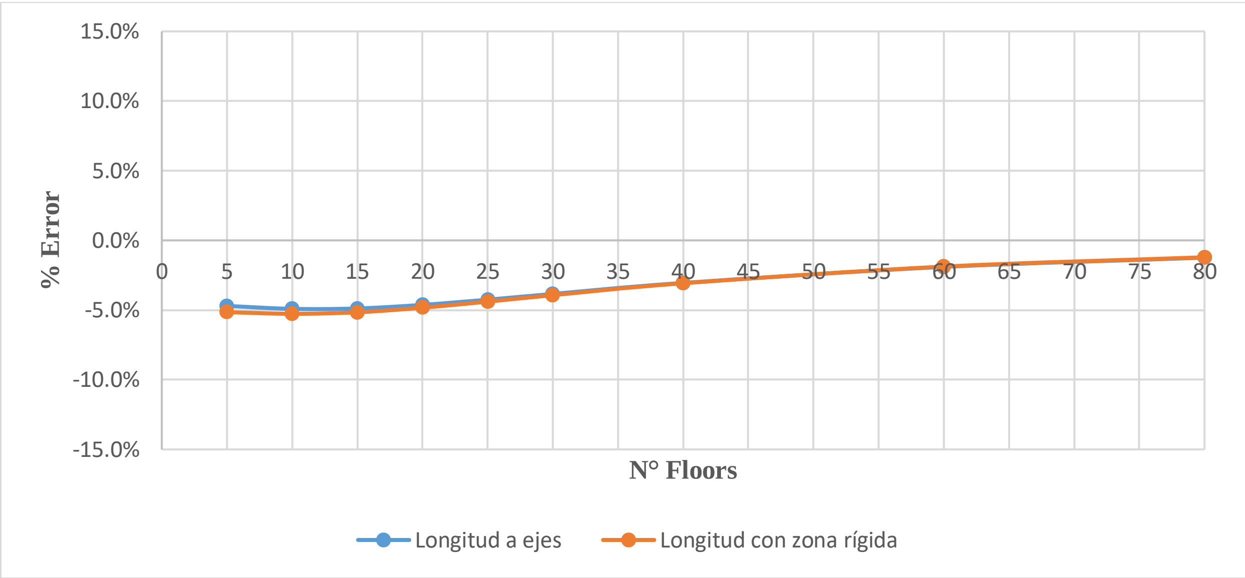


Figure 126. Accuracy of the maximum displacement of the frame Building.

4.8.6.2 Dynamic Analysis

The results are similar to the static case. Figure 127 demonstrates the accuracy of the approximate solution. In general, an average error of 1.18% was found for the case of axis length and 1.13% for the case of considering rigid zones in the nodes. It is important to mention that for both cases the error tends to stabilize and reduce as the height of the frame increases.

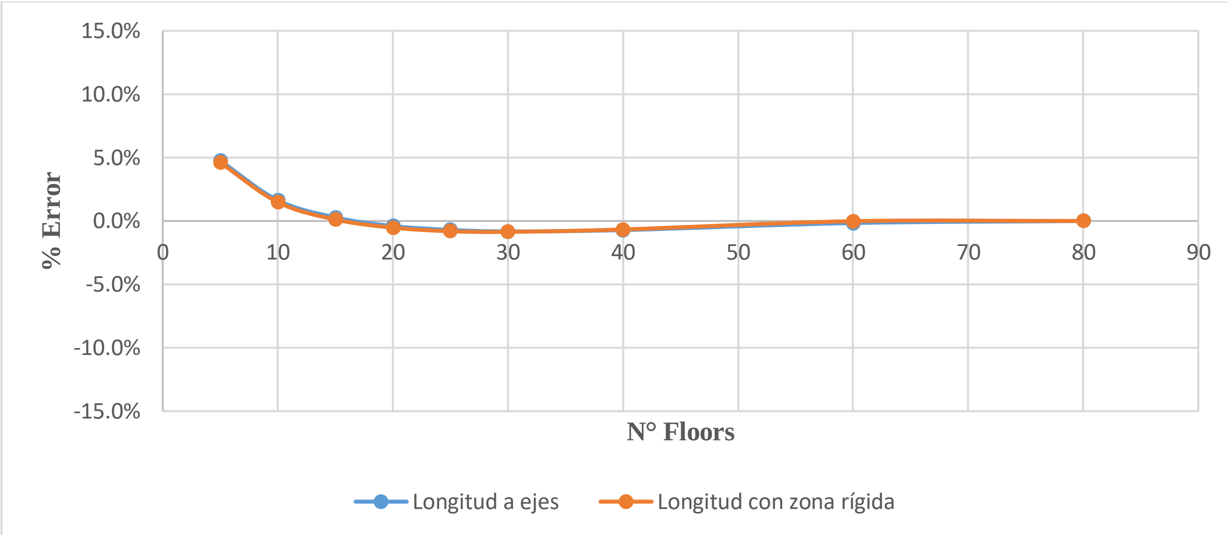


Figure 127. Precision of the fundamental period of the frame building

4.8.6.3 Stability Analysis

Figure 128 demonstrates the accuracy of the approximate solution. An average error of 6.74% was found for the case of axis length and 5.76% for the case of considering rigid zones in the nodes. It is important to mention that only four iterations were used for the calculation of the critical load, a higher number of iterations would lead to a smaller error. In addition, it was found that as the value of the parameter α grows, a greater number of iterations are needed for a value closer to the exact one.

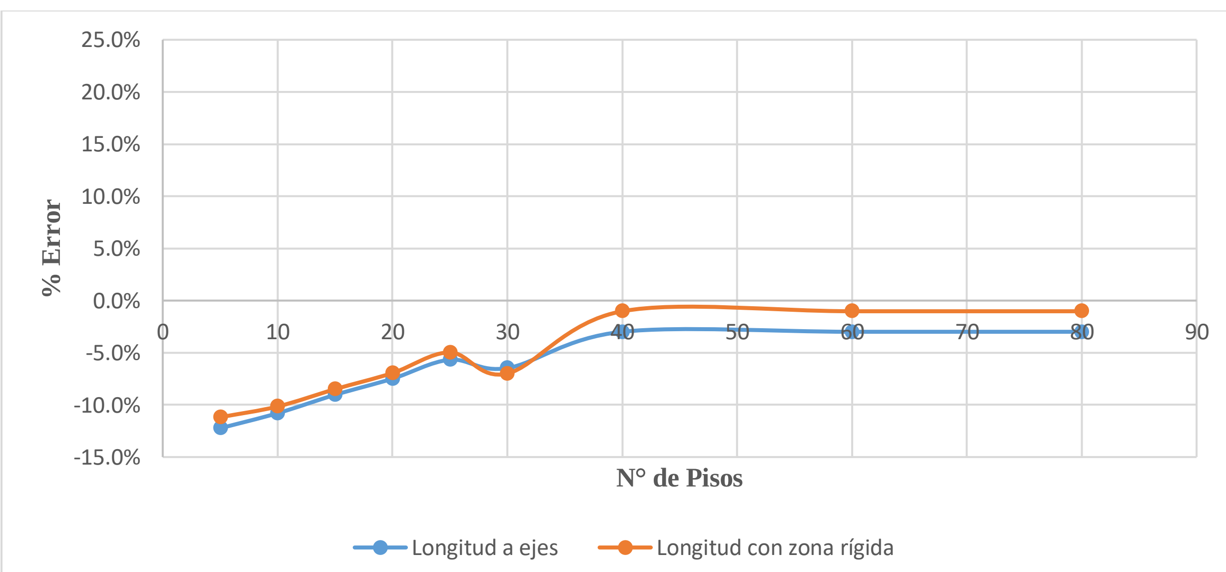


Figure 128. Precisión del periodo fundamental del edificio de pórticos

4.8.7 Dual Frame and Shear Wall Building

A total of 9 dual frame and shear wall buildings will be analyzed. The analysis consists of a dual building whose height will be varied from five to eighty stories (5, 10, 15, 20, 25, 30, 40, 6, 80 stories). The mechanical properties are summarized in table 29.

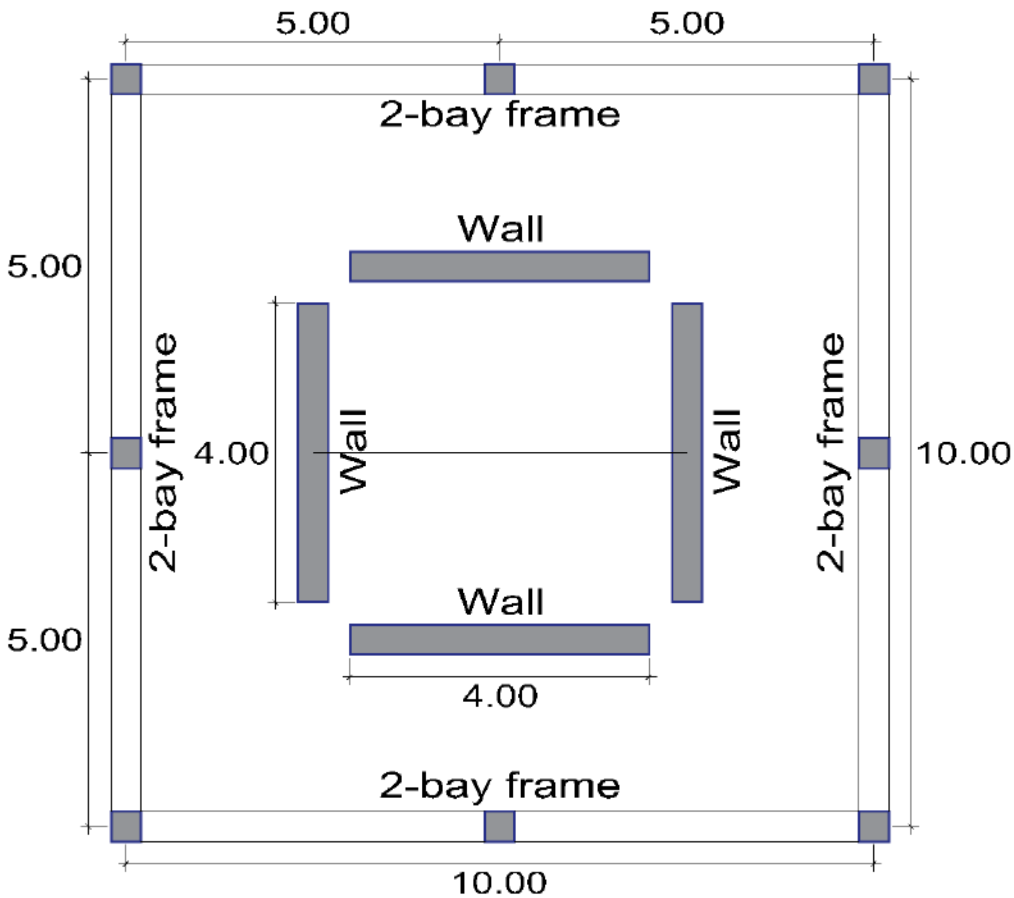


Figure 129. Dual frame and shear wall Building.

Tabla.29 *Structural properties and geometries of the dual frame and shear wall building.*

Floor height	3	m
Length to axes of columns	5	m
Beam depth	0.6	m
Beam width	0.4	m
Column depth	4.0	m
Column and shear wall width	0.4	m
Modulus of elasticity	25000000	kN/m2
Mass density per unit length	50	kN/m
Poisson's ratio	0.2	-
Uniformly distributed load	5	kN/m

4.8.7.1 Static Analysis

Figure 130 demonstrates the accuracy of the approximate solution. In general, it was found that the continuous solution leads to average estimates of 4.88% for the axis analysis of the beams and 5.32% for the case of considering the rigid zone at the nodes. It is important to mention that for both cases the error tends to stabilize and reduce as the height of the building increases.

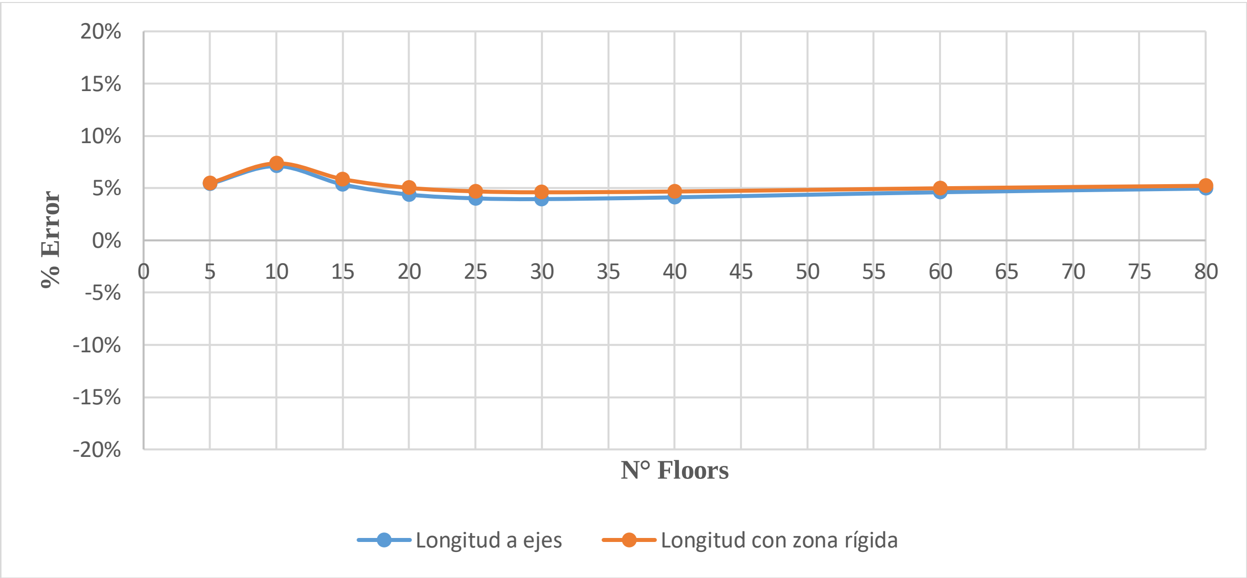


Figure 130. Accuracy of the maximum displacement of the dual building of frames and shear walls.

4.8.7.2 Dynamic Analysis

The results are similar to the static case. Figure 131 demonstrates the accuracy of the approximate solution. In general, an average error of 2.82% was found for the case of axis length and 3.06% for the case of considering rigid zones in the nodes. It is important to mention that for both cases the error tends to stabilize and reduce as the height of the building increases.

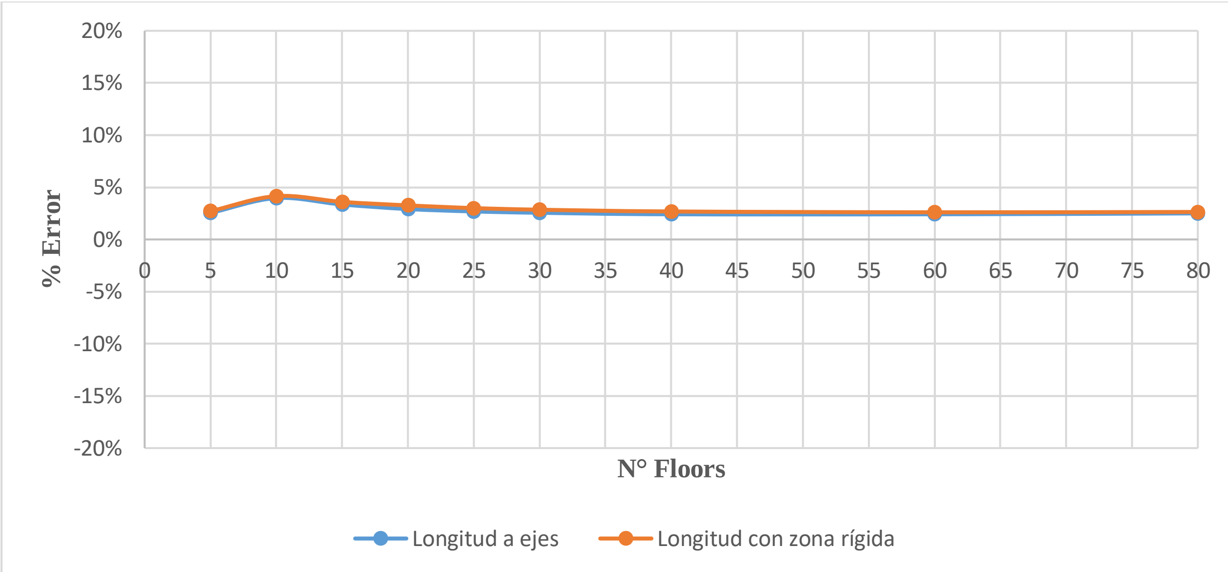


Figure 131. Accuracy of the fundamental period of the dual building of frames and shear walls.

4.8.7.3 Stability Analysis

Figure 132 demonstrates the accuracy of the approximate solution. Only four iterations were used and an average error of 4.10% was found for the case of axis length and 4.65% for the case of considering rigid zones in the nodes.

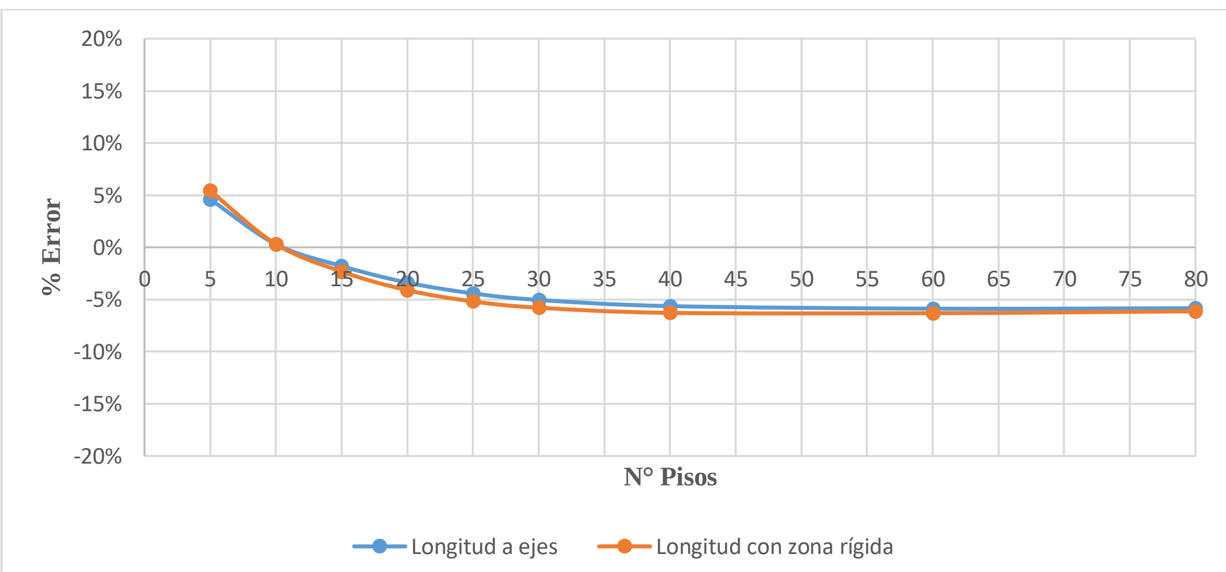


Figure 132. Precisión de la carga crítica del edificio dual de pórticos y muros de corte

5 DISCUSSION

5.1 DISCUSSION OF RESULTS

Nollet (1979) and Hoenderkamp (1983) worked on the static analysis of tall buildings using the continuous method. In the development of this research project it was found that deriving the replacement sandwich beam (SWB) equations using the continuous method with energy formulation lead to the same results.

Miranda (1999), used the replacement beam consisting of the parallel coupling of the bending and shear beam (CTB) and developed the static analysis subject to a general lateral load with a general profile that depends only on one parameter. In this research project, this approach was generalized to all replacement beams studied.

Potzta (2002) developed a replacement beam model for the entire building using a sandwich beam with an energetic approach. In this research project, this approach was generalized using the generalized sandwich replacement beam that additionally considers the local shear stiffness of the vertical elements.

Bozdoğan (2010), used the continuous method and the transfer matrix method for the static, dynamic and stability analysis of the tall building by modeling the building as a replacement generalized sandwich beam (GSB). In the development of this research project it was found that deriving the equations of the replacement generalized sandwich beam (GSB) using the continuous method and transfer matrix with energy formulation lead to the same results.

Hans (2002), Jigorel (2009), Chesnais (2010), Dinh (2020) and Franco (2021) developed the discrete periodic media homogenization method (HMPD) which consists of building a continuous global description of the structure based on its local properties (basic cell). In the development of this research project it was found that the dynamic analysis of some replacement beams leads to the same differential equations applying the method of homogenization of discrete periodic means (HMPD).

Moghadasi, H. (2015), proposed two replacement beams (GSB and GCTB) and analyzed them numerically using one-dimensional finite elements for static and dynamic analysis. In this research project he has continued with his investigation but solving the cases in an analytical, closed way and generalizing the external load profile for static analysis.

Zalka, K. (2020), developed a comprehensive study of the global structural analysis of regular tall buildings. In the development of this research project, it was found that deriving the differential equations of the replacement sandwich beam (SWB) using the continuous method with energetic formulation lead to the same differential equations developed in their research. Additionally, in this research project the structural analysis of regular and irregular buildings in height has been studied.

6 CONCLUSIONS AND RECOMMENDATIONS

6.1 CONCLUSIONS

In the formulation and development of this research, it was found very useful to use the energetic approach and the Hamilton principle for the deduction of the differential equations of the behavior of the replacement beams. It is important to mention that the methods of analytical calculations such as the one developed in this research project are fundamental because they allow to verify the methods considered exact and to know which structural parameters have an important influence on the behavior of the tall building to carry out a parametric study that in tall buildings are time consuming and expensive.

The analytical formulation allows for the analysis of tall buildings that are uniform in height and/or that may have stepwise structural changes in height. This is achieved using two approaches within the overall structural analysis of the tall building. Case 1 is based on the continuous method that leads to analytical and closed solutions, and consists of replacing the tall building with an equivalent replacement beam with its characteristic stiffnesses. Case 2 is based on the joint application of the continuous method and the transfer matrix method to replace the tall building with a stepped equivalent replacement beam, where an important advantage is that the transfer matrix remains constant at a size of 6x6 regardless of the number of floors in the building.

The main objective of this research is to develop a global structural analysis methodology for tall buildings using the continuous method and the transfer matrix method using an energy formulation. To this end, this research can contribute to four research areas in the structural analysis of tall buildings:

- Global structural analysis of replacement beams.
- Static structural analysis of the tall building,
- Dynamic structural analysis of the tall building, and
- Structural stability analysis of the tall building.

The main characteristics of this research are summarized as follows:

1. **Replacement beams for structural systems**: The static, dynamic and stability structural analysis of various replacement beams of one, two and three fields existing in the literature has been developed and solved analytically and in a closed way.

   It was found that the generalized sandwich beam (GSB) allows other replacement beams to be easily obtained as some characteristic stiffnesses are neglected and can be applied to any structural system. However, the sandwich beam (SWB), widely used in the literature due to its simplicity, turned out to be suitable for all the usual structural systems in structural engineering practice. In addition, more complex beams were also studied, such as the parallel coupling of a tensile Timoshenko beam and a 3-field Rotation Restraint Beam (GCTB), which makes it possible to reproduce very well the behavior of coupled shear walls that have high cant and where the local shear of the shear walls is important and cannot be neglected.

2. **Static structural analysis of the tall building**: The static structural analysis of the tall building has been developed and solved analytically and in a closed manner, defining control parameters such as the maximum displacement, interstory drift and global drift. The lateral load has a general profile that depends on a single parameter. The analytical solution allows to analyze tall buildings that are symmetric in plan (considered as buildings without torsion) and tall buildings that are asymmetric in one or two axes that have a strong lateral-torsional coupling.

3. **Dynamic structural analysis of the tall building**: The dynamic structural analysis of the tall building has been developed and resolved analytically and in a closed manner. The target control parameter of this section is the fundamental period of the building. Like static analysis it is also possible to analyze tall buildings with strong lateral-torsional coupling. In addition, rotational inertia was taken into account in the analytical procedure to evaluate its influence on dynamic behavior.

4. **Structural stability analysis of the tall building**: The structural stability analysis of the tall building has been developed and analytically solved. The objective control parameter of this section is the minimum critical load that leads to a global buckling of the building. Like static analysis it is also possible to analyze tall buildings with strong lateral-torsional coupling.

Numerical applications support that the analytical procedure proposed in this research project performs remarkably well for static, dynamic and stability structural analysis of tall building. With the purpose of generalizing the method, buildings with a minimum number of 5 floors were studied; however, the results improve markedly as we increase the height of the building.

6.2 PERSONAL CONTRIBUTIONS

The result of this research project will have a direct impact and benefit on the structural analysis of tall buildings, especially in the preliminary analysis and design phase where the structural engineer needs to make quick decisions, allowing the adoption of suitable replacement beam models without the need to resort to complex three-dimensional models that are impractical and expensive. The following contributions to the structural analysis of tall buildings are considered:

- Miranda (1999) proposed a general lateral load profile dependent on the parameter a and solved the static structural analysis of a classic CTB-type replacement beam. In this research project he has continued that research by performing static structural analysis of thirteen replacement beams from one, two and three fields. Therefore, the approximate static structural analysis of the tall building subjected to lateral load with general profile has been solved.

- Potzta (2002) proposed the equivalent sandwich beam to represent the tall building and developed the relationships that lead to the characteristic stiffnesses of the equivalent replacement beam. In this research project, the investigation has been continued and the equivalent generalized sandwich beam has been proposed to represent the tall building and the relationships that lead to its characteristic stiffnesses have been developed.

- Bozdogan (2010) developed the static, dynamic and stability structural analysis of a generalized sandwich beam replacement (GSB1) using the transfer matrix method and Moghadasi (2015) developed the static and dynamic structural analysis of the replacement beam generalized sandwich type (GSB1) using the one-dimensional finite element method. In this research project, the static, dynamic and stability structural analysis of the generalized sandwich-type replacement beam (GSB) has been solved in a closed way using the continuous method.

- Chesnais (2010) proposed and developed the dynamic structural analysis of the generalized sandwich beam replacement (GSB2). In this research project, the research has continued and the static, dynamic and stability structural analysis of the generalized sandwich beam replacement (GSB2) has been developed in a closed way with the continuous method and the transfer matrix method has been used for buildings with structural variability in height.
- Moghadasi (2015) proposed and developed the static structural analysis of one, two and three field CTB replacement beam subjected to uniformly distributed lateral load in height. In this research project, the investigation has continued and the static, dynamic and stability structural analysis of the CTB replacement beam of one, two and three fields has been developed in a closed form with the continuous method and the method of transfer matrix for buildings with structural variability in height.
- Static, dynamic and stability structural analysis of replacement beams known in the literature as bending beam (EBB), shear beam (SB), Timoshenko beam (TB), bending beam parallel coupling has been developed and shear beam (CTB) and sandwich beam (SWB). In addition, the static, dynamic and stability structural analysis of beams not well known in the literature but presented by Bozdogan (2010), Moghadasi (2015), Migliorati and Mangione (2015) and Chesnais (2010) as the generalized sandwich beam (GSB1 and GSB2), modified generalized sandwich beam (MGSB), modified parallel coupling of two beams (MCTB), generalized parallel coupling of two beams (GCTB) of one, two and three fields.
- The differential equations for the static analysis of the replacement beams and consequently of the tall building have been derived and solved. Generally, in the literature, equations are developed for each special case, such as the uniform load, the triangular load, etc.; however, in this research project the lateral load profile has been generalized.
- The differential equations for the dynamic analysis of the replacement beams have been derived and solved considering the rotational inertias. A subject little studied but that has an important influence for dynamic analysis.

- The differential equations for the stability analysis of the replacement beams and consequently of the tall building have been derived and solved. A subject little studied but very important due to the high slenderness of some tall buildings.
- A computer program has been developed to carry out the global structural analysis of tall buildings. This program performs the static, dynamic and stability analysis of the replacement beams and thus of the tall building. Subsequently, the efficiency of the program in numerical applications has been verified.

6.3 RECOMMENDATIONS

- Develop computer programs based on the global analysis of tall buildings, taking into account the soil-structure interaction.
- Perform a classification of the replacement beams based on the structural parameters of the building that allows the engineer to choose the appropriate replacement beam for the analysis. This is important for buildings that present some stiffnesses that are not of the same order of magnitude and that can be neglected.
- Evaluate the application of the continuous method and the transfer matrix method to the modeling of structures that have periodicity in their length such as railways, wooden structures, etc.
- The calculation of the global shear stiffness of structural and building systems is a delicate subject that requires studies to obtain closed equations. An approach developed by Franco (2021) is the construction of a single-story numerical model.
- Evaluate the application of static and dynamic structural analysis using the continuous method in confined masonry buildings with four and five floors, widely used in Peru.
- Carry out the global structural analysis of medium and tall buildings that have energy dissipation devices using the continuous method and the transfer matrix method.
- A fundamental hypothesis was to consider rigid diaphragms in the mezzanine slabs. It is suggested to extend the study to buildings that have mezzanine slabs with large openings and/or that are considered as flexible diaphragms.

- The global structural analysis of the replacement beams and the tall building has been developed in linear analysis. It is suggested to extend the study in the non-linear analysis.
- It is known in the literature that the centers of stiffness and shear in multi-story buildings are generally not coincident because their location depends not only on the geometric characteristics but also on the lateral forces. It is suggested to study the efficiency of the proposed analytical procedure by locating the center of stiffness and/or shear in the minimum torsion center defined as the fictitious reference place where the total sum of the squares of the rotations of the torsion floor of the building subjected to lateral inertial forces is minimal.

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8 ANNEXE

General Problem	Objectives	Variables	Dimensions	Methodology
	General	**Independent**		**Research type**
			• Characteristic stiffness.	• It is basic research.
• ¿It will be possible to develop a global structural analysis methodology for tall buildings by the continuous method and the transfer matrix method using an energetic formulation?	• Develop a global structural analysis methodology for tall buildings by the continuous method and the transfer matrix method using an energy formulation.	• Continuous method and transfer matrix method.	• Kinematic field.	• The method used is deductive.
			• Load.	• The level used is non-experimental.
Specific	**Specific**	**Dependents**		**Approach**
• ¿It will be possible to develop a static global structural analysis methodology of the tall building by the continuous method and the transfer matrix method using an energetic formulation?	• Develop a methodology for static global structural analysis of the tall building by the continuous method and the transfer matrix method using an energy formulation.		• Static analysis.	• In a first stage it will have a qualitative approach.
				• In a second stage it will have a quantitative approach.
• ¿It will be possible to develop a dynamic global structural analysis methodology of the tall building by the continuous method and the transfer matrix method using an energetic formulation?	• Develop a methodology for dynamic global structural analysis of the tall building by the continuous method and the transfer matrix method using an energy formulation.	• Global structural analysis of tall buildings.	• Dynamic analysis.	**Population**
				• The study population includes all tall buildings.
				Sample
• ¿It will be possible to develop a global structural stability analysis methodology of the tall building by the continuous method and the transfer matrix method using an energetic formulation?	• Develop a global structural analysis methodology for the stability of the tall building by the continuous method and the transfer matrix method using an energy formulation.		• Stability analysis.	• The study sample comprises a total of 1017 structural systems.